Unitary induction for locally compact groups

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1 Motivation: induction for finite groups

The history of induced representations begins with Frobenius, who introduced them for finite groups from a purely algebraic viewpoint. It is worthwhile to go over the original construction (or some of the several equivalent ones) before moving on to the more complicated general case (section 2). Accordingly, let the following be in force until the end of the section (unless explicitly stated otherwise):

Conventions 1.1. G denotes a finite group, H denotes a subgroup of G, and $G/H = \{gH : g \in G\}$ denotes the set of *left* cosets.

A "representation" of a group means a linear representation on a complex vector space.

Finally, (π, V) denotes a fixed but arbitrary representation of the subgroup H. $\diamond$

With the above conventions, set

$$h \cdot v \coloneqq \pi(h)v$$

for $h \in H$ and $v \in V$. The goal is to extend this to a rule that works for general $g \in G$ and again defines a left action by linear endomorphisms.

Naturally, any element of G lies in some left coset for H, so upon choosing a system of representatives $\{g_1, \dots, g_n\}$ for G/H, i.e.,

$$G = \coprod_{i=1}^{n} g_i H, \tag{1}$$

for arbitrary $g \in G$ and $v \in V$ it must hold that

$$g \cdot v = g_i \cdot (\underbrace{h \cdot v}_{\in V}) \qquad \text{where } g = g_i h \in g_i H$$

and thus it suffices to know how each g_i acts.

At this point clearly there is a choice involved. The most general ("universal"[1]) way to go about this now is to think of each vector space $g_iV := \{g_i \cdot v, v \in V\}$ as an independent copy of V. Then, by the defining properties of a left action of G, for an element $g \in G$ and an element $g_i \cdot v$ in some g_iV the equation

$$g \cdot (g_i \cdot v) = (g \cdot g_i) \cdot v = (g_k \cdot h) \cdot v = g_k \cdot \pi(h)v \in g_kV \tag{2}$$

should hold with the "obvious" choice of an index $k \in \{1, \ldots, n\}$ and an $h \in H$ (namely: k is such that $gg_i \in g_kH$, and then h is the element $g_k^{-1}gg_i \in H$). Observe in particular that none of the spaces g_iV is invariant under G in general: in fact, each g induces a permutation σ_g of the set of indices $\{1, \ldots, n\}$ via

$$gg_i \in g_{\sigma_g(i)}H.$$

Thus, in order to have a space invariant under the action in (2), one is led to consider the complex vector space

$$\bigoplus_{i=1}^{n} g_iV$$

(note the similarity with (1)) with the left action[2] of G given by

$$g \cdot \sum_{i=1}^{n} g_iv_i := \sum_{i=1}^{n} g_{\sigma_g(i)}\pi(\underbrace{g_{\sigma_g(i)}^{-1}gg_i}_{\in H})v_i \tag{3}$$

(cf. (2)). This is the construction of the induced representation found e.g. in Serre's *Linear representations of finite groups*, §3.3.

Realization A (Outer direct sum)
(First, choose a full system of representatives $\{g_1, \ldots, g_n\}$ for G/H.)
The representation space is

$$\bigoplus_{i=1}^{n} g_iV$$

and the action is given by (3) *supra*.

[1]Indeed, the induced representation as defined in this section can be shown to satisfy a universal property. As long as one works with algebraic representations, this universal property can be used to *define* the induced representation. Cf. also the remark at the end of this section.

[2]That this is indeed a left action boils down to associativity of the group operation on G.

A more natural way to look at the representation space $\bigoplus_{i=1}^n g_i V$ presents itself if one observes that the notation $\sum g_i v_i$ is really just an unorthodox way of writing a (set-theoretic) function

$$f : \{g_1, \dots, g_n\} \to V,$$

namely the one mapping g_i to v_i. Clearly this sets up an identification not only as sets, but even as complex vector spaces. It is then a trivial step to view the representation space simply as the space of (set-theoretic) functions $G/H \to V$. In this way, one obtains a different, but equivalent, realization of the induced representation:

Realization B (Functions on the quotient)
(First, choose a full system of representatives $\{g_1, \dots, g_n\}$ for G/H.)
The representation space is

$$V^{G/H} = \{f \mid f \text{ is a function } G/H \to V\}$$

and the action is again given by (3) *supra*.

Observe that, while this is a compact description *for the space*, the formula for the action still inevitably involves the g_i's and thus relies on a non-canonical choice. The next goal, therefore, is to give a more "intrinsic" realization.

First observe that, since the cosets $g_i H$ form a partition of G, a function $f : \{g_1, \dots, g_n\} \to V$, which may be viewed indifferently as $f : G/H \to V$, can be meaningfully extended to G. Since these functions go into a linear space, an extension would have to look something like

$$f(g_i h) = \underbrace{\text{"some linear endomorphism of } V\text{"}}_{=:A} \cdot f(g_i) \quad \forall i \, \forall h,$$

where the endomorphism $A = A(h, i)$ depends on h and possibly on i. Conversely – having fixed the endomorphism A – functions of this form are uniquely determined by their restriction to $\{g_1, \dots, g_n\}$, and so (for any A) they form a complex linear space isomorphic to the one of Realization B *supra*.

Since a group homomorphism $\pi : G \to \mathrm{GL}(V)$ is already given, the obvious choice for A would seem to be $A = \pi(h)$ (without any dependence on the index i). However, since the goal is to eliminate the dependence on the choice of $\{g_1, \dots, g_n\}$, it is necessary to take a group <u>anti</u>homomorphism. In fact, if $A : H \to \mathrm{GL}(V)$ is an antihomomorphism, then

$$f(g_i h) = A(h) f(g_i) \text{ for } i = 1, \dots, n \text{ and } h \in H$$

already implies

$$f(gh) = A(h)f(g) \text{ for } \underline{\text{all}}\ g \in G, h \in H,$$

as shown by the following computation: (write $g = g_i\widetilde{h}$)

$$\begin{aligned} f(gh) &= f(g_i\widetilde{h}h) \\ &= A(\widetilde{h}h)f(g_i) \\ &= A(h)A(\widetilde{h})f(g_i) \\ &= A(h)f(g_i\widetilde{h}) \\ &= A(h)f(g). \end{aligned}$$

It is then only natural to look at the antihomomorphism $A(h) := \pi(h^{-1})$.

To summarize the immediately preceding discussion, the elements of

$$V^\pi := \{f : G \to V : f(gh) = \pi(h^{-1})f(g) \text{ for all } g \in G, h \in H\}$$

have the property that, for any full system of representatives $\{g_1, \ldots, g_n\}$ for G/H, they are uniquely determined by their restriction to $\{g_1, \ldots, g_n\}$. Thus a choice of $\{g_1, \ldots, g_n\}$ sets up an identification (*a priori* as sets, but even as complex vector spaces) of V^π with the representation space $V^{G/H}$ from Realization B.

It is now straightforward to pull back the action of G as given in Realization B to the space V^π along any isomorphism $V^\pi \to V^{G/H}$. On the other hand, the space V^π, viewed as a subspace of the (complex linear) space V^G of *all* (set-theoretic) functions from G to V, is stable under the left action of G on V^G given by

$$(L(g)f)(x) := f(g^{-1}x)$$

(the *left regular representation*). Perhaps surprisingly, these two procedures lead to the very same action on V^π. Equivalently – and this is relatively easy to check – for any choice of $\{g_1, \ldots, g_n\}$ the linear isomorphism

$$\begin{aligned} V^\pi &\to \bigoplus_{i=1}^n g_i V \\ f &\mapsto \sum_{i=1}^n g_i f(g_i) \end{aligned}$$

intertwines the action L given above with the one of G given in Realization A. In conclusion, the following is yet another realization of the induced representation:

Realization C (Functions on G; left translation)
The representation space is

$$V^\pi := \{f : G \to V : f(gh) = \pi(h^{-1})f(g) \text{ for all } g \in G, h \in H\}$$

and the action $L : G \to \mathrm{GL}(V^\pi)$ is given by

$$(L(g)f)(x) := f(g^{-1}x)$$

for $g \in G$ and $x \in G$.

Observe that this is very sleek and elegant. However, aesthetically speaking, some might prefer to work with the *right regular representation*

$$(R(g)f)(x) := f(xg)$$

of G on V^G. This poses no real problems since L and R are equivariant representations: in fact, the map $f \mapsto \check{f}$, where

$$\check{f}(x) := f(x^{-1}) \qquad \text{for } x \in G,$$

is a linear isomorphism of V^G into itself which intertwines L and R, i.e., $R(g)\check{f} = (L(g)f)\check{}$. Thus, plainly, G operates on

$$\{\check{f} \text{ where } f : G \to V, f(gh) = \pi(h^{-1})f(g) \text{ for all } g \in G, h \in H\}$$

by right translation R. This yields the final realization of the induced representation:

Realization D (Functions on G; right translation)
The representation space is

$$\{f : G \to V : f(hg) = \pi(h)f(g) \text{ for all } g \in G, h \in H\}.$$

and the action R is given by

$$(R(g)f)(x) := f(xg)$$

for $g \in G$ and $x \in G$.

Of course, functions in the space of Realization D are uniquely determined by their values on a full system of representatives for *right* H-cosets, rather than left.

This is only a minor detail in the case of finite groups, but it will become apparent in the study of unitary induction (see section 2) that the initial choice of working with left actions, left cosets, &c. influences many other choices downstream and that, because of this, realizations where the action is by left translation turn out to be easier to construct (and to find references for) than ones where the action is by right translation.

Remark 1. Yet another way of defining the induced representation uses the isomorphism of categories between representations of a finite group and modules over its group ring. (See also Serre, *op. cit.*, §7.1; note, however, that there is not much there in the way of proofs.)

Recall that, for any group G (not necessarily finite) and any ring R (not necessarily commutative), the *group ring* $R[G]$ is the set

$$\begin{aligned} R[G] &:= \bigoplus_{g\in G} R \\ &= \{f : G \to R : f(g) \neq 0 \text{ for at most finitely many } g \in G\}, \end{aligned}$$

equipped with "component-wise" (in the first description) resp. "pointwise" (in the second description) addition, and with multiplication given by convolution: if $f_1, f_2 \in R[G]$ are viewed as functions, then $(f_1 \cdot f_2)(g) := \sum_{g=xy} f_1(x)f_2(y)$ for $g \in G$.

Clearly, if G is finite, then $R[G]$ is none other than the set of *all* (set-theoretic) functions from G to R. Let moreover $R = K$ be a field. Then it is a relatively standard fact that any linear representation of G on a K-linear space is also a $K[G]$-module in a canonical way, and viceversa. This can be checked to yield an isomorphism between the category $\mathbf{Rep}_K(G)$ (whose objects are linear representations of G on K-linear spaces and whose morphisms are K-linear G-equivariant maps) and the category $K[G]$-**mod** of left $K[G]$-modules.

It turns out that taking induced representations at the level of groups corresponds to extending scalars at the level of rings: if (π, V) is a linear representation of a subgroup $H \subseteq G$ on a $\mathbb{C}$-linear space, and V is viewed as a left $\mathbb{C}[H]$-module in the canonical way, then the induced representation of G is the one corresponding to the $\mathbb{C}[G]$-module $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$. This makes it less mysterious that the induced representation can be realized as a space of functions from G to V subject to some condition coming from π, or that it satisfies a universal property of sorts. It can also be checked that induction defines a functor $\mathbf{Rep}_K(H) \to \mathbf{Rep}_K(G)$ that is left adjoint to the "restriction" functor $\mathbf{Rep}_K(G) \to \mathbf{Rep}_K(H)$. This immediately sheds light on such phenomena as Frobenius reciprocity. . .

2 Unitary induction

2.1 Some context

For many purposes, including some applications to theoretical physics, one is confronted with the task of investigating (unitary) representations of infinite groups, prominently matrix groups over the reals or the complex numbers. Since these are naturally equipped with a topological (and even a smooth) structure, one is led to consider topological groups and their (unitary) continuous [see the Conventions below for clarification] representations.

A subclass of topological groups that is of particular interest is that of *locally compact* groups [see also the Conventions below], which includes all discrete groups, compact groups, Lie groups and p-adic groups. This is the subclass for which the theory is most highly developed (and arguably quite beautiful), due, among other factors, to the availability and essential uniqueness of invariant measures. The study and classification of irreducible unitary representations of locally compact groups can also be viewed as – and is often termed – (non-commutative) *harmonic analysis.*[3]

Remark 2. A further reason for restricting to unitary representations of locally compact groups is that the category they form (morphisms being, of course, intertwining operators) is equivalent to the category of representations of a certain "group algebra", whose role is thus entirely analogous to that of the group ring in Remark 1. The precise statement is as follows (all references will be to Wallach's two-volume book [1]).

Let G be a locally compact [see Conventions below], separable topological group. Define the C^**-algebra* (read: "C-star") of G as in 14.2 (see 14.1.6 for the general definiton of C^*-algebras). Consider now the category of nondegenerate representations of $C^*(G)$ (see 14.1.13–14). Then (Thm. 14.2.5) this category is equivalent to that of continuous unitary[4] representations of G.

[3] "Classical" harmonic analysis is Fourier analysis, including applications to time-frequency analysis, signal processing, &c. "Abstract" harmonic analysis is the generalization of Fourier analysis to arbitrary locally compact abelian (LCA) groups. For such groups, the unitary dual, i.e. the set (topological space) of irreducible unitary representations, is simply the group of unitary characters equipped with the compact-open topology, a.k.a. the *Pontrjagin dual* of the original group, and many classical results in Fourier theory such as the Fourier inversion formula, the Plancherel formula, &c. generalize to LCA groups.

[4] Wallach's "unitary" representations are by definition unitary *continuous* representations, see footnote 8 ahead.

2.2 Preliminaries and conventions

Unitary induction for locally compact groups was originally introduced by George Mackey under some countability assumptions on both the group at hand and the Hilbert spaces on which the group operated, cf. e.g. §5 of his survey [2] (as well as references **[77]-[80]** *ibid.* for the original papers). As Mackey himself admits in *loc. cit.*, his construction of the induced representation can really only be made "perspicuous" in the case that the space of (right) H-cosets has invariant measure [cf. Conventions below]. More "perspicuous" constructions will be given in the next subsection; the "optimal" one seems to be due to Blattner.

The main references for most of this section are [3, §§2.3–2.4] and [4, §6.1], which read quite similarly. It is often instructive to compare with Warner's book [5], esp. §5.1, but the reader should be aware that, in this book, there is *almost always an additional assumption that the group at hand be countable at infinity*[5], and in §5.1, the group G is *even assumed to be a Lie group* throughout.

Finally, an alternative construction is presented in Wallach's book [1, §1.5], but note that it is only valid under restrictive assumptions.

This being so, it is now a good time to lay down the conventions that will be in force until the end of the section.

Conventions 2.1 (Locally compact spaces, groups)**.** A topological space will be said to be *locally compact* if every point has a compact neighbourhood.

By this definition, locally compact spaces need not be Hausdorff. (*Caveat*: this contrasts with some authors' usage, most notably Bourbaki's!) However, following widespread convention, the phrase *locally compact group* will be reserved for topological groups whose topology is both locally compact and Hausdorff. ◇

Conventions 2.2 (G and H)**.** G will denote a fixed but arbitrary locally compact group, and H will denote a closed subgroup of G. (Observe that then H is again a locally compact group with the subspace topology.) ◇

Conventions 2.3 (Measures; $C_c(X)$)**.** In this note, there is no need for measure theory in greatest generality: it will only be necessary to consider (Borel) measures on locally compact Hausdorff (LCH) topological spaces.

Thus, following Bourbaki's *Integration* ([6, Chapter III, §1, No. 5]), a *(positive*

[5] i.e., that it is a countable union of compact subspaces; some use the term "σ-compact" instead.

real) measure on a LCH[6] topological space X is defined to be a positive linear functional on the space $C_c(X;\mathbb{R})$ of real-valued compactly supported continuous functions $X \to \mathbb{R}$.

Similarly (cf. *ibid.*, No. 3), a *complex measure* on X is by definition a continuous linear functional on $C_c(X)$ ($= C_c(X;\mathbb{C})$).

Caveat: What is called a "measure" in [6] and in this note is called a "Radon measure" or a "regular Borel measure" by other authors, cf. e.g. [4, pp. vii–viii] and [3, p. 2]. These authors then use some (unstated) version of the Riesz representation theorem (`https://en.wikipedia.org/wiki/Riesz-Markov-Kakutani_representation_theorem`) to obtain Radon measures out of positive linear functionals.

It is also worth noting that usage of the terms "Radon", "regular" and even "Borel" can be far from canonical, cf. `https://mathoverflow.net/questions/109505/about-the-definition-of-borel-and-radon-measures`. ◇

Conventions 2.4 (Haar measures)**.** Left Haar measures[7] on G and H are fixed now once and for all. When left Haar measure of G (resp., H) appears in integrals, it will be denoted simply by $\mathrm{d}g$ (resp., $\mathrm{d}h$), or $\mathrm{d}x$, depending on the integration variable. ◇

Remark 3 (Modular functions)**.** Recall that left Haar measure on a general locally compact group G need not be right-invariant. Instead, for every $g \in G$, there exists a positive real number $\Delta_G(g)$ such that (if μ denotes left Haar measure, then) $\mu(Eg) = \Delta_G(g)\mu(E)$ for every Borel subset $E \subseteq G$.

Similarly, for each $g \in G$ there is a positive real number $\delta_G(g)$ such that

$$\int_G f(xg)\,\mathrm{d}x = \delta_G(g) \int_G f(x)\,\mathrm{d}x$$

for any function f for which the integral makes sense. In fact, $\delta_G(g) = \Delta_G(g^{-1}) = \Delta_G(g)^{-1}$ for all $g \in G$. Thus, for any $g \in G$, the formulae

$$\int_G f(x)\,\mathrm{d}x = \Delta_G(g) \int_G f(xg)\,\mathrm{d}x$$

and

$$\int_G f(g^{-1})\,\mathrm{d}g = \int_G f(g)\Delta_G(g^{-1})\,\mathrm{d}g$$

hold for all f as above. Observe that, since left Haar measure is unique up to scalars, Δ_G, or equivalently δ_G, is truly unique.

[6] Recall that Bourbaki write "locally compact" in lieu of "locally compact Hausdorff" throughout *Integration*.

[7] By a well-known result of Haar, cf. [6, Chapter VII, §1, No. 2, Thm. 1], if G is locally compact group, then there exists a nonzero left-invariant measure [in the sense of the preceding Convention] on G, which is furthermore unique up to positive scalars; any such measure is called *(left) Haar measure* on G.

Conventions 2.5 (The modular function)**.** In the following, the phrase "modular function of G" (resp., H) will refer to Δ_G, resp. Δ_H.

(This is in line with the conventions and notations of [3]. In Warner's book [5], the "modular function", denoted by δ_G, is the same as "our" modular function, cf. *op. cit.*, p. 474. Beware however that, in Wallach's book [1], the "modular function", denoted δ_G, is by definition the same as "our" δ_G.) $\diamond$

Conventions 2.6 (The quotient G/H)**.** Consider the canonical map $q : G \to G/H$, where G/H denotes the set of left H-cosets. The image of $x \in G$ under q will also be denoted xH or $\dot{x}$. The quotient topology on G/H makes G/H into a locally compact Hausdorff space. A *section* $G/H \to G$ is a right inverse to q.

The quotient space G/H will be said to have "invariant measure" if there is a nonzero positive measure μ on G/H that is invariant with respect to multiplication on the left by elements of G, i.e., for each $g \in G$ the measure μ_g defined by

$$\mu_g(E) = \mu(gE) \qquad \text{for all Borel sets } E \subseteq G/H$$

is precisely equal to μ. Moreover, if such a μ exists, it is unique up to a constant factor ([4, Thm. 2.49]). $\diamond$

Conventions 2.7 (Representations; continuity; (π, V))**.** As in the previous section, a "representation" of a group means a linear representation of said group on a complex vector space. The representation spaces will always come equipped with the structure of Hausdorff topological vector spaces over $\mathbb{C}$ and, in fact, will as a general rule be Hilbert spaces.

The phrase "continuous representation" is to be intended as in [5, Vol. I, p. 219]. (If the representation space is barrelled, then this notion is equivalent with that of "strong continuity" as argued in *loc. cit.*, and if the space is additionally Banach, then both notions coincide with that of "weak" continuity, as proved in [5, Thm. 4.2.2.1].)

Phrases such as "Hilbert representation" or "Banach representation" are to be interpreted as "continuous representation on a Hilbert (resp., Banach) space".

Finally, (π, V) will denote a fixed but arbitrary *unitary* continuous representation of the subgroup H on a Hilbert space V.[8] The inner product on V will be denoted $\langle \cdot, \cdot \rangle_V$ if necessary. $\diamond$

[8] Note that, since V is assumed Hilbert and by the above discussion, "our" (= Warner's) unitary continuous representations are the same as Kaniuth-Taylor's ([3, p. 21]), Folland's ([4, p. 67f.]) and Wallach's ([1, §§1.1.1–1.1.2]).

2.3 The search for the optimal realization

The aim of this subsection is to get to the “best possible” definition of unitarily induced representation. Rather than a “top-down”, definition-theorem-proof approach, the idea here is to make the procedure seem “natural” and introduce the necessary concepts as they arise along the way. The “model” for the entirety of the discussion will be Realization C from section 1; in other words, the goal is to realize the induced representation on a Hilbert space of (essentially) functions from G to V, with the unitary action of G on the space being simply by left translation. (Having achieved this, alternative realizations will be presented which are more reminiscent of Realizations B and D from section 1.)

Since the representation space should be a Hilbert space made up (essentially) of functions from G to V, it is natural to draw inspiration from the standard construction of L^2 spaces. Of course, this entails pinpointing a precise notion of “measurability” for functions $G \to V$. We shall come back to this issue in the beginning of the next subsection, but for the moment, we shall work with measurability in the *Bochner* sense.

In conclusion, keeping in mind the construction of L^2 spaces as well as Realization C from the previous section, one might tentatively proceed as follows: consider the space of measurable functions from G to V satisfying

$$f(gh) = \pi(h^{-1})f(g) \qquad \text{for all } g \in G, h \in H. \tag{4}$$

Impose a condition

$$\int \ldots < \infty \tag{5}$$

that some integral involving f be finite (which will then, by definition, be equal to $\|f\|^2 = \langle f, f\rangle$), and subsequently identify functions that agree almost everywhere (with respect to the fixed left Haar measure on G). Then the space should become a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ (which arises from applying the polarization identity to $\|\cdot\|$). Additionally, the action of G on this space should be by left translation,

$$(L(g)f)(x) \coloneqq f(g^{-1}x)$$

(where as usual no distinction is made notationally between a function f and its equivalence class).

The next step is to pinpoint which integral should appear in (5). The key observation to this end is that, because of (4) and because π is unitary by assumption,

$$x \mapsto \langle f(x), f(x) \rangle_V$$

is a (nonnegative real-valued) function on G that is constant on left H-cosets, and hence descends to a function on G/H.

Thus, **if G/H has invariant measure** μ, it is only natural to choose (5) to read as follows:

$$\int_{G/H} \langle f(x), f(x) \rangle_V \, \mathrm{d}\mu(\dot{x}) < \infty. \tag{6}$$

(The notation should remind one that integration is with respect to the variable $\dot{x} = xH$ which runs over the elements of G/H.) The corresponding inner product is then

$$\langle f, g \rangle_{\mathrm{ind}} \coloneqq \int_{G/H} \langle f(x), g(x) \rangle_V \, \mathrm{d}\mu(\dot{x}) \tag{7}$$

and, by left invariance of μ, the operators $L(g)$ from above are, indeed, unitary for all $g \in G$. For convenience, the above process is summarized in the following box:

Realization C 2.0 (assuming invariant measure μ on G/H)
Choose a nonzero positive invariant measure μ on G/H, if possible.[9]
The Hilbert representation space is the space of measurable functions $f : G \to V$ satisfying

$$f(gh) = \pi(h^{-1}) f(g) \qquad \text{for all } g \in G, h \in H \tag{4↑}$$

and

$$\int_{G/H} \langle f(x), f(x) \rangle_V \, \mathrm{d}\mu(\dot{x}) < \infty, \tag{6↑}$$

with functions being identified if they agree almost everywhere (w.r.t. left Haar measure on G); the inner product on this space is given by

$$\langle f, g \rangle_{\mathrm{ind}} \coloneqq \int_{G/H} \langle f(x), g(x) \rangle_V \, \mathrm{d}\mu(\dot{x}) \,. \tag{7↑}$$

The action is by left translation precisely as in Realization C.

[9]Observe that replacing μ by another invariant measure on G/H, i.e. by a scalar multiple of μ [see the previous subsection], yields the same linear space and only changes the inner product by a scalar. Hence: while Realization 2.0 is not truly unique, its unitary equivalence class is.

While this already gives a complete solution *in the case where G/H has invariant measure*, the problem is that this does *not* happen very often. (In fact, G/H has invariant measure if and only if the restriction of Δ_G to H agrees with Δ_H, see also [4, Thm. 2.49].) It is, however, *always* the case that G/H can be equipped with a *(strongly) quasi-invariant* measure. For the purposes of this discussion, this existence result can be phrased as follows [but see also the Remark below a more systematic approach]: G and H being as above, there exist a nonzero positive measure μ on G/H and a real-valued function ρ on G with the following properties:

(a) $\rho(x) > 0$ for all $x \in G$;

(b) ρ is continuous;

(c) ρ satisfies:

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(x) \qquad \text{for all } x \in G, h \in H; \tag{8}$$

and finally,

(d) the equation

$$\int_{G/H} \int_H f(xh)\,\mathrm{d}h\,\mathrm{d}\mu(\dot{x}) = \int_G f(x)\rho(x)\,\mathrm{d}x \tag{9}$$

holds for all $f \in C_c(G)$.

We now fix such a μ and such a ρ and retain these notations for the rest of this subsection. About this choice, note the following facts:

- μ and ρ actually determine each other uniquely, and accordingly ρ will sometimes be called the "rho-function" corresponding to μ; cf. the remark immediately below.
- Any two quasi-invariant measures on G/H have the same null sets ([3, Corollary 1.23]; cf. also *op. cit.*, Lemma 1.22 with [4, Thm. 2.64]. Alternatively see [6, §2, No. 5, Thm. 1(a)]).
- The case of invariant measure, which was already "solved", will be contained as a special case of the discussion below: for if G/H has invariant measure μ, then μ is quasi-invariant and, upon scaling μ by a scalar, one may choose $\rho \equiv 1$ ([4, Thm. 2.49]).

Remark 4 (Quasi-invariant measures and rho-functions)**.** The contents of this remark are not essential to understanding the rest of the note and can safely be ignored. The background can be found in [3, §1.3], [4, §2.6] (which read very similarly to one another), or even in Chapter VII of Bourbaki's *Integration* [6]. Finally, some of this material is also discussed in Appendix 1 (pp. 474ff.) of Warner's book [5], under the usual additional assumption that all groups be countable at infinity. With these preliminaries out of the way, we can now dive in.

Recall from the previous subsection that a (nonzero positive) measure μ on G/H is said to be invariant if it is equal to all its "left translates" μ_g, $g \in G$. As a natural generalization, a nonzero measure μ on G/H is called *quasi-invariant* if, for all $g \in G$, μ and μ_g are *equivalent*, i.e., *mutually absolutely continuous*, which, by definition, means that they have the same null sets; see also [4, p. 58].

Existence of quasi-invariant measures can be proved in different ways. For instance, Warner sketches an argument due to Dieudonné on p. 474 of his book (here the assumption of countability at infinity seems to be crucial). On the other hand, the "usual" proof of existence, given e.g. in [4, Thm. 2.56], uses so-called "rho-functions"[10] and even establishes a slightly stronger result, which shall be discussed presently.

There is an arguably technical fact which underlies some points of the discussion below and which will even pop up outside of this remark. It is given here for convenience:

FACT. The map $C_c(G) \to C_c(G/H)$ sending f to

$$\dot{f} : \dot{x} \mapsto \int_H f(xh)\,\mathrm{d}h$$

is *surjective*. (Warning: The image of f is variously denoted Pf in [4], $f^\sharp$ in [3] and $f^\flat$ in [6]. A proof of surjectivity of the above rule is e.g. in [4, Prop. 2.48].)

Now for the proof of existence of quasi-invariant measures. All references in the following will be to [6], Chapter VII, §2 unless explicitly stated otherwise.

First, it is proved in No. 5, Lemma 4 that, if μ is a nonzero quasi-invariant measure on G/H, then there exists a function $\rho : G \to \mathbb{R}_{>0}$ which is locally integrable with respect to left Haar measure on G and such that the equation (9) relating μ and ρ holds. Moreover, any two such functions ρ and ρ' must agree outside of a set of measure zero. (In fact, ρ is "the" Radon-Nikodym derivative of a certain measure $\mu^\sharp$ on G with respect to left Haar measure on G, cf. also No. 2, Prop. 4 for the notation $\mu^\sharp$.)

Furthermore, a function ρ obtained in this fashion must satisfy (8) locally almost everywhere by No. 5, Lemma 5.

Finally, both implications are actually stated as equivalences in Bourbaki: going the other way, if ρ is a locally integrable function $G \to \mathbb{R}_{>0}$ that satisfies (8) locally almost everywhere, then ρ yields a quasi-invariant measure μ on G/H such that (9) holds. In fact,

$$\dot{f} \mapsto \int_G f(x)\rho(x)\,\mathrm{d}x$$

is a well-defined positive linear functional on $C_c(G/H)$, i.e. a measure on G/H, which is then quasi-invariant by the implication (c)$\Rightarrow$(a) in the aforementioned No. 5, Lemma 4. (For well-definedness, one has to keep in mind the above Fact and to check that, if $\dot{f}$ is identically zero, then also the right-hand side of the above rule is zero; this is [3, Prop. 1.13].)

In summary, proof of existence of quasi-invariant measure is reduced to proof of existence of functions ρ with the above properties. It now turns out that there even exist *continuous* functions

[10] There are different notions of "rho-functions" in the literature. Broadly speaking, they are functions satisfying properties (a)–(c) listed before this remark. Sometimes (b) is weakened to "local integrability" and/or (a) is weakened to $\rho \geq 0$ rather than > 0.

ρ with these properties (No. 5, Thm. 2(a), or [4, Prop. 2.54]). The corresponding μ is then not only quasi-invariant, but has the stronger property (No. 5, Thm. 2(c), or [4, Prop. 2.56]) that the function

$$\begin{aligned} G \times G/H &\to \mathbb{R}_{>0}, \\ (x, \dot{y}) &\mapsto \frac{\mathrm{d}\mu_x}{\mathrm{d}\mu}, \end{aligned}$$

where $\frac{\mathrm{d}\mu_x}{\mathrm{d}\mu}$ denotes the Radon-Nikodym derivative of the "translate" μ_x (see above) with respect to μ, is continuous. Following Folland, we call quasi-invariant measures with this additional property *strongly quasi-invariant.*

Finally, if μ is a strongly quasi-invariant measure on G/H, then there is a unique function ρ with properties (a)–(d) as listed before this remark. In fact, recall that there was already an argument for existence earlier in this remark (involving a certain measure $\mu^\sharp$ on G); however, to get (a)–(d) in their strong form above, strong quasi-invariance is needed, which is why it is worthwhile to look at the proof in [4, Thm. 2.59]. As for uniqueness, this follows from the fact that, among all Radon-Nikodym derivatives of $\mu^\sharp$ with respect to left Haar measure on G, there can be at most one which is continuous (because any two Radon-Nikodym derivatives differ at most on a set of measure zero).

Having fixed a strongly quasi-invariant measure μ (and a corresponding ρ), the next thing that comes to mind would be to check if the discussion leading up to Realization C 2.0 still applies *verbatim* to the general case.

This turns out to be the case as far as the representation *space* is concerned. Indeed, a closer look at the construction outlined for Realization C 2.0 shows that invariance of μ is not needed in order to construct the vector space or to define the inner product $\langle\cdot,\cdot\rangle_{\mathrm{ind}}$ on it. What *does* break down, unfortunately, is unitarity of the left-translation operators $L(g)$, $g \in G$.

To see this, it suffices to look at very special elements of the Hilbert space at hand. For let f be a function $G \to V$ which is continuous, satisfies

$$f(gh) = \pi(h^{-1})f(g) \qquad \text{for all } g \in G, h \in H \tag{4$\uparrow$}$$

and such that the image of the support of f under the projection $q : G \to G/H$ is compact in G/H. Then

$$\dot{x} \mapsto \langle f(x), f(x)\rangle_V$$

is a (well-defined) compactly supported (nonnegative real-valued) function on G/H, so its integral with respect to μ is certainly finite and so it (or more precisely its class w.r.t. equality a.e.) lies in the Hilbert space being discussed. Moreover, by the

Fact stated in the preceding Remark 4, this function can be written as $\dot{\Phi}$ for some $\Phi \in C_c(G)$. Thus, paraphrasing (9),

$$\int_{G/H} \langle f(x), f(x)\rangle_V \, \mathrm{d}\mu(\dot{x}) = \int_G \Phi(x)\rho(x)\,\mathrm{d}x$$

(and the common value is, by definition, $\langle f, f\rangle_{\mathrm{ind}}$).

Now consider

$$\begin{aligned}\langle L(g)f, L(g)f\rangle_{\mathrm{ind}} &= \int_{G/H} \langle (L(g)f)(x), (L(g)f)(x)\rangle_V \, \mathrm{d}\mu(\dot{x}) \\ &= \int_{G/H} \langle f(g^{-1}x), f(g^{-1}x)\rangle_V \, \mathrm{d}\mu(\dot{x})\,;\end{aligned}$$

by (9),

$$= \int_G \Phi(g^{-1}x)\rho(x)\,\mathrm{d}x\,,$$

and using left invariance of $\mathrm{d}x$ and again (9),

$$\begin{aligned}&= \int_G \Phi(x)\rho(gx)\,\mathrm{d}x \\ &= \int_{G/H} \langle f(x), f(x)\rangle_V \frac{\rho(gx)}{\rho(x)}\,\mathrm{d}\mu(\dot{x})\,,\end{aligned}$$

which is in general not the same as $\langle f, f\rangle_{\mathrm{ind}}$ (cf. above).

The silver lining is that this computation also immediately gives a way to fix the problem: it suffices to introduce a factor in the action of G in such a way that the ratio of ρ-values will cancel out. Keeping in mind also the change of variables in the above chain of equalities, one quickly arrives at the following:

Realization C+ρ

Fix a nonzero quasi-invariant Radon measure μ on G/H and the corresponding rho-function ρ.

The Hilbert representation space is the space of measurable functions $f : G \to V$ satisfying

$$f(gh) = \pi(h^{-1})f(g) \qquad \text{for all } g \in G, h \in H \tag{4$\uparrow$}$$

and

$$\int_{G/H} \langle f(x), f(x)\rangle_V \, \mathrm{d}\mu(\dot{x}) < \infty, \tag{6$\uparrow$}$$

with functions being identified if they agree almost everywhere (w.r.t. left Haar measure on G); the inner product on this space is given by

$$\langle f, g\rangle_{\mathrm{ind}} := \int_{G/H} \langle f(x), g(x)\rangle_V \, \mathrm{d}\mu(\dot{x}) \, . \tag{7$\uparrow$}$$

The action $\widetilde{L}$ is by left translation adjusted by a factor:

$$(\widetilde{L}(g)f)(x) := \left(\frac{\rho(g^{-1}x)}{\rho(x)}\right)^{\frac{1}{2}} f(g^{-1}x). \tag{10}$$

By rewriting the very last equation as

$$\rho(x)^{\frac{1}{2}}(\widetilde{L}(g)f)(x) = \rho(g^{-1}x)^{\frac{1}{2}} f(g^{-1}x)$$

it should become apparent that "multiplication by $\rho(\cdot)^{1/2}$ intertwines the modified left translation $\widetilde{L}$ with standard left translation L". To make this more precise, consider

$$\Big\{x \mapsto \rho(x)^{\frac{1}{2}} f(x), f \text{ is in the representation space of Realization C+}\rho\Big\}.$$

(We're being a little nonchalant here with the identification between functions and their equivalence classes, but this really poses no problems.)

It is not hard to define this space more intrinsically, building upon the properties that elements of the previous representation space are known (required) to have. For instance, let ϕ correspond to f, i.e., $\phi(x) = \rho(x)^{\frac{1}{2}} f(x)$ for all $x \in G$. Then clearly – as long as the notion of measurability is a sane one – ϕ is measurable if and only if f is. Moreover, the integral in (6) reads

$$\int_{G/H} \rho(x)^{-1} \langle \phi(x), \phi(x)\rangle_V \, \mathrm{d}\mu(\dot{x}) \, . \tag{11}$$

Finally (4) implies

$$\begin{aligned}\phi(gh) &= \rho(gh)^{\frac{1}{2}} f(gh) \\ &= \left[\rho(g)\frac{\Delta_H(h)}{\Delta_G(h)}\right]^{\frac{1}{2}} \pi(h^{-1})f(g) \\ &= \left(\frac{\Delta_H(h)}{\Delta_G(h)}\right)^{\frac{1}{2}} \pi(h^{-1})\phi(g).\end{aligned}$$

for all $g \in G$ and $h \in H$. In summary, here is a further realization of the induced representation (where we return to denoting functions by f rather than ϕ):

Realization C+Δ
Fix a nonzero quasi-invariant Radon measure μ on G/H and the corresponding rho-function ρ.
The Hilbert representation space is the space of measurable functions $f : G \to V$ satisfying

$$f(gh) = \left(\frac{\Delta_H(h)}{\Delta_G(h)}\right)^{\frac{1}{2}} \pi(h^{-1})f(g) \qquad \text{for all } g \in G, h \in H \tag{12}$$

and[11]

$$\int_{G/H} \rho(x)^{-1}\langle f(x), f(x)\rangle_V \,\mathrm{d}\mu(\dot{x})\,. \tag{11↑}$$

with functions being identified if they agree almost everywhere (w.r.t. left Haar measure on G); the inner product on this space is given by

$$\langle f, g\rangle_{\text{ind}} := \int_{G/H} \rho(x)^{-1}\langle f(x), g(x)\rangle_V \,\mathrm{d}\mu(\dot{x})\,. \tag{13}$$

The action L is by "regular" left translation:

$$(L(g)f)(x) = f(g^{-1}x).$$

For all the advantages of Realization C+Δ with respect to previous attempts, it might have been noted that it still involves a choice of a quasi-invariant measure μ or, equivalently, of a (continuous, everywhere strictly positive) rho-function ρ. But

[11]Observe that, because of (12) and (8) (and, of course, unitarity of π), the integrand in (11) is indeed a well-defined function on G/H.

even this “non-canonicity” can be eliminated, making the resulting realization truly intrinsic.

The first observation to be made is that μ and ρ now only appear in the condition (11), and not in the “functional equation” (12) or in the equation describing the action of G. Thus, it is only a matter of replacing the “finiteness of some integral” condition (cf. (5) from the very beginning) by a more instrinsic one. This can be done relatively easily using the Fact stated in Remark 4 *supra*.

Thus, let f be a measurable function $G \to V$ satisfying (12). An assumption needs to be made here, the reason for which will become apparent presently: it will be assumed that $\|f(\cdot)\|^2$, i.e., the mapping

$$\begin{aligned} G &\to \mathbb{C} \\ x &\mapsto \langle f(x), f(x) \rangle_V, \end{aligned}$$

is a locally integrable function on G. This entails that the product of this function with any $\Phi \in C_c(G)$ is integrable (in the sense that the integral exists and is finite).

This being so, it can be shown ([5, Lemma 5.1.1.1]) that the rule

$$\dot{\Phi} \mapsto \int_G \langle f(x), f(x) \rangle_V \Phi(x) \, \mathrm{d}x$$

is well-defined (cf. also the Fact stated in Remark 4) and thus yields a positive linear functional on $C_c(G/H)$, i.e., a measure μ_f on G/H. One sets

$$\begin{aligned} (f, f) &\coloneqq \mu_f(G/H) \\ &= \int_{G/H} \mathrm{d}\mu_f \end{aligned}$$

provided this is finite. Then, if g is another function satisfying the same assumptions as f and if both $\mu_f(G/H)$ and $\mu_g(G/H)$ are finite, one can define the appropriate inner product (f, g) by an application the polarization identity, obtaining

$$\begin{aligned} (f, g) &= \mu_{f,g}(G/H) \\ &= \int_{G/H} \mathrm{d}\mu_{f,g} \end{aligned}$$

where $\mu_{f,g}$ is the complex measure on G/H corresponding to the linear functional

$$\dot{\Phi} \mapsto \int_G \langle f(x), g(x) \rangle_V \Phi(x) \, \mathrm{d}x$$

on $C_c(G/H)$ (see again [5, Lemma 5.1.1.1]; in passing, note that $\mu_f = \mu_{f,f}$). In summary,

Realization C-opt (no choices, totally intrinsic, optimal)
The Hilbert representation space is the space of measurable functions $f : G \to V$ satisfying

$$f(gh) = \left(\frac{\Delta_H(h)}{\Delta_G(h)}\right)^{\frac{1}{2}} \pi(h^{-1}) f(g) \qquad \text{for all } g \in G, h \in H \tag{12↑}$$

and such that $\|f(\cdot)\|^2$ is locally integrable and

$$\int_{G/H} \mathrm{d}\mu_f < \infty \tag{14}$$

(where μ_f is constructed as in the discussion above), with functions being identified if they agree almost everywhere (w.r.t. left Haar measure on G); the inner product on this space is given by

$$(f, g) := \int_{G/H} \mathrm{d}\mu_{f,g} \, . \tag{15}$$

(see discussion above).
The action L is by "regular" left translation:

$$(L(g)f)(x) = f(g^{-1}x).$$

(We will comment on the proof of unitary equivalence between Realization C-opt and the other realizations in the next subsection.)

2.4 References and recap

For starters, we ought to give at least one reference for at least one of the above realizations. Hence, let it be said that Realization C-opt as given above is *verbatim* as in Warner's book [5, pp. 366–368]. This source also includes complete proofs (see e.g. pp. 371–372 for the proof of unitarity). A couple of remarks on the word "verbatim":

- In the whole section §5.1 of Warner's book, G is assumed to be a <u>Lie</u> group, countable at infinity. However, it is quite likely that this is not used substantially anywhere in the proofs.

- By the conventions laid out on p. 365f., Warner's phrase "d_G-measurable" means "Bochner (strongly) measurable w.r.t. left Haar measure on G". This is precisely in accordance with the notion of measurability adopted throughout the previous subsection of this note.

Next, Realization C+Δ as given above is "almost verbatim" the one given in Warner's book at the beginning of p. 374. In more detail,

- For Warner, G is still a Lie group.
- Warner's "measurability" condition here reads (in our notation): "for every $v \in V$, the function $G \to \mathbb{C}$, $x \mapsto \langle f(x), v\rangle_V$ is a Borel function on G".
- Warner's rho-function (ρ_H in his notation) is assumed to be normalized so that $\rho_H(1) = 1$, which, by (8), implies $\rho_H(h) = \frac{\Delta_H(h)}{\Delta_G(h)}$ for any $h \in H$. Moreover, ρ_H is assumed to be smooth on G, but it is also claimed on the same page that this is not essential.

Finally, Realization C+ρ as given above is verbatim the one given in Kaniuth and Taylor's book [3] in the middle part of p. 73. The only problem here is (again) the possibly differing notion of measurability: this book *often uses the word "measurable"* in reference to functions $G \to V$ *without ever spending a word on what notion of measurability is meant.*

It is precisely to "finesse most technical problems associated with the study of measurable vector-valued functions" ([4, p. 154]) that the representation spaces of Realizations C+ρ, C+Δ and C-opt are often constructed not starting with measurable functions and then "cutting down" to suitable subspaces to ensure finiteness of the inner product, but rather starting with *continuous* function with compact support (in a suitable sense), for which inner products are *automatically* finite, and then taking Hilbert space completions.

For the sake of clarity, we illustrate this by giving the alternative construction for the space of Realization C-opt; in this way we shall obtain a "new" realization of the unitarily induced representation (which differs from C-opt only in how the representation space is concretely constructed), to be denoted by $\overline{\text{C-opt}}$ in reference to the common notation for completions. Of course, it is not yet obvious at this point that the representation spaces of C-opt and $\overline{\text{C-opt}}$ are identical, or at least isometric; references for this are given below.

Thus, let $\mathcal{F}$ denote the space of continuous functions $f : G \to V$ such that

$$\text{the image of } \operatorname{supp} f \text{ under } q : G \to G/H \text{ is compact} \tag{16}$$

(we say that f is "*compactly supported mod* H"[12]) and which satisfy

$$f(gh) = \left(\frac{\Delta_H(h)}{\Delta_G(h)}\right)^{\frac{1}{2}} \pi(h^{-1}) f(g) \qquad \text{for all } g \in G, h \in H. \tag{12↑}$$

For any $f, g \in \mathcal{F}$, it makes sense to define the measure $\mu_{f,g}$ on G/H precisely as in the discussion preceding Realization C-opt, and it holds that

$$\int_{G/H} \mathrm{d}\mu_{f,g} < \infty. \tag{17}$$

It is then easily checked that

$$(f, g) \coloneqq \int_{G/H} \mathrm{d}\mu_{f,g}\,. \tag{15↑}$$

turns $\mathcal{F}$ into a pre-Hilbert space.

Realization $\overline{\textbf{C-opt}}$
The Hilbert representation space is given as the completion of the space $\mathcal{F}$ w.r.t. the inner product (15) (see discussion above).
The action L is by "regular" left translation,

$$(L(g)f)(x) = f(g^{-1}x), \qquad f \in \mathcal{F},$$

which extends unitarily to the completion of $\mathcal{F}$.

In a totally analogous manner one can, of course, look at Realizations $\overline{\text{C 2.0}}$, $\overline{\text{C+}\rho}$ and $\overline{\text{C+}\Delta}$. For more details, see the following references:

- Realization $\overline{\text{C 2.0}}$ is given in [4, p. 152f.].
- Realization $\overline{\text{C+}\rho}$ is handled in [4, p. 153f.] and [3, pp. 70–72].
- Realization $\overline{\text{C-opt}}$ is the subject of [3, §2.3] and [4, p. 155f.].

Moreover, at the end of the previous subsection we left open the question of the unitary equivalence of Realization C-opt and the other ones. In our references, one can find the following:

- In [4, p. 157f.], it is proved that $\overline{\text{C+}\rho}$ is unitarily equivalent to $\overline{\text{C-opt}}$ via $f \mapsto \sqrt{\rho} f$.
- At the beginning of [3, §2.4], it is proved that:

[12]This condition had already appeared in the discussion preceding Realization C+ρ.

- ◇ $f \mapsto \sqrt{\rho} f$ is a linear isomorphism between the space (pre-completion) of $\overline{\text{C}+\rho}$ and the space (pre-completion) of $\overline{\text{C-opt}}$;
- ◇ pulling back the inner product on the latter space to the former space yields precisely (6); and
- ◇ pulling the unitary action L of G on the space (post-completion) of $\overline{\text{C-opt}}$ back to the space (post-completion) of $\overline{\text{C}+\rho}$ yields precisely (10).

Implicitly, this should also essentially prove that the inner products on the spaces of $\overline{\text{C}+\Delta}$ and $\overline{\text{C-opt}}$ agree.

Of course, this still leaves the question of whether the spaces of Realization C-opt (resp. C+ρ) and $\overline{\text{C-opt}}$ (resp. $\overline{\text{C}+\rho}$) are "the same". On this note:

- It is the subject of [3, p. 72f.] that the spaces of Realizations C+ρ and $\overline{\text{C}+\rho}$ can be identified. Recall however that in [3] the notion of measurability is never made precise.
- In [4, Remark 1, p. 154f.], it is explained how to view elements of $\overline{\text{C}+\rho}$ as equivalence classes (w.r.t. equality a.e.) of functions $G \to V$ (defined a.e.). This might shed some light into what notions of measurability can be used in these identifications.
- As for Realizations C-opt and $\overline{\text{C-opt}}$, Lemma 5.1.1.5 in Warner's book [5] shows that the space $\mathcal{F}$ discussed above (which he denotes ${}^{L}C(G; E)$) embeds as a dense subspace in the representation space of Realization C-opt (which he denotes E^L), and a comparison of [5, Lemma 5.1.1.1] and [3, p. 64, Prop. 2.20] shows that the inner products on both sides of the embedding are "the same". Thus, the spaces of C-opt and $\overline{\text{C-opt}}$ can be identified by essential uniqueness of Hilbert space completions.

Summarizing essentially the entire content of the previous subsection and this subsection thus far, one can say:

Summary. *Given the data G, H (with fixed left Haar measures) and (π, V) as per subsection 2.2,*

(a) the induced representation can be defined intrinsically (= without any further data) to be the one constructed as in C-opt*, or equivalently as in* $\overline{\text{C-opt}}$.

(b) For any choice of a strongly quasi-invariant measure μ on G/H and corresponding rho-function ρ,

(1) C+Δ *(resp.* $\overline{\text{C}+\Delta}$ *) is precisely the same as* C-opt *(resp.* $\overline{\text{C-opt}}$ *) as a Hilbert space and as a representation of G.*

(2) C+ρ *(resp.* $\overline{\text{C+}\rho}$ *) is unitarily equivalent to* C-opt *(resp.* $\overline{\text{C-opt}}$ *) via (essentially)* $f \mapsto \sqrt{\rho} f$*.*

(c) If G/H has invariant measure μ, scaled in such a way that $\rho \equiv 1$, then all of C+ρ, C+Δ *and* C-opt *are identical as Hilbert spaces and as representations of G and any of these yields back Realization* C 2.0 *from the beginning of subsection 2.3. (Moreover, the analogous statements hold if all Realizations are replaced by their overlined* alter ego*'s.)*

Furthermore, while this was not mentioned yet, it is an easy exercise to check that, if G is a finite group (in particular, $\Delta_G = \Delta_H \equiv 1$, hence G/H has invariant measure) and both G and H are equipped with the counting measure, then:

(d) the inner product $(f, g) = \mu_{f,g}(G/H)$ from Realization C-opt is defined and finite for any set-theoretic functions $f, g : G \to V$, hence the space of Realization C-opt (or equivalently, by (c), C 2.0) coincides with the space of Realization C from section 1. Explicitly the inner product on the induced representation is given by

$$(f, g) = \frac{1}{|H|} \sum_{x \in G} \langle f(x), g(x) \rangle_V .$$

(To understand where the factor $1/|H|$ comes from, observe that, if the surjective linear map $C_c(G) \twoheadrightarrow C_c(G/H)$ from the Fact in Remark 4 is denoted $\Phi \mapsto P(\Phi)$, then for any subset $A \subseteq G/H$,

$$\chi_A = P\left(\frac{1}{|H|} \chi_{q^{-1}(A)} \right),$$

where $\chi_\bullet$ denotes the characteristic function.)

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