Exercises for the „PS Advanced Topics in global Number Theory“
– Session on the 24th of April

Recall that a function $\psi : (a,b) \to \mathbb{R}$ on an open interval $-\infty \leq a < b \leq \infty$ is called ***convex***, if it satisfies the inequality

$$\psi((1-\lambda)x + \lambda y) \leq (1-\lambda)\psi(x) + \lambda\psi(y)$$

for all $x, y \in (a,b)$ and $0 \leq \lambda \leq 1$.

Exc. 21) Show that a differentiable function is convex, if and only if $a < s < t < b$ implies $\psi'(s) \leq \psi'(t)$ and deduce that the exponential $\psi(x) = e^x$ is convex on $\mathbb{R} = (-\infty, \infty)$.

Exc. 22) Is every convex function continuous? If so, prove it. If not, provide a counterexample.

Exc. 23) Let X be a measured space with positive measure $d\mu$. Assume that $\int_X d\mu = 1$. i.e., X has volume equal to 1 with respect to $d\mu$. Show that, if $f \in \mathcal{L}^1(X)$ is real-valued, such that $f(X) \subseteq (a,b)$ for an open interval $-\infty \leq a < b \leq \infty$, and if $\psi : (a,b) \to \mathbb{C}$ is a convex function, then $\psi\left(\int_X f(x) d\mu\right) \leq \int_X \psi(f(x)) d\mu$.

Exc. 24) Person A claims: “$(\mathbb{R}, +)$ *and* $(\mathbb{R}^2, +)$ *are isomorphic as additive, abstract groups, but not as topological groups*.”
Person B contradicts: “*If* $(\mathbb{R}, +)$ *and* $(\mathbb{R}^2, +)$ *were isomorphic as additive, abstract groups, then they would also be isomorphic as real vector spaces, hence this is wrong. But they are homeomorphic, since there are continuous space-filling curves.*”

Who (if any) of the two persons is right? And why?

Hope, you will enjoy these exercises!