Unexpected Properties of Thales' Circle, Revealed by Dynamic Geometry Experiments

By **Hans Humenberger**

Recently I read an interesting article by Hartmut Müller-Sommer (in German; 2019). Here some geometric phenomena regarding the famous Thales' Circle (see Note) are presented using two topics of advanced mathematics: Stewart's Theorem from geometry (probably not part of school geometry) and Pell's Equation from algebraic number theory (surely not part of mathematics school curricula). On the one hand, these phenomena can be described easily using terms met in school mathematics, on the other hand they can be vividly visualized by using dynamic geometry. So I thought of possibilities to integrate them into mathematics classes in a way that does not rely on Stewart's Theorem and Pell's Equation. Since there are such ways, this short note was written to share this idea with interested teachers (and students).

The following activities can be described by asking *what if?* questions which in the *process* of *doing mathematics* are typical. Being interested in "what is the result, if some preconditions are changed?" is specifically important for discovering new phenomena in mathematics. This applies to both mathematical research and the teaching of mathematics, where students should also experience the process of doing mathematics, not only consuming and applying new formulas and the like. Here dynamic geometry programs can help a lot, and the complexity of the challenges presented to the students should be chosen rather carefully, in order that students are not 'overcharged' and frustrated.

Phenomenon 1

Consider the first phenomenon involving Thales' circle. Let $c = AB$ be a fixed line segment and C a point on the circle with diameter AB. Then Thales' theorem states that $\angle\, ACB = 90°$. Furthermore, by the Pythagorean theorem we know that $a^2 + b^2$ is a constant, namely c^2 (where a, b denote the legs of the right triangle ABC, Fig. 1).

A possible *what if* question is now: What if C does not move on the Thales' circle with diameter AB but on another *concentric* circle to it (same centre)?

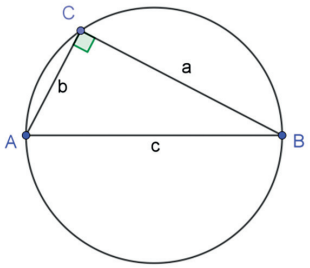

Fig. 1: Thales' Theorem, Pythagoras' Theorem

What can be said about $e^2 + f^2$ in this situation (Fig. 2)? It is rather clear intuitively (and formally by the converse of the Pythagorean Theorem) that if the circle is smaller than Thales' circle we will have $e_1^{\,2} + f_1^{\,2} < c^2$ (Fig. 2a) and that if the circle is larger than Thales' circle we will have $e_2^{\,2} + f_2^{\,2} > c^2$ (Fig. 2b).

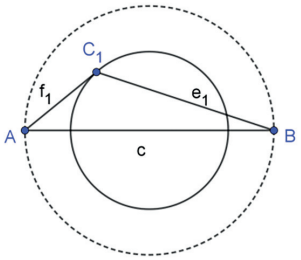

Fig. 2a: Smaller circle

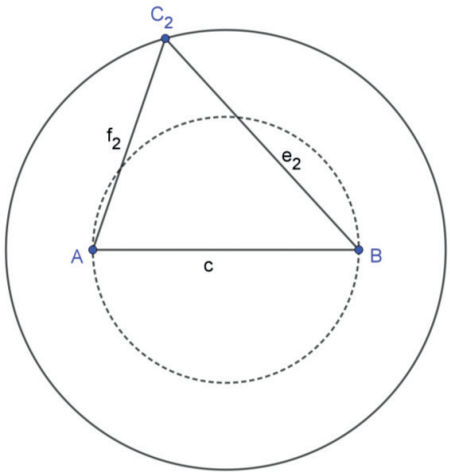

Fig. 2b: Larger circle

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But surprisingly, the sum $e^2 + f^2$ is constant in both cases for all C on the smaller/larger circle! Using dynamic geometry, moving C on the respective circle, and calculating $e^2 + f^2$ gives an impressive empirical validation. Although most students will not doubt *that* this is true after such experiments, the question *why* it is true cannot be touched by computer experiments. Here we need other arguments (proofs), and this is a crucial part of *doing mathematics as a process*, too. And the proof merely relies on the Pythagorean theorem, so it can readily be found by students autonomously in groups (possible hint: "use the Pythagorean theorem!"). It is also well possible that in class one case is done together with the teacher, the other case autonomously by the students. We will deal with the second case (Fig. 3b), the first one works analogously, and a corresponding sketch should suffice here (Fig. 3a).

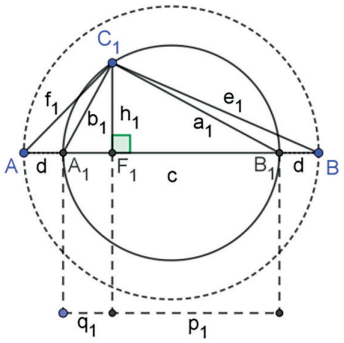

Fig. 3a: Smaller circle

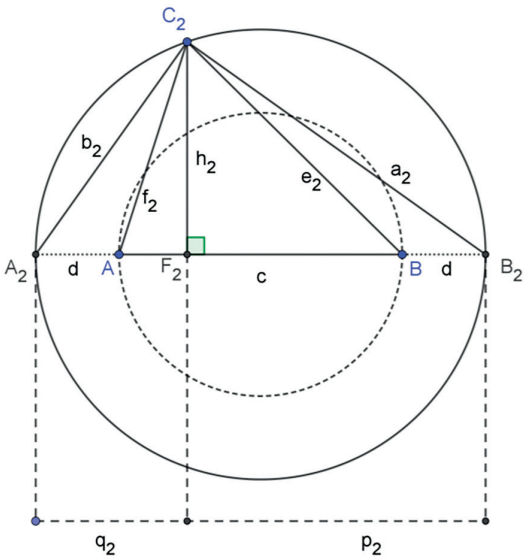

Fig. 3b: Larger circle

Let F_2 be the foot of the perpendicular from C_2 to AB, h_2 the corresponding altitude, d the increase of the radius, so that the diameter of the larger circle is $c + 2d$, $p_2 = F_2B_2$, and $q_2 = F_2A_2$ (Fig. 3b). Then, by the Pythagorean theorem, we can establish

$$\begin{aligned} e_2^2 &= (p_2 - d)^2 + h_2^2 \\ &= p_2^2 - 2dp_2 + d^2 + h_2^2 \\ &= a_2^2 - 2dp_2 + d^2, \\ f_2^2 &= (q_2 - d)^2 + h_2^2 \\ &= q_2^2 - 2dq_2 + d^2 + h_2^2 \\ &= b_2^2 - 2dq_2 + d^2, \end{aligned}$$

which gives by addition

$$\begin{aligned} e_2^2 + f_2^2 &= (c + 2d)^2 - 2d(c + 2d) + 2d^2 \\ &= (c + 2d)c + 2d^2 \\ &= (c + d)^2 + d^2. \end{aligned}$$

Since this expression does not depend on p_2 and q_2 (in other words on the position of C_2 on the larger circle) we have proved hereby the claimed constancy. This fits also perfectly to the possible positions of C_2 at A_2 or B_2, where we can immediately read off the distances $c + d$ and d for e_2 and f_2, respectively. The result in case of the smaller circle (Fig. 3a) is

$$e_1^2 + f_1^2 = (c - d)^2 + d^2,$$

again independent of the position of C_1 on the smaller circle.

Remark: Although I did not know this result before reading the mentioned paper, it seems to have been known for a very long time. It is reported in the above-mentioned paper that this even dates back to the Persian mathematician Al-Sijzī in the 10th century.

Phenomenon 2

Another *what if* question is: What if C moves on a circle which touches the Thales' circle in its interior at point A (we could also take point B instead)? Let us assume that the centre M of this smaller circle divides the diameter $c = AB$ in the ratio $1{:}k$. What can be said about $e^2 + kf^2$ (Fig. 4a, where $k = 3$; next page)?

Using dynamic geometry we can see perhaps with surprise that $e^2 + 3f^2 = c^2$ seems to hold for every point C on the small circle. That means, the value of $e^2 + 3f^2$ is not only somehow constant, but the constant for the small circle is the same as that for the Thales' circle!

Again, we want to avoid Stewart's theorem and Pell's Equation; the proof will rely only on the Pythagorean theorem and an enlargement centred at A, which is a common geometric transformation for changing the size of figures.

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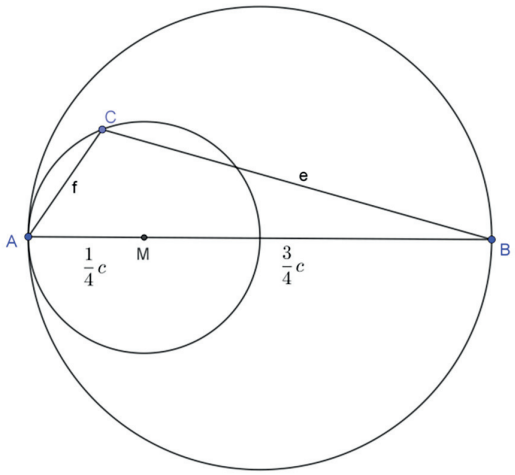

Fig. 4a: Towards phenomenon 2 with $k = 3$

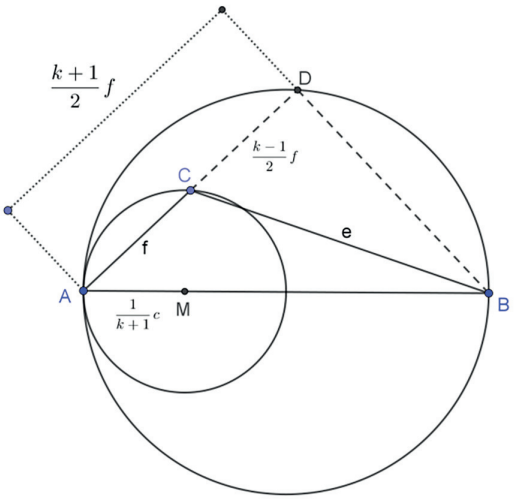

Fig 4b: Phenomenon 2 with general k

The first issue one must figure out is the corresponding *factor* of the mentioned central dilation that maps the small circle to the larger one (Thales' circle). With $k = 3$ (see Fig. 4a) the factor of the dilation would be 2. In the general case the radius of the smaller circle $\frac{1}{k+1}\,c$ must be mapped to the radius of the larger circle $\frac{1}{2}\,c$, and this is obviously done by the factor $\frac{k+1}{2}$. We can use this to calculate some distances in Fig. 4b: $AD = \frac{k+1}{2}\,f$, and subtracting $f = AC$ from it yields $CD = \frac{k-1}{2}\,f$. Next, we can calculate

$$BD^2 = AB^2 - AD^2 = c^2 - \left(\frac{k+1}{2}\right)^2 f^2,$$

and with this information in the right triangle CBD we can use the Pythagorean theorem once more to get

$$\begin{aligned} e^2 &= BD^2 + CD^2 \\ &= c^2 - \left(\frac{k+1}{2}\right)^2 f^2 + \left(\frac{k-1}{2}\right)^2 f^2 \\ &= c^2 - kf^2, \end{aligned}$$

from which follows our claim $e^2 + kf^2 = c^2$.

In both cases dynamic geometry is used to explore a geometric situation, to come to a conjecture by empirically looking at values that obviously are constant. Then the question *why* this is the case is answered by a proof. These proofs do not rely on rather complex mathematics, but only on the Pythagorean theorem and in the second case also on a central dilation. This is the reason why these phenomena and problems have good chances to be dealt with at school with students. Altogether, this is another vivid example of the *verification* function and *explanation* function of proof (De Villiers 1999).

Phenomenon 3

Given: two fixed points $A \neq B$ in the plane and a point C in this plane not on AB. Every such point C makes a triangle together with A and B and a corresponding incentre I. What is the possible range ("locus") of the incentre I? First, make experiments with dynamic geometry and observe the trace of I when dragging C!

The result of such an experiment can be seen in Fig. 5a. It will not be very long until some students have the conjecture that, for all possible positions of C, the incentre I must lie in the interior of Thales' circle with diameter AB (fixed). This would mean that Thales' circle is so to speak the border of the region in which I can lie. Since the incentre is the intersection of the angle bisectors and every interior angle is smaller than 180° it is clear that I must lie in the horizontal region between A and B (shaded grey in Fig. 5b, next page), but how far above AB can I lie? Obviously, when C lies above AB and moves away from B on a fixed straight line through B the distance of the incentre I to AB increases, because $\frac{\angle A}{2}$ increases and $\frac{\angle B}{2}$ stays unchanged. Thus, to answer the question how far above (or beneath) AB can I lie, we imagine that C is – so to speak as limit case – "infinitely far away" in one specific direction (no matter which), then CA and CB are parallel, as shown in Fig. 5c. Since in that case $\alpha + \beta = 180°$ we know that $\frac{\alpha}{2} + \frac{\beta}{2} = 90°$, from which we can conclude that $\angle AIB = 90°$ (Fig. 5c, next page). Thus, in this "limit case" I must lie on Thales' circle.

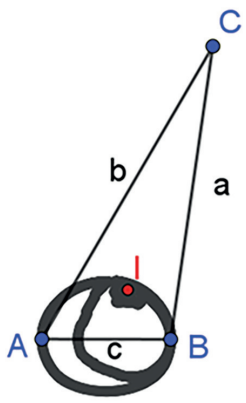

Fig. 5a Traces of I when dragging C

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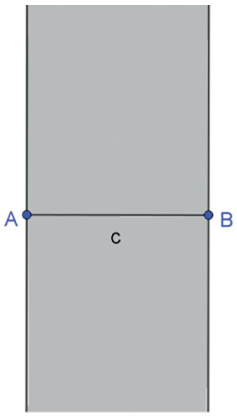

Fig. 5b horizontal region for *I*

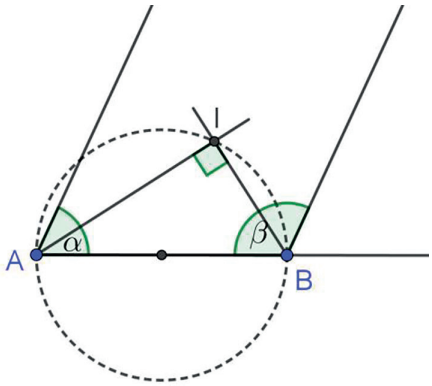

Fig. 5c Limit case for incentre *I*

Also, phenomenon 3 fits with the first two phenomena; on the one hand regarding the content (dynamic geometry at the beginning, relating to Thales' circle), and on the other hand regarding the source, which I first came across in another paper (again in German) of Hartmut Müller-Sommer (2021). This is a further interesting and substantial example of students' problem-solving activities (exploring a situation, discovering a phenomenon, reasoning), letting students experience the process of doing mathematics. Furthermore, the underlying phenomena (all three) involving Thales' Circle definitely deserve to be better known.

Notes

This is the circle mentioned in Thales' Theorem. For brevity we call it *Thales' Circle*. In German language this term is widely spread and used. In English language this seems not to be the case, thus we are pointing out this term explicitly.

The case of *C* and *I* lying beneath *AB* can be dealt with analogously.

References

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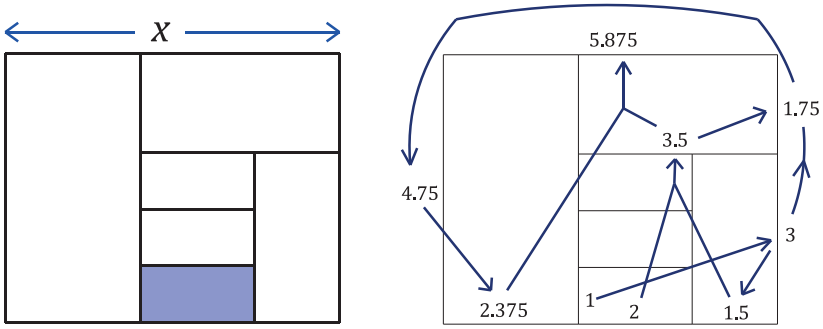
DOMINO PROBLEM 5: SOLUTION

By **Chris Pritchard**

Let the blue domino be 1 by 2. Then follow the arrows to find

$x = 5\frac{7}{8}$ or 5.875.



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