Special values of automorphic L-functions

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1 Motivation

Imagine you're a French lawyer in the 17^{th} century with a strong interest in mathematics. What *we* know and sometimes thoughtlessly dismiss as "undergraduate mathematics" is in its infancy, so even just idly playing around with natural numbers can lead to new and unexpected discoveries. Apparently the holiday season is an especially idle time, because on Christmas day, you go and you write a letter to your friend, telling him, among other things, about the following observation you've made:

1.1. *For an odd prime p, TFAE:*

(i) $p = x^2 + y^2$ *for some* $x, y \in \mathbb{Z}$;
(ii)
(iii) $p \equiv 1 \pmod 4$.

So the equivalent condition is well-known, I think, but there is another one which I would like to mention, and I'm going to write them in reverse order, just to keep it interesting. The middle condition is then

(ii) the polynomial $f(t) = t^2 + 1 \in \mathbb{Z}[t]$ *has two distinct roots in* $\mathbb{F}_p$.

which of course you could also phrase succinctly as saying that -1 is a quadratic residue modulo p, or in symbols:

(ii) ... i.e., $\left(\frac{-1}{p}\right) = 1$ *(Legendre symbol).*

Now this is only the most superficial occurrence of an incredibly pervasive pattern. In fact, if we consider primes which can be written as the sum of a square and n times a square, for many small values of n, we mostly get similar statements. For instance,

1.2. *For an odd prime $p \neq 7$, TFAE:*

(i) $p = x^2 + 7y^2$ *for some* $x, y \in \mathbb{Z}$*;*
(ii) $\left(\frac{-7}{p}\right) = 1$*;*
(iii) $p \equiv 1, 9, 11 \pmod{14}$.

The way these statements are proved is similar, even though the specifics differ. In both cases, the bottom equivalence is essentially quadratic reciprocity.

(ii)⇔(iii): quadratic reciprocity.

I think quadratic reciprocity sometimes doesn't get the acclaim it deserves, because it is actually not at all obvious that a statement of the form "q is a quadratic residue modulo p" can be turned into a congruence condition for p mod q (or in these cases, mod $4q$). But if we understand "reciprocity" as meaning, more generally, that we switch the roles of two objects (and keep the equivalence), then the top half of the equivalence can also be seen as a form of reciprocity.

(i)⇔(ii): Let $K \subset L$ *be a Galois extension of number fields,* $L =$ *splitting field of some* $f \in O_K[x]$*. Then a prime* $\mathfrak{p} \subset O_K$ *splits completely in* L *iff* f *splits completely mod* $\mathfrak{p}$ *(with finitely many exceptions).*

(Keep in mind, of course, that the meaning of "splits completely" is different.)

So things become *really* interesting when you keep varying this number n here, because you don't always get the same type of results. For instance,

1.3. *For an odd prime* $p \neq 23$*, TFAE:*

(i) $p = x^2 + 23y^2$ *for some* $x, y \in \mathbb{Z}$*;*
(ii) $\left(\frac{-23}{p}\right) = 1$ *and* $f(t) = t^3 - t - 1$ *splits completely mod* p*;*
(iii) $p \equiv$???

As it turns out, we cannot express this in terms of congruences anymore, essentially because the splitting field of this $f(t)$ over $\mathbb{Q}$ is not an abelian extension of $\mathbb{Q}$. cross out (iii). So we have to come up with something different if we want a "reciprocity law" for this kind of polynomial.

To see how we can do this, let us go back to the examples from before. It is not hard to see:

1.4. *For any prime* p*,* $\left(\frac{-1}{p}\right) = 1$ *iff the coefficient of* X^p *in*

$$X(1 - X^2)(1 - X^4)^{-1} = X - X^3 + X^5 - X^7 \pm \cdots$$

is equal to 1*.*

This is because

the coeff. of X^m *is* $\left(\frac{-1}{m}\right)$ *(Jacobi symbol, completely multiplicative).*

In part., $\#\{t \in \mathbb{F}_p : t^2 + 1 = 0\} = 1 + \left(\frac{-1}{p}\right)$.
Similarly for $n = 7$.

For $n = 23$, we get the following statement instead:

1.5. *For any prime* p,

$$\#\{t \in \mathbb{F}_p : t^3 - t - 1 = 0\} = 1 + a_p$$

where

$$\sum_{n=1}^{\infty} a_n X^n = X \prod_{k=1}^{\infty} (1 - X^k)(1 - X^{23k}).$$

In fact,

$$a_p = \begin{cases} 2, & \text{if } \left(\frac{p}{23}\right) = 1 \text{ and } p = x^2 + 23y^2, \\ 1, & \text{if } p = 23, \\ 0, & \text{if } \left(\frac{p}{23}\right) = -1, \\ -1, & \text{if } \left(\frac{p}{23}\right) = 1 \text{ but } p \neq x^2 + 23y^2. \end{cases}$$

Also, $a_{mn} = a_m a_n$ *when* $\gcd(m, n) = 1$.

Now we may start to get a hint that there is something deep going on here. From what I gather, this was actually proved by Hecke, so we are around three centuries later than when we started.

2 Cusp forms

Hecke's insight was that such power series expansions $\sum a_n X^n$ correspond to certain complex-valued functions which, if we wanted to be precise, we might opt to call *normalized primitive Hecke eigenforms*, which sit inside a space of *cusp forms* (for some level N and some weight k and some character χ). Now classically, cusp forms are thought of as functions on the upper half-plane subject to automorphy condition, plus holomorphy and growth conditions. In this case, the function corresponding to $\sum a_n X^n$ is essentially $\mathfrak{H} \ni \tau \mapsto \sum a_n (\mathrm{e}^{2\pi \mathrm{i} \tau})^n$.

A more sophisticated viewpoint is that cusp forms are functions on $G = \mathrm{GL}_2(\mathbb{R})$ which are invariant under $\Gamma =$ (some subgroup of) $\mathrm{GL}_2(\mathbb{Z})$, subject again to holomorphy and growth conditions and a condition of *finiteness under* $K = \mathrm{O}_2(\mathbb{R})$, the standard maximal compact subgroup of G. (The last condition corresponds to the existence of an integral weight k in the classical setting.)

The modern viewpoint for cusp forms is that they are functions on $G = \mathrm{GL}_2(\mathbb{A})$ which are invariant under $\Gamma = \mathrm{GL}_2(\mathbb{Q})$, again + conditions.

This modern viewpoint is more flexible because it allows us to define cusp forms (more precisely *cuspidal automorphic forms*) for any reductive algebraic group G which is defined over $\mathbb{Q}$ (or even more generally over a number field F). For GL_2 over $\mathbb{Q}$, one essentially recovers the original definition. For GL_1 again over $\mathbb{Q}$ for simplicity, the cusp forms are precisely finite linear combinations of Hecke characters. (Hecke characters include the classical Dirichlet characters, an example of which is the Jacobi symbol.)

Moreover, cuspidal automorphic forms are related to…

- …the representation theory of $G(\mathbb{A})$: they "generate" "representations" of this group, accordingly called *cuspidal automorphic representations*.
- …the cohomology of arithmetic subgroups of G. (key word: Eichler–Shimura isomorphism)

Bottom line: cusp forms, and more generally *cuspidal automorphic representations*, are deep and ubiquitous objects which (sometimes) contain rich arithmetic information, including about the splitting of primes.

3 Analytic reformulation

Earlier, we saw a statement which could be seen as a "*reciprocity law*" in a similar way to quadratic reciprocity. We could say the same about the correspondence we saw later on between "$f(t)$ splits completely mod p" and "the p-th coefficient of $\sum a_n X^n$ is equal to $\deg f - 1$".

All these "reciprocity laws" can be conveniently expressed as equalities of certain analytic or meromorphic functions attached to the objects in question, which are called *zeta functions* or *L-functions* (depending largely on convention). There is no uniform criterion for a certain complex-valued function to be an L-function (or zeta function); rather, there are classes of objects *to which* such functions can be attached, and common properties that the functions are then known—or expected—to have. Here are some important classes of zeta functions and L-functions:

- Splitting of primes in a number field $F/\mathbb{Q} \rightsquigarrow$ Dedekind zeta function.
- Primes in congruence classes $\rightsquigarrow$ Dirichlet L-function $L(s, \chi)$, where χ is a Dirichlet character.
- Coefficients of a Hecke eigenform $\rightsquigarrow$ modular L-function; more generally:
- Cuspidal representation $\rightsquigarrow$ automorphic L-function.
- Points of a smooth projective variety $X/\mathbb{Q}$ mod $p \rightsquigarrow$ Hasse–Weil zeta function / L-function.

Some of these classes are contained in others and there is in general a lot of overlap. E.g., the Riemann zeta function, or, in some instances, its corresponding *completed L-function*

$$\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s),$$

can be seen as:

- the Dedekind zeta function $\zeta_{\mathbb{Q}}(s)$ of the rational field $\mathbb{Q}$;
- the Dirichlet L-function of the trivial character $\chi \equiv 1$;
- the automorphic L-function associated to the trivial (cuspidal) automorphic representation of $\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})$; or
- the Hasse–Weil zeta function of a variety consisting of a single point.

Going back to the idea/claim that reciprocity laws can be expressed as equalities of certain L-functions, it turns out that what we saw in the first part of the talk can be expressed as follows. (Technically, neither variety we are considering here is projective, but the necessary modification is straightforward.)

3.1. *Examples.*

(1) If X is the vanishing set of $t^2+1 \in \mathbb{Z}[t]$, then $L(s,X) = L(s,\chi)$ for the cusp form $\chi(\cdot) = \left(\frac{-1}{\cdot}\right)$ for GL_1.
(2) If X is the vanishing set of $t^3-t-1 \in \mathbb{Z}[t]$, then $L(s,X) = L(s,f)$ for a certain Hecke eigenform f ($\rightsquigarrow$ cusp form for GL_2). $\diamond$

Interestingly, there is another "reciprocity law" which can be formulated in a similar way:

3.2. *Examples.*

(3) If $X = E$ is an elliptic curve over $\mathbb{Q}$, then $L(s,E) = L(s,f)$ for a certain Hecke eigenform f. $\diamond$

This is, in fact, the *modularity theorem*, proved building on work by Wiles and Taylor–Wiles.

It is conjectured that there is a more general form of these "reciprocity laws" that holds for even more general objects than algebraic varieties. More precisely,

3.3. Conjecture (after Clozel). *Let F be an algebraic number field. There is a correspondence between motives M over F (+ conditions), on one side, and regular algebraic (a.k.a.: cohomological) cuspidal representations Π of $\mathrm{GL}_n(\mathbb{A}_F)$, on the other, such that the partial L-functions agree up to a shift: $L^\infty(\Pi,s) = L^\infty(M, s+\frac{1-n}{2})$.*

Here,

- ... to be *regular algebraic* or *cohomological* means to contribute nontrivially to the cohomology of arithmetic subgroups of GL_n in a certain precise way. This is a natural generalization of the properties of holomorphic cuspidal forms which were mentioned in §2.

- ... a *motive* is an abstraction of an algebraic variety, which captures the various realizations that a variety over $\mathbb{Q}$ (or an algebraic number field F) has in terms of cohomology groups.
- ... $L^\infty(\cdot, s)$ denotes the *partial* L-function, where we exclude the factors corresponding to the archimedean places.

4 Special L-values

I think it is a pretty obvious question, once you have defined yourself a function, to want to compute its values exactly at some points in the domain. Here the point of course is not to approximate the value numerically to arbitrary precision, which I believe should be relatively easy to do once you have the infinite expansion and the functional equation, but to actually prove closed formulae relating the value of the function to other known constants. The interesting thing here is that, just like you get functions that behave almost identically from objects which are very different and each time with different methods, so too you get similar results about their special values but with completely different methods.

The easiest example is, as usual, the Riemann zeta function. In this case, it is known that

$$\zeta(n) \begin{cases} \in \mathbb{Q}^\times, & \text{if } n < 0 \text{ odd}, \\ = 0, & \text{if } n < 0 \text{ even}, \\ \in \pi^n \mathbb{Q}^\times, & \text{if } n \geq 0 \text{ even}. \end{cases}$$

(Of course these are related by the functional equation.) Note that this leaves out precisely the positive odd integers. With the exception of $s = 1$ (where $\zeta(s)$ has a pole), the reason for this has to do with the poles of the Γ factors in the functional equation:

$$\Lambda(s) = \Lambda(1 - s),$$

where

$$\Lambda(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

So we know that ζ is analytic for $\mathrm{Re}(s) > 1$, but the Γ function has a pole at negative integers, so ζ must have simple zeros at *even* negative integers to compensate for those poles, but then the functional equation doesn't help us with the values at *odd* positive integers.

More generally, L-functions (either provably or conjecturally) always have an Euler product ranging over all primes of the ground field *as well as* a global functional equation relating values at s and (up to choices of normalization) $1 - s$. This global functional equation comes from local functional equations at each (finite or infinite) prime. Putting together the local functional equations for the infinite primes, we have something of the form

$$L_\infty(s, M) = (\text{some factor}) \cdot L_\infty(1-s, M^\vee)$$

(we wrote this down for motives but it is true also e.g. for automorphic L-functions). We can then again hope to say something about the values of the L-function at those points where neither side has a pole, just as we did for the Riemann zeta function (which is the partial L-function of the "unity motive" $\mathbb{Z}(0) = H^*(\text{point})$).

In fact, Pierre Deligne formulated the following definition in the motivic context. We define a *critical point* for $L(s, M)$ to be an integer $s_0 \in \mathbb{Z}$ for which both sides of the above equation are holomorphic in s_0. (In the case of the Riemann zeta function, these are precisely the positive even integers and the negative odd integers.) Then, he formulated the following

4.1. Conjecture (Deligne). *Let M be a motive defined over $\mathbb{Q}$, and let $s_0 \in \mathbb{Z}$ be a critical point for $L(s, M)$. Then $L(s_0, M)$ is, up to an element in a precise number field $E = E(M)$ dependent (only) on M, equal to some other complex number (a "*period*") coming exclusively from the realizations of M in cohomology.*

(Historically, *periods* are integrals of algebraic cycles over algebraic chains. There is a reason for this nomenclature, because there is, at least in some important special cases, a connection with something being periodic, but it is a long story.)

The periods themselves are only defined up to a factor in $E^\times$. For instance, if we specialize to the Riemann zeta function, then the period corresponding to the critical point $s_0 = m$ can be taken to be $(2\pi i)^m$ if m is a positive even integer and 1 if m is a negative odd integer, and $E(M)$ is simply the rational field $\mathbb{Q}$.

Now, since motivic L-functions and automorphic L-functions are conjectured to be the same up to a shift, it is sensible to posit the following definition.

4.2. Let Π be (say) a unitary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, for some number field F and some positive integer n. We define a *critical point* for $L(s, \Pi)$ to be a number $s_0 \in \frac{n-1}{2} + \mathbb{Z}$ for which both sides of the "archimedean functional equation"

$$L_\infty(s, \Pi) = (\text{some factor}) \cdot L_\infty(1-s, \Pi^\vee)$$

are holomorphic in s_0.

(Observe that, because of the shift, the critical points for automorphic L-functions are then possibly half-integers.) In view of Deligne's conjecture, we might expect that, with Π and s_0 as above, the *special value* $L(s_0, \Pi)$ is, up to an element in a precise number field $E = E(\Pi)$ dependent only on Π, equal to a quantity—an invariant, if you will—coming exclusively from a representation-theoretic study of Π. Results which confirm this expectation in concrete cases can serve, on top of their intrinsic value, to provide "evidence" for Deligne's conjecture.

5 My dissertation topic

Having discussed motivation and setup at length, finally I come to addressing my own dissertation topic within this larger framework.

In short, the goal is to obtain rationality results for quotients of special values of Rankin–Selberg automorphic L-functions.

Recall that, to Π, Π' globally generic, unitary automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, $\mathrm{GL}_m(\mathbb{A}_F)$ respectively (F a number field), with $1 \le m \le n$, one can associate $L(s, \Pi \times \Pi')$, the *Rankin–Selberg automorphic L-function.*

(This is a generalization of the "standard" L-function: when for Π' we take the trivial representation of $\mathrm{GL}_m = \mathrm{GL}_1$, we obtain the standard L-function of Π, i.e. the L-function $L(s, \Pi)$ we saw in the previous discussion.)

In my dissertation project,

- F is totally real,
- n is odd, m is even,
- Π and Π' are additionally assumed (unitary) cuspidal and *cohomological.* (Hence, conjecturally, related to motives.)

What we ultimately want is a rationality result for quotients of the form

$$\frac{L(s_0, \Pi \times \Pi')}{L(1 + s_0, \Pi \times \Pi')}, \tag{1}$$

where s_0 and $s_0 + 1$ are consecutive critical points of the Rankin–Selberg L-function $L(s, \Pi \times \Pi')$. Results like these are well-understood in the case $m = n - 1$ by the works of Raghuram–Shahidi, Grobner–Harris, Grobner–Raghuram, Grobner–Lin, Li–Liu–Sun among others, based on earlier work by Shimura, Harder, Mahnkopf among others. In the case where m is not $n - 1$, we can draw on an observation of Mahnkopf and use *isobaric sums* to reduce to the known case.

More precisely, to obtain our result we will introduce an auxiliary automorphic representation of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$. Concretely, this representation, which will be denoted by Σ, will be obtained as an isobaric sum

$$\Sigma := \Pi' \boxplus \Pi_1 \boxplus \cdots \boxplus \Pi_r,$$

where r denotes the nonnegative integer $\frac{n-m-1}{2}$ and the summands $\Pi_1, \ldots, \Pi_r$ on the right-hand side will be suitably-chosen unitary cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_F)$.

The point of introducing the auxiliary representation Σ is, again, that we already have a rationality result for special values of the Rankin–Selberg L-function $L(s, \Pi \times \Sigma)$ with Π and Σ (and F) as above, namely Thm. 1.4 from the recent preprint by Li, Liu and Sun. On the other hand, we have that

$$L(s, \Pi \times \Sigma) = L(s, \Pi \times \Pi') \cdot \prod_{i=1}^{r} L(s, \Pi \times \Pi_i)$$

since Σ is an isobaric sum. The proof will then be completed by a result of Harder and Raghuram (see Thm. 7.40 in their paper *Eisenstein cohomology for* GL_N *and ratios of critical values of Rankin–Selberg L-functions*) on quotients of consecutive special values of Rankin–Selberg L-functions for $\mathrm{GL}_n \times \mathrm{GL}_2$ (with n odd) over a totally real field, which we will apply to each of the L-functions $L(s, \Pi \times \Pi_i)$. Naturally, to ensure that one can choose the remaining summands $\Pi_1, \ldots, \Pi_r$ in such a way that the aforementioned results by Li–Liu–Sun and Harder–Raghuram can be applied in our setting, we will have to impose some (mild) conditions on Π'. Note that a very similar technique can be found in Grobner–Sachdeva, but over a *CM-field* rather than a totally real one and with the parities reversed (i.e., n even and m odd).

On the other hand, since Harder–Raghuram's aforementioned Thm. 7.40 concerns more generally ratios of special values of Rankin–Selberg L-functions for $\mathrm{GL}_n \times \mathrm{GL}_m$, with nm even, it can also be applied to (1) directly. What arises from this, together with the previous discussion, is an equality (up to nonzero scalars in some concrete number field) relating two different types of periods: on the one hand, the *Whittaker periods* that appear in the aforementioned Thm. 1.4 by Li–Liu–Sun, and on the other hand, the *relative periods* introduced by Harder–Raghuram that appear in the statement of their Thm. 7.40. Working out some consequences of this relation will be the next step in my research!