Highly entangled quantum spin chains and their extensions by semigroups

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Center for Theoretical Physics of the Universe, Institute for Basic Science

Workshop on “Matrix Models for Noncommutative Geometry and String Theory”
Erwin Schrödinger Institute (ESI), July 12, 2018
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F.S. and V. Korepin, arXiv:1806.04049
Outline

Introduction

Motzkin spin model

Colored Motzkin model

SIS Motzkin model

Colored SIS Motzkin model

Rényi entropy

Rényi entropy of Motzkin model

Summary and discussion
Quantum entanglement

- Most surprising feature of quantum mechanics,
  No analog in classical mechanics
Quantum entanglement

- Most surprising feature of quantum mechanics, No analog in classical mechanics
- From pure state of the full system $S$: $\rho = |\psi\rangle\langle\psi|$, reduced density matrix of a subsystem $A$: $\rho_A = \text{Tr}_{S-A} \rho$ can become mixed states, and has nonzero entanglement entropy

$$S_A = -\text{Tr}_A [\rho_A \ln \rho_A].$$

This is purely a quantum property.
Area law of entanglement entropy

- Ground states of quantum many-body systems with local interactions typically exhibit the area law behavior of the entanglement entropy: $S_A \propto (\text{area of } A)$

- Gapped systems in 1D are proven to obey the area law. [Hastings 2007]
Introduction 2

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- For gapless case, (1 + 1)-dimensional CFT violates logarithmically: $S_A = \frac{c}{3} \ln (\text{volume of } A)$. [Calabrese, Cardy 2009]
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- Belief for gapless case in $D$-dim. (over two decades): $S_A = O(L^{D-1} \ln L) \quad (L:\text{ length scale of } A)$
Area law of entanglement entropy

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- For gapless case, (1 + 1)-dimensional CFT violates logarithmically: $S_A = \frac{c}{3} \ln \text{(volume of } A\text{)}. \quad [\text{Calabrese, Cardy 2009}]
- Belief for gapless case in $D$-dim. (over two decades) : $S_A = O(L^{D-1} \ln L)$ ($L$: length scale of $A$)
- Recently, 1D solvable spin chain model which exhibit extensive entanglement entropy have been discussed.
  - Beyond logarithmic violation: $S_A \propto \sqrt{\text{(volume of } A\text{)}}$
    [Movassagh, Shor 2014], [Salberger, Korepin 2016]
    Counterexamples of the belief!
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Summary and discussion
Motzkin spin model 1

- 1D spin chain at sites $i \in \{1, 2, \cdots, 2n\}$
- Spin-1 state at each site can be regarded as up, down and flat steps;
  
  $|u\rangle \Leftrightarrow \uparrow$, \quad $|d\rangle \Leftrightarrow \downarrow$, \quad $|0\rangle \Leftrightarrow \rightarrow$
Motzkin spin model

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- Spin-1 state at each site can be regarded as up, down and flat steps;
  \[
  |u\rangle \Leftrightarrow \uparrow, \quad |d\rangle \Leftrightarrow \downarrow, \quad |0\rangle \Leftrightarrow \rightarrow
  \]
- Each spin configuration \( \Leftrightarrow \) length-2\( n \) walk in \((x, y)\) plane

Example)

\[
\begin{array}{ccccccc}
|u\rangle_1 & |0\rangle_2 & |d\rangle_3 & |u\rangle_4 & |u\rangle_5 & |d\rangle_6 \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow
\end{array}
\]
Motzkin spin model 2

Hamiltonian: \( H_{\text{Motzkin}} = H_{\text{bulk}} + H_{\text{bdy}} \)

- **Bulk part:** \( H_{\text{bulk}} = \sum_{j=1}^{2n-1} \Pi_{j,j+1} \),

\[
\Pi_{j,j+1} = |D\rangle_{j,j+1}\langle D| + |U\rangle_{j,j+1}\langle U| + |F\rangle_{j,j+1}\langle F|
\]

(local interactions) with

\[
|D\rangle \equiv \frac{1}{\sqrt{2}} (|0, d\rangle - |d, 0\rangle),
\]

\[
|U\rangle \equiv \frac{1}{\sqrt{2}} (|0, u\rangle - |u, 0\rangle),
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$\Leftrightarrow \sim$
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[Bravyi et al 2012]

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\]

\[
\Leftrightarrow \quad \Leftrightarrow \quad \sim \quad \sim \quad \sim \quad \Leftrightarrow \quad \Leftrightarrow \quad \sim \quad \sim \quad \sim \quad \text{“gauge equivalence”}.
\]
Motzkin spin model 3

Hamiltonian: $H_{\text{Motzkin}} = H_{\text{bulk}} + H_{\text{bdy}}$

$\text{Boundary part: } H_{\text{bdy}} = |d\rangle_1 \langle d| + |u\rangle_{2n} \langle u|$

$\Downarrow$

[Bravyi et al 2012]
Motzkin spin model 3

Hamiltonian: \( H_{Motzkin} = H_{bulk} + H_{bdy} \)

- Boundary part: \( H_{bdy} = |d\rangle_1 \langle d| + |u\rangle_{2n} \langle u| \)

\( \Downarrow \)

- \( H_{Motzkin} \) is the sum of projection operators.
  \( \Rightarrow \) Positive semi-definite spectrum

- We find the unique zero-energy ground state.
Motzkin spin model 3

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Hamiltonian: $H_{\text{Motzkin}} = H_{\text{bulk}} + H_{\text{bdy}}$

- Boundary part: $H_{\text{bdy}} = |d\rangle_1 \langle d| + |u\rangle_{2n} \langle u|$

$\Rightarrow$ Positive semi-definite spectrum

- $H_{\text{Motzkin}}$ is the sum of projection operators.

- We find the unique zero-energy ground state.
  - Each projector in $H_{\text{Motzkin}}$ annihilates the zero-energy state.
    $\Rightarrow$ Frustration free

- The ground state corresponds to random walks starting at $(0, 0)$ and ending at $(2n, 0)$ restricted to the region $y \geq 0$ (Motzkin Walks (MWs)).
In terms of $S = 1$ spin matrices

\[
S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S_\pm \equiv \frac{1}{\sqrt{2}} (S_x \pm iS_y) = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

\[
H_{\text{bulk}} = \frac{1}{2} \sum_{j=1}^{2n-1} \left[ 1_j 1_{j+1} - \frac{1}{4} S_z j S_z j+1 - \frac{1}{4} S^2_j S_j S_{j+1} + \frac{1}{4} S_j S^2_{j+1} \\
- \frac{3}{4} S^2_{zj} S^2_{zj+1} + S_+ j (S_z S_-)_{j+1} + S_- j (S_+ S_z)_{j+1} - (S_- S_z)_j S_+ j+1 \\
- (S_+ S_z)_j S_- j+1 - (S_+ S_z)_{j+1} - (S_z S_+)_j (S_z S_-)_{j+1} \right],
\]

\[
H_{\text{bdy}} = \frac{1}{2} (S^2_z - S_z)_1 + \frac{1}{2} (S^2_z + S_z)_{2n}
\]

Quartic spin interactions
Motzkin spin model 5

Example) $2n = 4$ case,
MWs:

\[
\begin{align*}
|P_4\rangle &= \frac{1}{\sqrt{9}} \left[ |0000\rangle + |ud00\rangle + |0ud0\rangle + |00ud\rangle \\
&\quad + |u0d0\rangle + |0u0d\rangle + |u00d\rangle + |udud\rangle \\
&\quad + |uudd\rangle \right].
\end{align*}
\]
Motzkin spin model 6

[Bravyi et al 2012]

Note
Forbidden paths for the ground state

1. Path entering $y < 0$ region

2. Path ending at nonzero height

Forbidden by $H_{bdy}$
Entanglement entropy of the subsystem $A = \{1, 2, \cdots, n\}$:

- Normalization factor of the ground state $|P_{2n}\rangle$ is given by the number of MWs of length $2n$: $M_{2n} = \sum_{k=0}^{n} C_k \binom{2n}{2k}$.

$$C_k = \frac{1}{k+1} \binom{2k}{k}: \text{Catalan number}$$
Motzkin spin model 7

Entanglement entropy of the subsystem \( A = \{1, 2, \cdots, n\} \):

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\[
C_k = \frac{1}{k+1} \binom{2k}{k} : \text{Catalan number}
\]

- Consider to trace out the density matrix \( \rho = |P_{2n}\rangle \langle P_{2n}| \) w.r.t. the subsystem \( B = \{n+1, \cdots, 2n\} \).

Schmidt decomposition:

\[
|P_{2n}\rangle = \sum_{h \geq 0} \sqrt{p_{n,n}^{(h)}} |P_{n}^{(0 \rightarrow h)}\rangle \otimes |P_{n}^{(h \rightarrow 0)}\rangle
\]

with \( p_{n,n}^{(h)} \equiv \left( \frac{M_{n}^{(h)}}{M_{2n}} \right)^2 \).

\[\uparrow\]
Paths from \((0, 0)\) to \((n, h)\)
Motzkin spin model 8

- $M_n^{(h)}$ is the number of paths in $P_n^{(0\rightarrow h)}$.
  For $n \to \infty$, Gaussian distribution

\[ p_{n,n}^{(h)} \sim \frac{3\sqrt{6}}{\sqrt{\pi}} \frac{(h+1)^2}{n^{3/2}} e^{-\frac{3}{2} \frac{(h+1)^2}{n}} \times [1 + O(1/n)]. \]

- Reduced density matrix

\[ \rho_A = \text{Tr}_B \rho = \sum_{h \geq 0} p_{n,n}^{(h)} |P_n^{(0\rightarrow h)}\rangle \langle P_n^{(0\rightarrow h)}| \]

- Entanglement entropy

\[ S_A = - \sum_{h \geq 0} p_{n,n}^{(h)} \ln p_{n,n}^{(h)} \]

\[ = \frac{1}{2} \ln n + \frac{1}{2} \ln \frac{2\pi}{3} + \gamma - \frac{1}{2} \quad (\gamma: \text{Euler constant}) \]

up to terms vanishing as $n \to \infty$. 

[Bravyi et al 2012]
Notes

- The system is critical (gapless).
  \( S_A \) is similar to the \((1 + 1)\)-dimensional CFT with \( c = 3/2 \).
Notes

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  \[ S_A \] is similar to the \((1 + 1)\)-dimensional CFT with \( c = 3/2 \).
- But, gap scales as \( O(1/n^z) \) with \( z \geq 2 \).
  The system cannot be described by relativistic CFT.

Lifshitz type ?
Different \( z \) depending on excited states (Multiple dynamics) ?

[Chen, Fradkin, Witczak-Krempa 2017]
Motzkin spin model 9

[Bravyi et al 2012]

Notes

- The system is critical (gapless).
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- But, gap scales as \( O(1/n^z) \) with \( z \geq 2 \).
  The system cannot be described by relativistic CFT.
  Lifshitz type?
  Different \( z \) depending on excited states (Multiple dynamics)?
  [Chen, Fradkin, Witczak-Krempa 2017]
- Excitations have not been much investigated.
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Rényi entropy of Motzkin model

Summary and discussion
Colored Motzkin spin model 1

- Introducing color d.o.f. $k = 1, 2, \cdots, s$ to up and down spins as

$$ |u^k\rangle \leftrightarrow \begin{array}{c} k \end{array}, \quad |d^k\rangle \leftrightarrow \begin{array}{c} k \end{array}, \quad |0\rangle \leftrightarrow \begin{array}{c} \rightarrow \end{array} $$

Color d.o.f. decorated to Motzkin Walks
Colored Motzkin spin model 1

- Introducing color d.o.f. $k = 1, 2, \cdots, s$ to up and down spins as

$$
|u^k\rangle \leftrightarrow \uparrow, \quad |d^k\rangle \leftrightarrow \downarrow, \quad |0\rangle \leftrightarrow \to
$$

Color d.o.f. decorated to Motzkin Walks

- Hamiltonian $H_{cMotzkin} = H_{bulk} + H_{bdy}$
  - Bulk part consisting of local interactions:

$$
H_{bulk} = \sum_{j=1}^{2n-1} \left( \Pi_{j,j+1} + \Pi_{j,j+1}^{cross} \right),
$$

$$
\Pi_{j,j+1} = \sum_{k=1}^{s} \left[ |D^k\rangle_{j,j+1}\langle D^k| + |U^k\rangle_{j,j+1}\langle U^k| + |F^k\rangle_{j,j+1}\langle F^k| \right]
$$

with
Colored Motzkin spin model 2

\[ D^k \equiv \frac{1}{\sqrt{2}} \left( |0, d^k\rangle - |d^k, 0\rangle \right), \]
\[ U^k \equiv \frac{1}{\sqrt{2}} \left( |0, u^k\rangle - |u^k, 0\rangle \right), \]
\[ F^k \equiv \frac{1}{\sqrt{2}} \left( |0, 0\rangle - |u^k, d^k\rangle \right), \]

and

\[ \prod_{j,j+1}^{\text{cross}} = \sum_{k \neq k'} |u^k, d^{k'}\rangle_{j,j+1} \langle u^k, d^{k'}|. \]

⇒ Colors should be matched in up and down pairs.

Boundary part

\[ H_{\text{bdy}} = \sum_{k=1}^{s} \left( |d^k\rangle_1 \langle d^k| + |u^k\rangle_{2n} \langle u^k| \right). \]
Colored Motzkin spin model 3

- Still unique ground state with zero energy
Colored Motzkin spin model 3

- Still unique ground state with zero energy
- Example) $2n = 4$ case,

\[ |P_4\rangle = \frac{1}{\sqrt{1 + 6s + 2s^2}} \left[ |0000\rangle + \sum_{k=1}^{s} \left\{ |u^k d^k 00\rangle + \cdots + |u^k 00 d^k \rangle \right\} ight. 
+ \sum_{k,k' = 1}^{s} \left\{ |u^k d^k u^k' d^k' \rangle + |u^k u^k' d^k d^k' \rangle \right\}. \]
Entanglement entropy

- Paths from $(0, 0)$ to $(n, h)$, $P_n^{(0\rightarrow h)}$, have $h$ unmatched up steps.

Let $\tilde{P}_n^{(0\rightarrow h)}(\{\kappa_m\})$ be paths with the colors of unmatched up steps frozen.

\[(\text{unmatched up from height } (m - 1) \text{ to } m) \rightarrow u^{\kappa_m}\]

- Similarly,

\[P_n^{(h\rightarrow 0)} \rightarrow \tilde{P}_n^{(h\rightarrow 0)}(\{\kappa_m\}),\]

\[(\text{unmatched down from height } m \text{ to } (m - 1)) \rightarrow d^{\kappa_m}\]

- The numbers satisfy $M_n^{(h)} = s^h \tilde{M}_n^{(h)}$. 
Example

$2n = 8$ case, $h = 2$
Colored Motzkin spin model 6

- Schmidt decomposition

\[
| P_{2n} \rangle = \sum_{h \geq 0} \sum_{\kappa_1 = 1}^{s} \cdots \sum_{\kappa_h = 1}^{s} \sqrt{p_{n,n}^{(h)}} \\
\times | \tilde{P}_{n}^{(0 \rightarrow h)}(\{\kappa_m\}) \rangle \otimes | \tilde{P}_{n}^{(h \rightarrow 0)}(\{\kappa_m\}) \rangle
\]

with

\[
p_{n,n}^{(h)} = \left( \frac{\tilde{M}_{n}^{(h)}}{M_{2n}} \right)^2.
\]

- Reduced density matrix

\[
\rho_A = \sum_{h \geq 0} \sum_{\kappa_1 = 1}^{s} \cdots \sum_{\kappa_h = 1}^{s} p_{n,n}^{(h)} \\
\times | \tilde{P}_{n}^{(0 \rightarrow h)}(\{\kappa_m\}) \rangle \langle \tilde{P}_{n}^{(0 \rightarrow h)}(\{\kappa_m\}) |.
\]
For $n \to \infty$, 

$$p_{n,n}^{(h)} \sim \frac{\sqrt{2} s^{-h}}{\sqrt{\pi} (\sigma n)^{3/2}} (h + 1)^2 e^{-\frac{(h+1)^2}{2\sigma n}} \times [1 + O(1/n)]$$

with $\sigma \equiv \frac{\sqrt{s}}{2\sqrt{s+1}}$. Note: Effectively $h \lesssim O(\sqrt{n})$.

Entanglement entropy

$$S_A = - \sum_{h \geq 0} s^h p_{n,n}^{(h)} \ln p_{n,n}^{(h)}$$
Colored Motzkin spin model 7

For \( n \to \infty \),

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Entanglement entropy

\[
S_A = - \sum_{h \geq 0} s^h p_{n,n}^{(h)} \ln p_{n,n}^{(h)}
\]

\[
= (2 \ln s) \sqrt{\frac{2\sigma n}{\pi}} + \frac{1}{2} \ln n + \frac{1}{2} \ln(2\pi\sigma) + \gamma - \frac{1}{2} - \ln s
\]

up to terms vanishing as \( n \to \infty \).

Grows as \( \sqrt{n} \).
Comments

Matching color $\Rightarrow s^{-h}$ factor in $p^{(h)}_{n,n}$
$\Rightarrow$ crucial to $O(\sqrt{n})$ behavior in $S_A$
Colored Motzkin spin model 8

Comments

- Matching color $\Rightarrow s^{-h}$ factor in $p_{n,n}^{(h)}$
  $\Rightarrow$ crucial to $O(\sqrt{n})$ behavior in $S_A$

- Typical configurations:

\[ h = O(\sqrt{n}) \]

+ (equivalence moves).
Colored Motzkin spin model 8

[Movassagh, Shor 2014]

Comments

Matching color \( \Rightarrow \) \( s^{-h} \) factor in \( p_{n,n}^{(h)} \)
\( \Rightarrow \) crucial to \( O(\sqrt{n}) \) behavior in \( S_A \)

Typical configurations:

For spin 1/2 chain (only up and down), the model in which similar behavior exhibits in colored as well as uncolored cases has been constructed. (Fredkin model)  

[Salberger, Korepin 2016]
Correlation functions

\[
\langle S_z, 1 S_z, 2n \rangle_{\text{connected}} \rightarrow -0.034... \times \frac{s^3 - s}{6} \neq 0 \quad (n \rightarrow \infty)
\]

⇒ Violation of cluster decomposition property for \( s > 1 \)

(Strong correlation due to color matching)
Correlation functions

\[ \langle S_z, 1S_z, 2n \rangle_{\text{connected}} \to -0.034... \times \frac{s^3 - s}{6} \neq 0 \quad (n \to \infty) \]

⇒ Violation of cluster decomposition property for \( s > 1 \)
   (Strong correlation due to color matching)

Deformation of models to achieve the volume law behavior
(\( S_A \propto n \))

Weighted Motzkin/Dyck walks

[Movassagh, Shor 2014]

[Dell’Anna et al, 2016]

[Zhang et al, Salberger et al 2016]
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Symmetric Inverse Semigroups (SISs)

- **Inverse Semigroup** (⊂ Semigroup):
  An unique inverse exists for every element.
  But, no unique identity (partial identities).
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  Semigroup version of the symmetric group $S_k$

  $S_p^k \ (p = 1, \ldots, k)$
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  Semigroup version of the symmetric group $S_k$
  $$S_p^k \ (p = 1, \cdots, k)$$

- $x_{a,b} \in S_1^k$ maps $a$ to $b$. ($a, b \in \{1, \cdots, k\}$)
  Product rule: $x_{a,b} \ast x_{c,d} = \delta_{b,c} x_{a,d}$

  - $x_{1,2} \ast x_{2,1} = x_{1,1}$, $x_{2,1} \ast x_{1,2} = x_{2,2}$
  - $(x_{1,2})^{-1} = x_{2,1}$

  (partial identities) (unique inverse)
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- $(x_{1,2})^{-1} = x_{2,1}$ (unique inverse)

- $x_{a_1,a_2; b_1,b_2} \in S_2^k$ etc, ...
Symmetric Inverse Semigroups (SISs)

- **Inverse Semigroup** (⊂ Semigroup):
  An unique inverse exists for every element.
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  $S^k_p$ ($p = 1, \cdots, k$)

- $x_{a,b} \in S^k_1$ maps $a$ to $b$. ($a, b \in \{1, \cdots, k\}$)
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  \[ x_{1,2} \ast x_{2,1} = x_{1,1}, \quad x_{2,1} \ast x_{1,2} = x_{2,2} \]

  (partial identities)

  \[ (x_{1,2})^{-1} = x_{2,1} \]

  (unique inverse)

- $x_{a_1,a_2; b_1,b_2} \in S^k_2$ etc, ...

  $S^k_k \equiv S_k$
Change the spin d.o.f. as $|x_{a,b}\rangle$ with $a, b \in \{1, 2, \cdots, k\}$.

$\begin{align*}
&\text{a < b case: ‘up’ } \iff \quad a \quad \quad \quad \quad \quad \quad b \\
&\text{a > b case: ‘down’ } \iff \quad a \quad \quad \quad b \\
&\text{a = b case: ‘flat’ } \iff \quad a \quad \quad \quad b
\end{align*}$
Change the spin d.o.f. as $|x_{a,b}\rangle$ with $a, b \in \{1, 2, \cdots, k\}$.

- $a < b$ case: ‘up’ $\iff$ [Diagram: $a \uparrow b$]
- $a > b$ case: ‘down’ $\iff$ [Diagram: $a \downarrow b$]
- $a = b$ case: ‘flat’ $\iff$ [Diagram: $a \leftrightarrow b$]

We regard the configuration of adjacent sites $|(x_{a,b})_j\rangle |(x_{c,d})_{j+1}\rangle$ as a connected path for $b = c$.

c.f.) Analogous to the product rule of Symmetric Inverse Semigroup ($S_1^k$):

$x_{a,b} \ast x_{c,d} = \delta_{b,c} x_{a,d}$

$a, b$: semigroup indices

Inner product: $\langle x_{a,b} | x_{c,d} \rangle = \delta_{a,c} \delta_{b,d}$

Let us consider the $k = 3$ case.
Maximum height is lower than the original Motzkin case.
Hamiltonian $H_{S_{31} Motzkin} = H_{bulk} + H_{bulk, disc} + H_{bdy}$

- $H_{bulk}$: local interactions corresponding to the following moves:

  (Down) $a \to b \sim a \to b \quad (a > b)$

  (Up) $a \to b \sim a \to b \quad (a < b)$

  (Flat) $a \to a \to a \sim a \to b \quad (a < b)$

  (Wedge) $3 \to 1 \to 3 \sim 3 \to 2 \to 3$
SIS Motzkin model 4

\[ H_{\text{bulk, disc}} \text{ lifts disconnected paths to excited states.} \]

\[ \Pi|\psi\rangle: \text{projector to } |\psi\rangle \]

\[ H_{\text{bulk, disc}} = \sum_{j=1}^{2n-1} \sum_{a,b,c,d=1; b \neq c}^{3} \Pi |(x_{a,b})_j, (x_{c,d})_{j+1}\rangle \]
$H_{bulk, disc}$ lifts disconnected paths to excited states.

\[ H_{bulk, disc} = \sum_{j=1}^{2n-1} \sum_{a,b,c,d=1; b \neq c}^{3} \Pi |(x_{a,b})_j, (x_{c,d})_{j+1}\rangle \]

$H_{bdy} = \sum_{a<b} \Pi |(x_{a,b})_1\rangle + \sum_{a>b} \Pi |(x_{a,b})_{2n}\rangle$

\[ + \Pi |(x_{1,3})_1, (x_{3,2})_2, (x_{2,1})_3\rangle + \Pi |(x_{1,2})_{2n-2}, (x_{2,3})_{2n-1}, (x_{3,1})_{2n}\rangle \]

The last 2 terms have no analog to the original Motzkin model.
Ground states correspond to connected paths starting at 
\((0, 0)\), ending at \((2n, 0)\) and not entering \(y < 0\).
Ground states correspond to connected paths starting at 
(0, 0), ending at (2n, 0) and not entering y < 0.

The ground states have 5 fold degeneracy according to the 
initial and final semigroup indices:
(1, 1), (1, 2), (2, 1), (2, 2) and (3, 3) sectors
The (3, 3) sector is trivial, consisting of only one path:
\[ X_{3,3}X_{3,3} \cdots X_{3,3}. \]
Ground states correspond to connected paths starting at $(0, 0)$, ending at $(2n, 0)$ and not entering $y < 0$. The ground states have 5 fold degeneracy according to the initial and final semigroup indices: $(1, 1), (1, 2), (2, 1), (2, 2)$ and $(3, 3)$ sectors. The $(3, 3)$ sector is trivial, consisting of only one path: \( x_{3,3}x_{3,3} \cdots x_{3,3} \).

The number of paths can be obtained by recursion relations. For length-$n$ paths from the semigroup index $a$ to $b$ ($P_{n,a\rightarrow b}$),

\[
P_{n,1\rightarrow 1} = x_{1,1}P_{n-1,1\rightarrow 1} + x_{1,2} \sum_{i=1}^{n-2} P_{i,2\rightarrow 2} x_{2,1} P_{n-2-i,1\rightarrow 1}
\]

\[
+ x_{1,3} \sum_{i=1}^{n-2} P_{i,3\rightarrow 3} x_{3,1} P_{n-2-i,1\rightarrow 1}
\]

\[
+ x_{1,3} \sum_{i=1}^{n-2} P_{i,3\rightarrow 3} x_{3,2} P_{n-2-i,2\rightarrow 1}, \quad \text{etc.}
\]
Result

- The entanglement entropies $S_{A,1\rightarrow 1}$, $S_{A,1\rightarrow 2}$, $S_{A,2\rightarrow 1}$ and $S_{A,2\rightarrow 2}$ take the same form as in the case of the Motzkin model.

Logarithmic violation of the area law

- The form of $p_n^{(h)} \sim \frac{(h+1)^2}{n^{3/2}} e^{-\text{const.} \frac{(h+1)^2}{n}}$ is universal.

- $S_{A,3\rightarrow 3} = 0$. 
There are excited states corresponding to disconnected paths. Example) One such path in $2n = 6$ case,
There are excited states corresponding to disconnected paths. Example) One such path in $2n = 6$ case,

Corresponding excited state: $|P_{3,1\rightarrow1}\rangle \otimes |P_{3,2\rightarrow1}^{(1\rightarrow0)}\rangle$

Each connected component has no entanglement with other components. “2nd quantization” of paths
There are excited states corresponding to disconnected paths. Example) One such path in $2n = 6$ case,

Corresponding excited state: $|P_{3,1\rightarrow 1}\rangle \otimes |P_{3,2\rightarrow 1}^{(1\rightarrow 0)}\rangle$

Each connected component has no entanglement with other components. “2nd quantization” of paths

$\Rightarrow$ 2pt connected correlation functions of local operators belonging to separate connected components vanish.

$\Rightarrow$ Localization!
Introduction

Motzkin spin model

Colored Motzkin model

SIS Motzkin model

Colored SIS Motzkin model

Rényi entropy

Rényi entropy of Motzkin model

Summary and discussion
The SIS $S_2^3$

- 18 elements $x_{ab,cd}$ with $ab \in \{12, 23, 31\}$ and $cd \in \{12, 23, 31, 21, 32, 13\}$ satisfying

$$x_{ab,cd} \ast x_{ef,gh} = \delta_{c,e} \delta_{d,f} x_{ab,gh} + \delta_{c,f} \delta_{d,e} x_{ab,hg}.$$ 

- can be regarded as 2 sets of $S_1^3$. ⇒ color d.o.f.
Colored SIS Motzkin model 1

The SIS $S^3_2$

- 18 elements $x_{ab,cd}$ with $ab \in \{12, 23, 31\}$ and $cd \in \{12, 23, 31, 21, 32, 13\}$ satisfying

$$x_{ab,cd} \times x_{ef,gh} = \delta_{c,e} \delta_{d,f} x_{ab,gh} + \delta_{c,f} \delta_{d,e} x_{ab,hg}.$$ 

- can be regarded as 2 sets of $S^3_1$. ⇒ color d.o.f.

- Spin variables: $x^s_{a,b}$ ($s = 1, 2$) ($a, b = 1, 2, 3$)

- The new moves (C moves) introduced to the Hamiltonian.

$\cancel{a \rightarrow a} \sim \cancel{a \rightarrow a}$
Colored SIS Motzkin model 2

Hamiltonian: $H_{cs31\text{Motzkin}} = H_{bulk} + H_{bulk,\text{disc}} + H_{bdy}$

- In $H_{bulk}$, (Down), (Up) and (Flat) are essentially the same as before.

(Down) \[ a \xrightarrow{s} a \xrightarrow{s} b \sim a \xrightarrow{s} b \xrightarrow{s} b \quad (a > b) \]

(Up) \[ a \xrightarrow{s} \quad a \xrightarrow{s} b \sim a \xrightarrow{s} b \quad (a < b) \]

(Flat) \[ a \xrightarrow{s} a \xrightarrow{s} a \sim a \xrightarrow{s} b \xrightarrow{s} a \quad (a < b) \]
Colored SIS Motzkin model 3

- **Wedge move:**

\[
(Wedge) \quad \begin{array}{c}
3 \quad s \quad s' \\
\downarrow 1 \\
3
\end{array} \sim \begin{array}{c}
3 \quad s \quad s' \\
\downarrow 2 \\
3
\end{array}
\]

- **Cross move:**

\[
(Cross)_{j,j+1} = \sum_{b > a,c} \left[ \prod |(x_{a,b}^1)_{j} (x_{b,c}^2)_{j+1} \rangle + \prod |(x_{a,b}^2)_{j} (x_{b,c}^1)_{j+1} \rangle \right]
\]

forbids unmatched up and down steps in ground states.

\[
H_{bulk} = \mu \sum_{j=1}^{2n} C_j + \sum_{j=1}^{2n-1} [(Down)_{j,j+1} + (Up)_{j,j+1}
+ (Flat)_{j,j+1} + (Wedge)_{j,j+1} + (Cross)_{j,j+1}]
\]
Colored SIS Motzkin model 4

\[ H_{\text{bulk, disc}} = \sum_{j=1}^{2n-1} \sum_{a,b,c,d=1}^{3} \sum_{b \neq c}^{2} \prod |(x_{a,b}^s)_{j}, (x_{c,d}^t)_{j+1} \rangle \]

\[ H_{\text{bdy}} = \sum_{a>b}^{2} \prod |(x_{a,b}^s)_{1} \rangle + \sum_{a<b}^{2} \prod |(x_{a,b}^s)_{2n} \rangle \]

\[ + \sum_{s,t=1}^{2} \prod |(x_{1,3}^s)_{1}, (x_{3,2}^s)_{2}, (x_{2,1}^t)_{3} \rangle \]

\[ + \sum_{s,t=1}^{2} \prod |(x_{1,2}^s)_{2n-2}, (x_{2,3}^t)_{2n-1}, (x_{3,1}^t)_{2n} \rangle \]
5 ground states of (1, 1), (1, 2), (2, 1), (2, 2), (3, 3) sectors

Quantum phase transition between $\mu > 0$ and $\mu = 0$ in the 4 sectors except (3, 3).

- For $\mu > 0$,

$$S_A = (2 \ln 2) \sqrt{\frac{2\sigma n}{\pi}} + \frac{1}{2} \ln n + \frac{1}{2} \ln(2\pi\sigma) + \gamma - \frac{1}{2} + \ln \frac{3}{2^{1/3}}$$

with $\sigma \equiv \frac{\sqrt{2} - 1}{9\sqrt{2}}$.

- For $\mu = 0$, colors 1 and 2 decouple.

$$S_A \propto \ln n.$$
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Rényi entropy

Rényi entropy of Motzkin model

Summary and discussion
Rényi entropy

- Rényi entropy has further importance than the von Neumann entanglement entropy:

\[ S_{A, \alpha} = \frac{1}{1 - \alpha} \ln \text{Tr} A \rho_A^\alpha \quad \text{with} \quad \alpha > 0 \quad \text{and} \quad \alpha \neq 1. \]
Rényi entropy

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  \[ S_{A,\alpha} = \frac{1}{1 - \alpha} \ln \text{Tr}_A \rho_A^{\alpha} \]

  with \( \alpha > 0 \) and \( \alpha \neq 1 \).

• Generalization of the von Neumann entanglement entropy:
  \[ \lim_{\alpha \rightarrow 1} S_{A,\alpha} = S_A \]
Rényi entropy

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- Reconstructs the whole spectrum of the entanglement Hamiltonian \( H_{\text{ent},A} \equiv -\ln \rho_A \).
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- For \( S_{A, \alpha} \) (0 < \( \alpha < 1 \)), the gapped systems in 1D is proven to obey the area law.

[Huang, 2015]
Rényi entropy

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[Huang, 2015]

Here, I give a review of Motzkin spin chain and analytically compute its Rényi entropy of half-chain.

New phase transition found at \( \alpha = 1 \)!
Introduction

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Colored Motzkin model

SIS Motzkin model

Colored SIS Motzkin model

Rényi entropy

Rényi entropy of Motzkin model

Summary and discussion
What we compute is the asymptotic behavior of

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What we compute is the asymptotic behavior of

$$S_{A, \alpha} = \frac{1}{1 - \alpha} \ln \sum_{h=0}^{n} s^h \left( \rho_{n,n}^{(h)} \right)^\alpha.$$ 

For colorless case ($s = 1$), we obtain

$$S_{A, \alpha} = \frac{1}{2} \ln n + \frac{1}{1 - \alpha} \ln \Gamma \left( \alpha + \frac{1}{2} \right)$$

$$- \frac{1}{2(1 - \alpha)} \left\{ (1 + 2\alpha) \ln \alpha + \alpha \ln \frac{\pi}{24} + \ln 6 \right\}$$

up to terms vanishing as $n \to \infty$. 
What we compute is the asymptotic behavior of

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up to terms vanishing as \( n \to \infty \).

- Logarithmic growth
- Reduces to \( S_A \) in the \( \alpha \to 1 \) limit.
- Consistent with half-chain case in the result in [Movassagh, 2017]
Réyni entropy of Motzkin model 2

Colored case ($s > 1$)

- The summand $s^h \left( p_{n,n}^{(h)} \right)^\alpha$ has a factor $s^{(1-\alpha)h}$. 
Réyni entropy of Motzkin model 2

Colored case ($s > 1$)

- The summand $s^h \left(p_{n,n}^{(h)}\right)\alpha$ has a factor $s^{(1-\alpha)h}$.
  
  For $0 < \alpha < 1$, exponentially growing (colored case ($s > 1$)).
  
  $\Rightarrow$ Saddle point value of the sum: $h_* = O(n)$
Réyni entropy of Motzkin model 2

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- Saddle point analysis for the sum leads to

\[
S_{A,\alpha} = n \frac{2\alpha}{1 - \alpha} \ln \left[ \sigma \left( s^{\frac{1-\alpha}{2\alpha}} + s^{-\frac{1-\alpha}{2\alpha}} + s^{-1/2} \right) \right] + \frac{1 + \alpha}{2(1 - \alpha)} \ln n + C(s, \alpha)
\]

with \(C(s, \alpha)\) being \(n\)-independent terms.
Réyni entropy of Motzkin model 2

[ F.S., Korepin, 2018 ]

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  \]

  with \(C(s, \alpha)\) being \(n\)-independent terms.

- The saddle point value is \(h_* = n \left[ \frac{1}{s^{\frac{1}{2\alpha}}} - \frac{1}{s^{-\frac{1}{2\alpha}}} \right] \frac{1}{s^{\frac{1}{2\alpha}} + s^{-\frac{1}{2\alpha}} + 1} + O(n^0)\).
Réyni entropy of Motzkin model 2

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- The saddle point value is \(h_* = n \frac{s^{\frac{1}{2\alpha}} - s^{\frac{1}{2\alpha}} - s^{-\frac{1}{2\alpha}}}{s^{\frac{1}{2\alpha}} + s^{\frac{1}{2\alpha}} + 1} + O(n^0)\).

- Linear growth in \(n\).
Réyni entropy of Motzkin model 2

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- The summand \(s^h \left( p_{n,n}^{(h)} \right)^\alpha\) has a factor \(s^{(1-\alpha)h}\).

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with \(C(s, \alpha)\) being \(n\)-independent terms.

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- Linear growth in \(n\).

- Note: \(\alpha \to 1\) or \(s \to 1\) limit does not commute with the \(n \to \infty\) limit.
Rényi entropy for $\alpha > 1$

- For $\alpha > 1$, the factor $s^{(1-\alpha)h}$ in the summand $s^h \left( \rho_{n,n}^{(h)} \right)^{\alpha}$ exponentially decays.
Rényi entropy for $\alpha > 1$

- For $\alpha > 1$, the factor $s^{(1-\alpha)h}$ in the summand $s^h \left( p_n^{(h)} \right)^\alpha$ exponentially decays.
  
  $\Rightarrow h \lesssim O \left( \frac{1}{(\alpha-1) \ln s} \right) = O(n^0)$ dominantly contributes to the sum.
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- The result:

$$S_{A,\alpha} = \frac{3\alpha}{2(\alpha - 1)} \ln n + O(n^0).$$
Rényi entropy for $\alpha > 1$

▶ For $\alpha > 1$, the factor $s^{(1-\alpha)h}$ in the summand $s^h \left( p^{(h)}_{n,n} \right)^\alpha$ exponentially decays.

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▶ Logarithmic growth
Rényi entropy for $\alpha > 1$

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- Logarithmic growth

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Rényi entropy of Motzkin model 4

Phase transition

- $S_{A\alpha}$ grows as $O(n)$ for $0 < \alpha < 1$ while as $O(\ln n)$ for $\alpha > 1$. 
Réyni entropy of Motzkin model 4

Phase transition

- $S_{A\alpha}$ grows as $O(n)$ for $0 < \alpha < 1$ while as $O(\ln n)$ for $\alpha > 1$.
  - Non-analytic behavior at $\alpha = 1$ (Phase transition)

In terms of the entanglement Hamiltonian, $\text{Tr}_A \rho_\alpha A = \text{Tr}_A e^{-\alpha H_{\text{ent}}}$, $A_\alpha$: "inverse temperature"

- $0 < \alpha < 1$: "high temperature" (Height of dominant paths $h = O(n)$)
- $\alpha > 1$: "low temperature" (Height of dominant paths $h = O(\sqrt{n})$)
Phase transition

- $S_A \alpha$ grows as $O(n)$ for $0 < \alpha < 1$ while as $O(\ln n)$ for $\alpha > 1$.
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Réyni entropy of Motzkin model 4

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Réyni entropy of Motzkin model 4

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  - $\alpha > 1$: “low temperature”
    (Height of dominant paths $h = O(n^0)$)

- The transition point $\alpha = 1$ itself forms the third phase.

\[ S_{A,\alpha} : \quad O(\ln n) \quad O(\sqrt{n}) \quad O(n) \]

\[ h : \quad O(n^0) \quad O(\sqrt{n}) \quad O(n) \]
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Rényi entropy of Motzkin model

Summary and discussion
Summary and discussion 1

Summary

- We have reviewed the (colored) Motzkin spin models which yield large entanglement entropy proportional to the square root of the volume.
Summary and discussion 1

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- We have reviewed the (colored) Motzkin spin models which yield large entanglement entropy proportional to the square root of the volume.
- We have extended the models by introducing additional d.o.f. based on Symmetric Inverse Semigroups.
  - Quantum phase transitions
    - In uncolored case ($S_1^3$), log. violation v.s. area law $O(1)$ for $S_A$
    - In colored case ($S_2^3$), $\sqrt{n}$ v.s. $\ln n$ for $S_A$. 

"2nd quantized paths"
Summary and discussion 1

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- We have reviewed the (colored) Motzkin spin models which yield large entanglement entropy proportional to the square root of the volume.
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- Semigroup extension of the Fredkin model

[Padmanabhan, F.S., Korepin 2018]
Summary and discussion 1

Summary

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- Semigroup extension of the Fredkin model
  [Padmanabhan, F.S., Korepin 2018]
- As a feature of the extended models, Anderson-like localization occurs in excited states corresponding to disconnected paths.
  - “2nd quantized paths”.
Summary and discussion 2

Summary

▶ We have analytically computed the Rényi entropy of half-chain in the Motzkin model.
  ▶ Phase transition at \( \alpha = 1 \) (New phase transition!)
    No analog for other spin chains investigated so far (XX, XY, AKLT,...).
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Summary and discussion 3

Future directions

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Summary and discussion

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Thank you very much for your attention!
By adding the balancing term to the Hamiltonian

\[ \lambda_2 \sum_{j=1}^{2n-1} \left( \prod \left| (x_1,3)_j, (x_3,2)_{j+1} \right> + \prod \left| (x_2,3)_j, (x_3,1)_{j+1} \right> \right) \]

with \( \lambda_1 \) put to the term, quantum phase transition takes place in the 4 sectors except (3, 3):

\[ \lambda_1 \quad S_A \propto \ln n \]

\[ 0 \quad S_A = O(1) \text{ (area law)} \]

\( \lambda_1, \lambda_2 > 0 \) is not frustration free (here, we do not consider).