A proof of Theorem 1 based on Euclidean Geometry

Theorem 1: Given $\triangle ABC$ and a point $G_i$ not on its circumcircle $\Sigma$, let $\triangle A_iB_iC_i$ be its circumcevian triangle w.r.t. $G_i$. Then $G_i$ is the centroid of $\triangle A_iB_iC_i$ if, and only if, it is a focus of the Steiner inellipse of $\triangle ABC$.

In this proof we want to restrict the methods essentially to Euclidean geometry, because this is obviously the area of Theorem 1.

We found in the www a helpful statement with solution (Lemma 1) and we build on this “basis” a proof of Theorem 1 with formulating two further lemmas (2 and 3). At the first glance these lemmas have very little to do with Theorem 1, the things will be brought together at the end. The reader needs a bit of patience (especially after reading the one page proof in the printed article), because in all three following lemmas one cannot see immediately their connection to Theorem 1. I would be pleased to hear or read a shorter and more elementary proof from a reader, using only Euclidean geometry.

Lemma 1\textsuperscript{1}: The circumcenter $O$ of $\triangle ABC$ is the centroid of the antipedal triangle $\triangle A'B'C'$ of the symmedian point\textsuperscript{2} $L$ of $\triangle ABC$ (see Fig. 3).

We found this statement and a solution in (there one can find also a short proof):

https://artofproblemsolving.com/community/c6h440326

\textsuperscript{1} For reasons of clarity one can split up this Lemma 1 into the Lemmas 1a – 1d (see below for interested readers). Fig. 1 and 2 are in the printed version (The Mathematical Gazette, March 2020), therefore the list of figures here in this online appendix starts with number 3.

\textsuperscript{2} This point is also called Lemoine point, therefore the letter L.
Fig. 3: $O$ is the centroid of $\triangle A'B'C'$

**Lemma 2:** Let $\triangle PQR$ be a triangle with centroid $G_i$ and symmedian point $L$. The circum-medial triangle is $\triangle ABC$ and the pedal triangle of $G_i$ w.r.t. $\triangle ABC$ is $\triangle XYZ$. Then $G_i$ is the symmedian point of $\triangle XYZ$ (see Fig. 4).

Fig. 4: $G_i$ is symmedian point of $\triangle XYZ$, centroid $G$ of $\triangle ABC$ is circumcenter of $\triangle XYZ$
$\triangle PQR$ is the circumcevian triangle of $\triangle ABC$ w.r.t. $G_1$. We will show that $\triangle RPL \sim \triangle ZXG_1$ and $\triangle PQL \sim \triangle XYG_1$. Then we have the similarity $\triangle XYZ \sim \triangle PQR$ and $G_1$ is the symmedian point.

For $\triangle RPL \sim \triangle ZXG_1$ we have to prove that $\angle XZG_1 = \angle PRL$ and $\angle G_1XZ = \angle LPR$.

From the cyclic quadrilateral $ZBXG_1$ and the inscribed angle theorem we get $\angle XZG_1 = \angle XBG_1 = \angle CBQ = \angle CRQ = \angle G_1RQ = \angle PRL$, analogously we get $\angle G_1XZ = \angle G_1BZ = \angle QBA = \angle QPA = \angle QPG_1 = \angle LPR$.

Analogously one can prove $\triangle PQL \sim \triangle XYG_1$.

**Lemma 3:** It is well known and easy to see by analytical means that the locus of the points $X$ with the property “the tangents from $X$ onto an ellipse and the line segment from $X$ to a focus of the ellipse are perpendicular at $X$” is a circle with radius = semi-major axis of the ellipse and center = center of the ellipse (see Fig. 5, this circle is also called the *pedal curve* of the ellipse w.r.t. the *focus*, also *pedal circle*).

Fig. 5: Pedal curve w.r.t. $F_1$, pedal circle
Now we put these three lemmas – seemingly unconnected to the matter in hand – together and can **prove Theorem 1**: Let $\Delta PQR$ be a ("unknown") triangle with centroid $G_1$, symmedian point $L$ and the property "its circum-medial triangle is $\Delta ABC". Then by Lemma 2 we know that $G_1$ is the symmedian point of $\Delta XYZ$ (Fig. 4). The antipedal triangle of $G_1$ w.r.t. $\Delta XYZ$ is $\Delta ABC$ by construction. Then Lemma 1 implies that the centroid $G$ of $\Delta ABC$ is the circumcenter of $\Delta XYZ$ (Fig. 4), i.e. the center of the pedal circle of $G_1$ w.r.t. $\Delta ABC$. According to Lemma 3 $G_1$ is a focus of an ellipse with center $G$ and tangent to the sides $AB, AC, BC$. Therefore, it must be the Steiner inellipse of the triangle $\Delta ABC$ tangent to its sides through their midpoints (an ellipse tangent to the triangle sides with center $G$ can be mapped by an affine transformation to the incircle of an equilateral triangle). The possible points for $G_1$ are the two foci of this special ellipse. For the "unknown" triangle $\Delta PQR$ there are two "solutions", the *circumcevian* triangles of $G_1$ and $G_2$ (see Fig. 2 in printed article, The Mathematical Gazette, March 2020, here again):

![Fig. 2: $G_1, G_2$ are the foci of the Steiner inellipse of $\Delta ABC$](image-url)
Here a split up version of Lemma 1, in order to make the short solution that can be found at https://artofproblemsolving.com/community/c6h440326 more easily to follow.

**Lemma 1a:** A point $X$ lies on the $C$-median of a triangle if, and only if, the distances to the sides $BC$ and $AC$ are reciprocal to the side lengths themselves: $\frac{a}{b} = \frac{j}{i}$ (see Fig. 6)

![Fig. 6: Point on the $C$-median](image)

The proof follows immediately by the fact that the median bisects the area of the triangle and $\frac{n}{m} = \frac{j}{i}$.

**Lemma 1b:** Given a triangle $\triangle ABC$. The pedal triangle $\triangle XYZ$ of the centroid $G$ and the antipedal triangle $\triangle A'B'C'$ of the symmedian point $L$ are homothetic (see Fig. 7).

![Fig. 7: Homothetic triangles](image)

This is clear because $XZ$ is perpendicular to $BL$ (see [5, p. 14f, Theorem 6], see also [6, p. 64f]), therefore $XZ \parallel A'C'$ (analogous for the other sides).
Lemma 1c: Let $\triangle ABC$ be a triangle, $P$ an arbitrary interior point and $X$, $Y$, $Z$ the orthogonal projections of $P$ onto the triangle sides. Then the following equation holds:

$$\frac{|AB|}{|AC|} \cdot \frac{|PC|}{|PB|} = \frac{|XY|}{|XZ|} \quad \text{(see Fig. 8)}$$

![Fig. 8: Cyclic quadrilateral](image)

$ZBXP$ is a cyclic quadrilateral, $PB$ is a diameter of the circle and the angle $\angle ZBX$ is an inscribed angle to the chord $ZX$. From the inscribed angle theorem we get

$$\sin(\angle ZBX) = \frac{|XZ|}{|PB|},$$

and analogously

$$\sin(\angle ZBX) = \frac{|XY|}{|PC|}.$$  

According to the law of sines we have

$$\frac{\sin(\angle ZBX)}{\sin(\angle ZBY)} = \frac{|AB|}{|AC|} \quad \text{and this yields the equation we want to prove.}$$

Lemma 1d: Let $G, O, L$ be the centroid, circumcenter and symmedian point of $\triangle ABC$. $E, F$ are the midpoints of $AC$, $AB$ and $\triangle A'B'C'$ is the antipedal triangle of the symmedian point $L$ of $\triangle ABC$. Then we have $\angle OBA' = \angle BEA$ and $\angle OCA' = \angle CFA$ (see Fig. 9).
Fig. 9: Lemma 1d

The angle bisector at $B$ meets the perpendicular bisector of $AC$ at the point $H$ on the circumcircle of $\triangle ABC$. Then we have

$$\angle OBA' = 90^\circ - (\angle OBH + \angle HBL) =$$

$$= 90^\circ - (\angle OHB + \angle EBH) =$$

$$= 90^\circ - \angle OEB = \angle BEA$$

and analogously $\angle OCA' = \angle CFA$.

Then we can prove Lemma 1 step by step: With $d(O, A'C')$ we denote the distance between the point $O$ and the line segment $A'C'$:

$$\frac{d(O, A'C')} {d(O, A'B')} = \frac{\sin(\angle GEY)} {\sin(\angle GFZ)} = \frac{|GY|}{|GZ|} \frac{|GF|}{|GE|} = \frac{|AB|}{|AC|} \frac{|GC|}{|GB|} = \frac{XY}{XZ} = \frac{|A'B'|}{|A'C'|}$$

With Lemma 1a we can conclude that $A'O$ is the $A'$-median of $\Delta A'B'C'$. Similarly $B'O$, $C'O$ are the medians issuing from $B'$, $C'$. Thus we have proven that $O$ is the centroid of $\Delta A'B'C'$.

References: