Balanced areas in quadrilaterals – on the way to Anne's Theorem

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This article gives an example for using Dynamic Geometry Software to encourage making conjectures, reasoning, and experiencing mathematics as a process.

When teaching geometry, students can use Dynamic Geometry Software (DGS[1] such as: *GeoGebra*, *Sketchpad*, *Cindarella*, *Cabri*) making the learning more dynamic, interesting, vivid, and process oriented. Students have the chance to discover some phenomena by themselves and this provides opportunities for discussion about reasons and proofs in a more natural way. The corresponding questions arise out of the situation, not merely from the teacher or the text book.

DGS programs can measure (lengths of line segments, areas, angles, etc.) and carry out operations (sums, differences, products, etc.) with the measurements. In this article we give an example of a learning environment in which students should discover the underlying phenomena (by using the measurement features) and look for corresponding reasoning. The proofs are quite simple; therefore, students have a good chance of finding them independently. If they don't find a formal proof they at least have made some experiences of finding some conjectures which is also an important part of mathematics as a process.

We do not give much guidance for implementing the following problems in classrooms because the corresponding questions depend on the teacher, the institution, the learning group, the proficiency levels, and the year level. Our aim is primarily to present a series of problems that has a high potential for teaching processes in the sense we described above. We start with a simple problem and then ask for generalisations[2].

Problem 1

Construct a square *ABCD* and mark an arbitrary point *I* in it. Connect *I* with the four vertices *A*, *B*, *C*, *D* and take the same colour for opposite triangles (in Figure 1 grey and white). What do you notice for the sum of grey areas and the sum of white areas?

Here students can use DGS for calculating the sum of the areas and explore the situation before thinking of formal mathematics and reasoning. Moving the point *I* in the square, they will notice that the sum of the grey areas

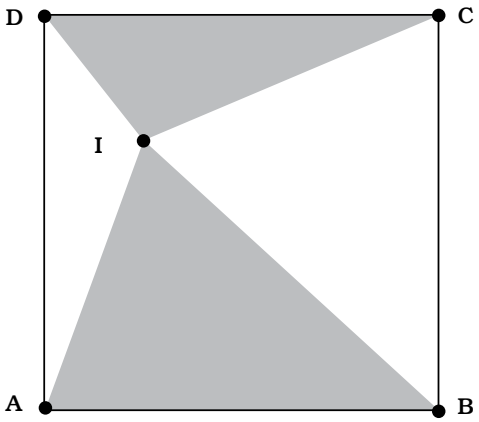

Figure 1. Square—grey and white areas.

and the sum of the white areas do not change and are always identical to each other. Therefore, it seems that the areas of grey and white are always half of the area of the square because together they make up the complete square.

Now the question of mathematical reasoning arises, which is very simple in this case using the formula for triangle areas. With a = side length of the square and noticing that the sum of the altitudes is constant $h_1 + h_2 = a$ we get (see Figure 2):

$$\text{Area}_{\text{gray}} = a \cdot \frac{h_1}{2} + a \cdot \frac{h_2}{2} = \frac{a}{2} \cdot \underbrace{(h_1 + h_2)}_{a} = \frac{a^2}{2} = \frac{\text{Area}_{\text{square}}}{2}$$

Having dealt with this elementary problem it is very natural to think of variations. One possibility is formulated in Problem 2.

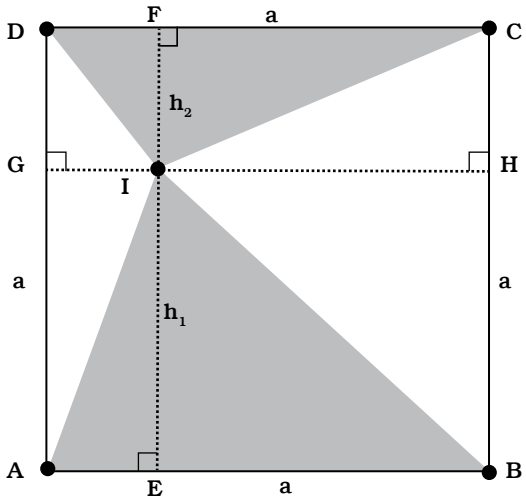

Figure 2. Square —constant sum of altitudes $h_1 + h_2 = a$.

Problem 2

Are there other quadrilaterals with this property (balanced area for grey and white for every point)? Which ones? Can you give reasons?

Students can do the same DGS experiments with other quadrilaterals like rectangles and parallelograms with the same results and reasoning (Figure 3).

The answer to the questions of Problem 2 is: Every parallelogram has this property. What about other quadrilaterals like trapeziums, kites, etc.? Students can explore this question in the same way: first using DGS and exploring the situation, then using mathematical reasoning.

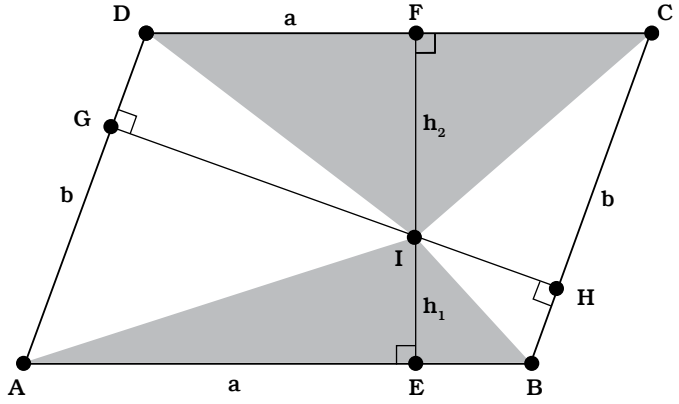

Figure 3. Parallelogram—constant sum of altitudes $h_1 + h_2$.

Problem 3

What do you notice when you take a general trapezium? Can you give reasons for the phenomena you observe?

Exploring the situation with DGS one will notice that the situation is different here. When moving the point I the sum of the grey and the white areas are not constant. Are there points I of the trapezium that lead to "balanced areas" (the sum of grey areas equals the sum of white areas)?

1. Here we used GeoGebra. But also, the other mentioned DGS programs would work.
2. One may think that for some learners it would be better the other way around: Pose the problem in its general form (general quadrilaterals), then realise that this is quite a tough problem. Using the problem-solving strategy ("solve an easier problem", "deal with special cases") students may come up with cooperative strategies like: One group deals with parallelograms, one with trapeziums, one with kites, ... But one problem with this way is that the geometrical locus of the point I (see the problems below) is in some cases the whole quadrilateral, and in other cases only a special line segment. Therefore, probably this way of dealing with this phenomenon will only work well with rather gifted learners.

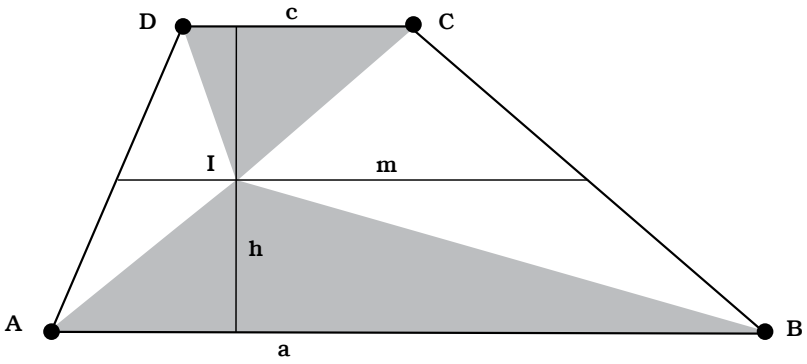

Figure 4. For *I* on *m* the grey and white triangles seem to have balanced areas.

A good way to examine that is to create a new variable d = difference between the grey area and the white area. When moving I in the trapezium one must look if the calculated value d vanishes at some positions[3].

Experimenting with DGS it is not too difficult to find that d seems to be 0 for points I on the midsegment m of the trapezium (Figure 4).

Again, the proof is easy because both grey triangles have altitude $h/2$ (where h denotes the altitude of the trapezium):

$$\text{Area}_{\text{gray}} = \frac{a \cdot \frac{h}{2}}{2} + \frac{c \cdot \frac{h}{2}}{2} = \frac{(a+c) \cdot h}{4} = \frac{\text{Area}_{\text{trapezium}}}{2}$$

Another question arises: Are there some other points in the interior of the trapezium—not lying on m—with this property (balanced areas)? It may well be that students do not ask this question by themselves because the DGS experiments[4] "show" that one probably will not find such points. But this observation is of course not a proof, it only leads to a "strong" conjecture (Figure 5).

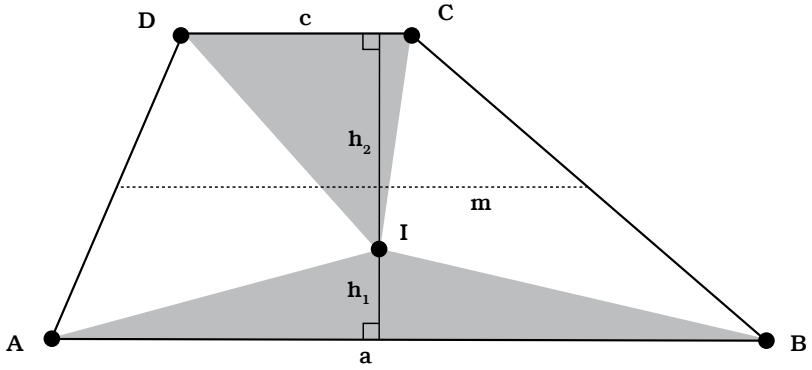

Figure 5: Trapezium: *I* not on *m*.

Conjecture: Except on the midsegment m there are no other points with the property of balanced areas. The nearer the point I lies to the side a, (i.e. the smaller the altitude h_1 is) the smaller the grey area sum is. The nearer the point I lies to the side c, (i.e. the longer the altitude h_1 is) the bigger is the grey area sum.

But why is that so? What are mathematical arguments (or even a formal proof) for this phenomenon?

3. It may well happen that students will have some difficulties to make d exactly zero (when measurements are taken with high precision), or they may observe that a range of locations of I result in measurements that round d to 0 (when measurements are taken with low precision).
4. When I leaves m then $d > 0$ on the one side of the line and $d < 0$ on the other side.

At first an explanation based on plausibility (not a formal proof) for $a > c$: If I leaves m towards a then the altitude h_2 becomes longer by a certain amount but h_1 becomes shorter by the same amount, but this getting shorter has a greater effect because of $a > c$; therefore it is clear that the sum of the grey areas becomes less during this process.

The same idea can be formalised. For $h_1 < h_2$ and $a > c$ we get (x denotes the distance from I to the midsegment m, see Figure 5).

$$\text{Area}_{\text{gray}} = \frac{a \cdot \overbrace{\left(\frac{h}{2} - x\right)}^{h_1}}{2} + \frac{c \cdot \overbrace{\left(\frac{h}{2} + x\right)}^{h_2}}{2} = \underbrace{\frac{(a+c)\cdot h}{4}}_{\frac{\text{Area}_{\text{trapezium}}}{2}} - \underbrace{\frac{x}{2}\cdot(a-c)}_{>0} < \frac{\text{Area}_{\text{trapezium}}}{2}$$

In the other cases $h_1 > h_2$ or $a < c$ one can argue analogously.

So, we can summarise for trapeziums: A point I of the trapezium has the balanced area property if and only if it lies on the midsegment m. In other words: The midsegment m is the locus of the points with the balanced area property.

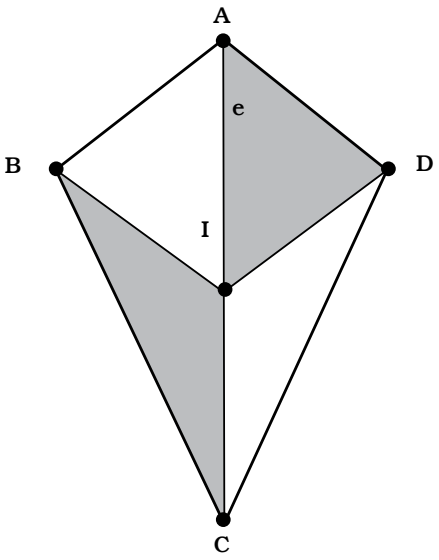

Figure 6. Kite with axis of symmetry $e = AC$.

Problem 4

What do you notice when you use kites[5] instead? Can you give reasons for the phenomena you observe?

Here students may start with DGS experiments and observe: When I lies on the diagonal $e = AC$ (axis of symmetry) then we have area balance (Figure 6). This may be also clear without DGS experiments because when I lies on e the left-hand triangles are congruent to the right hand ones, therefore the area balance is evident in the case of I lying on e.

The proof here (kites and axis of symmetry; congruent triangles) is even easier than in the case of the trapezium and the midsegment m. But in the case of the trapezium, the proof of "there are no other points" was easy enough that students can discover it independently (the basis of the proof was the area formula for triangles).

Now the proof of "there are no other points" is a bit more difficult and not so straight forward. It does not matter if students don't find such a proof independently, when they just say: "Due to our DGS experiments we are sure that there exists no other area balance points except on the axis of symmetry", the teacher may give some hints or show a possible proof. Here we will present two possibilities.

Without loss of generality we may assume that we have a kite with $|AB| < |BC|$ (and therefore $|AR| < |RC|$, cf. Figure 7a).

Possibility 1: We assume that I lies somewhere "down to the right" (in the other cases the reasoning is analogous). We reflect I through e ($\rightarrow I'$) and are interested in the changes of the grey area sum (Figure 7a). It is clear that the white triangles with vertex I (ΔICD and ΔIAB) are congruent to the grey ones with vertex I' ($\Delta I'BC$ and $\Delta I'DA$),

5. Here we mean kites that are not rhombuses.

i.e. comparing the sum of the grey areas with the vertices I and I' respectively is equivalent to comparing the sum of the grey areas and the sum of the white areas with vertex I.

The following triangles go from grey to white due to the reflection $I \to I'$ (in Figure 7a hatched): $\Delta CII'$, $\Delta I'IB$, $\Delta I'ID$ (in the triangle $\Delta I'ID$ the triangle $\Delta I'IE$ is included here wrongly, but we will consider it—again wrongly—when dealing with the area that changes from white to grey; then this "mistake" will be equalised).

The following triangle goes from white to grey due to the reflection $I \to I'$ (in Figure 7a with checkerboard pattern): $\Delta I'IA$.

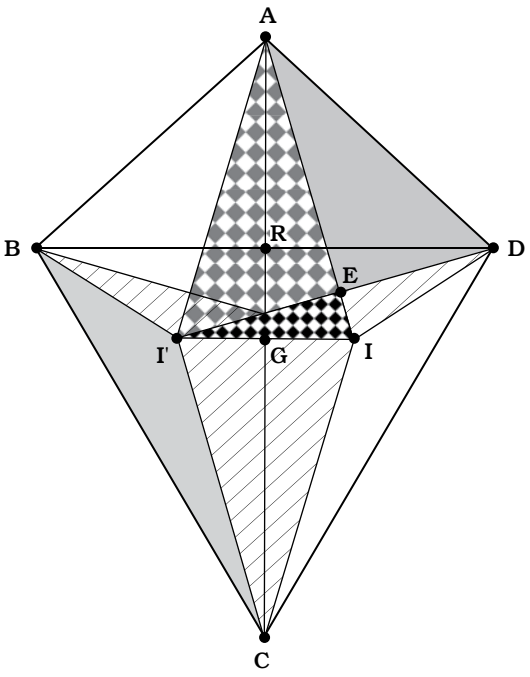

Figure 7a. Changes of color (grey ↔ white). due to the reflection $I \to I'$.

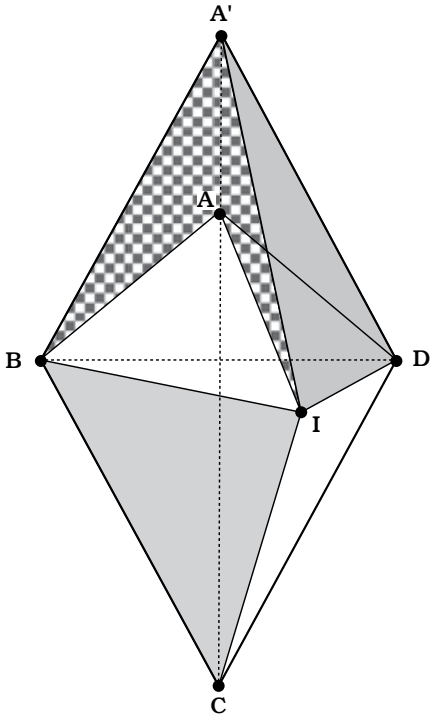

Figure 7b. Change of the white area in the process $A \leftrightarrow A'$ (checkerboard pattern).

With $d = |I'I|$ the resulting change of the sum of grey areas due to the reflection $I \to I'$ is:

$$\frac{d \cdot (|AR| + |RG|)}{2} - \frac{d \cdot |GC|}{2} - 2 \cdot \frac{d \cdot |RG|}{2} = \frac{d}{2} \cdot \underbrace{(|AR| - |RC|)}_{<0} < 0$$

But this change would have to be zero in the case that I had the balanced area property (see above).

Possibility 2[6]: We assume that I lies somewhere in the right hand side of the kite (in the left hand side the reasoning is analogous). We reflect C through BD ($\to A'$) and get a rhombus $A'BCD$. We know that in a rhombus every point I has the balanced area property.

$$\underbrace{|\Delta BCI| + |\Delta A'ID|}_{\text{gray}} = \underbrace{|\Delta BIA'| + |\Delta ICD|}_{\text{white}} = \frac{\text{Area}_{\text{rhombus}}}{2}$$

The difference of the rhombus and the kite $ABCD$ is the quadrilateral (kite) $BADA'$, and the change of the white area when looking at the kite ($A'BCD$) instead of the rhombus

6. Idea by B. Szöke-Ogrutan (teacher student of the author, Vienna); she presented this idea in a problem-solving seminar at the University of Vienna.

(checkerboard pattern in Figure 7b) is more than $|\Delta BAA'|$, i.e. more than the half of the difference area $|BADA'|$.

Therefore, the point I cannot have the balanced areas property with respect to the kite $ABCD$ (I has this property with respect to the rhombus; and if I had this property also with respect to the kite, the change of the white area would have to be exactly half of $|BADA'|$).

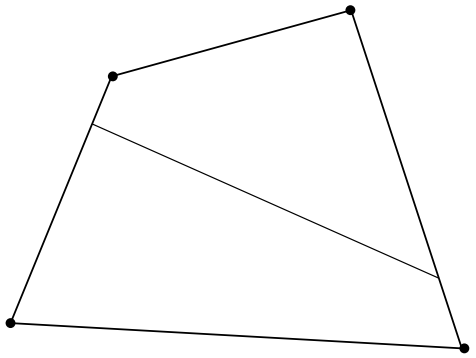

Figure 8. General quadrilateral and a special straight line as the locus of the points with the balanced areas property.

Thus we have proven that a point I not lying on the axis of symmetry cannot have the balanced areas property. There are also other possibilities to show that, and students are doing well if they find one independently.

Now the question arises: What about the general case of a quadrilateral? Are there always points I in a (convex) quadrilateral that have the mentioned balanced areas property? Does the locus of such points always have a special shape? If yes which shape?

This is—of course—too difficult for students to give mathematical proofs independently. But they can do DGS experiments with a general quadrilateral and explore the situation (Figure 8; the line through the quadrilateral shows the conjectured locus of the points I having the balanced area property). Maybe they can come to a conjecture and maybe some of them find some ideas for a proof.

Looking for points I with the property that the area difference between grey and white vanishes, one can reach the conjecture that these points lie on a special straight line (Figure 8). This is really the case and leads to 'Anne's Theorem'.

Anne's Theorem

Let $ABCD$ be a convex quadrilateral (not a parallelogram[7]). Then the locus of the points I with area balance between grey and white is the straight line $g = MN$ through the midpoints of the diagonals (Figure 9).

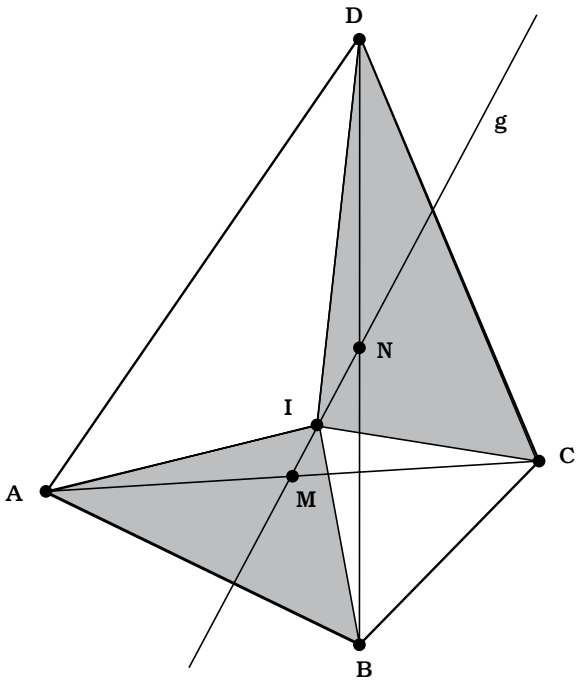

Figure 9. Anne's Theorem

This not very well known theorem may be proven analytically (Alsina & Nelsen, 2010, p.116f; Honsberger, 1991, p.174f) in a very short and elegant way. But we doubt that this was the original idea of Pierre Leon Anne (French mathematician, 1806–1850). Unfortunately, we have not managed to get an original reference from Anne. The mentioned line $g = MN$ is called "Newton's line" (one can prove the so called "Newton's Theorem" by using Anne's Theorem; see http://www.cut-the-knot.org/Curriculum/Geometry/NewtonTheorem.shtml). From Anne's Theorem it is clear that the midpoints of the diagonals have the area/balance property. But it is not so easy to discover

7. Because in case of a parallelogram the mentioned locus is the whole parallelogram—as seen above.

independently that the mentioned points have a special role when considering balanced areas as we did it here: starting with squares and asking for generalisations (heuristic strategies are important here!).

Possible heuristic strategy with DGS in our approach: Moving the point I one will notice that in some situations the difference d of the area sums will vanish. Noticing this phenomenon at several situations, one will come to the conjecture that these points seem to lie on a special line. How can this conjecture be confirmed? Mark two points where d vanishes and construct a straight line through these points (say g). Now stick the point I to this line g and move it on it: one will see that d vanishes for all points of g; if I leaves the straight line g then $d \neq 0$. Thus, Anne's theorem can be conjectured, "seen", and dynamically experienced.

More details concerning Anne's Theorem (especially a proof) are not crucial to this short paper—they are dealt with in a more mathematical paper (Humenberger & Schuppar).

Here we wanted to stress primarily other items which may give arguments to deal with this topic in mathematics teaching (possible at several levels):

- Starting at a very elementary problem for students (square) the problem is generalised step by step. Thus mathematics as a vivid process (vs. mathematics as a finished product) can be shown and experienced by students.
- This process oriented way coming close to Anne's Theorem is—in terms of the teaching process—more important than the theorem itself (or a possible proof of it).
- Heuristic strategies play an important role. How does one come to conjectures? How can they be confirmed by DGS experiments?
- Easy DGS experiments can be done, using DGS primarily as a means of measuring areas. These experiments (exploring the situation) lead quite naturally to mathematical reasoning and proving. And these proofs (square, rectangle, parallelogram; trapezium, kite[8]) are simple enough that students can discover them by themselves.
- A nearly forgotten theorem can be revived and discovered again based on own experiments with DGS.
- Typical ways of operating and thinking in mathematics can be stressed and discussed; for instance if you want to show that a certain line is a special "locus" then you have to deal with two directions: (1) All points on the line have a certain property; (2) No other point has this property. This principle is also important for the "perpendicular line bisector" and the "angle bisector" (as the well-known loci).

References

Alsina, C., Nelsen, R. B. (2010). Charming Proofs: A Journey Into Elegant Mathematics. *Dolciani Mathematical Expositions 42*; MAA.

Honsberger, R. (1991). More Mathematical Morsels. *Dolciani Mathematical Expositions 10*; MAA.

Humenberger, H., Schuppar, B. (n.d.). Balanced areas in quadrilaterals—Anne's Theorem and its unknown origin. To appear in: *Australian Senior Mathematics Journal* (ASMJ).

8. Here we mean: I lying on $e \rightarrow I$ is a point with area balance; the other direction '$\leftarrow$' is a bit more complicated, see above.