Stokes' Theorem is cool!

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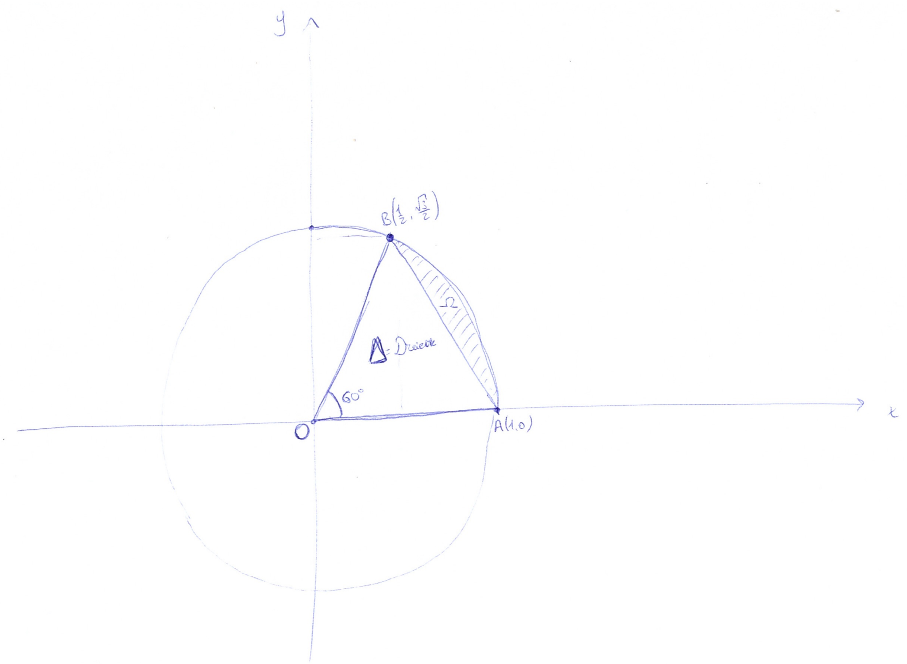
22 June 2018

As the title may suggest, I just had the sudden realization that *Stokes' Theorem is actually pretty cool*! If any decent mathematician is reading this, they're probably scoffing at me right now, but let me walk you through the example that triggered my epiphany.

This story begins with me trying to write a problem for a class on "Analysis II for physicists". For this problem, I absolutely wanted to have my students integrate some function $f = f(x, y)$ over a circular segment, i.e. the part of a circle enclosed by an arc and the chord connecting the endpoints A and B of the arc. The idea was to determine the integral of f over the segment Ω as the integral of f over the corresponding *sector* (to be computed using polar coordinates, of course!) *minus* the integral of f over the triangle OAB, where O denotes the centre of the circle, which for us is the origin.



As if this problem didn't already require the students to compute two separate integrals with two quite different methods, I wanted to also throw Stokes' Theorem in there. So I took my function f to be a very simple one,

$$f(x, y) = x,$$

and tried to find a vector field $\vec{v} = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$ whose curl $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$ was exactly f, because then Stokes' Theorem would tell me that

$$\int_{\partial\Omega} \vec{v} \cdot \mathrm{d}\vec{s} = \iint_{\Omega} f(x, y)\, \mathrm{d}\mu\,.$$

My choice fell on

$$\vec{v} = \frac{1}{2}\begin{pmatrix} x^3 \\ x^2 - y^2 \end{pmatrix}.$$

As for the segment Ω, I still had to specify at least the endpoints, so I chose $A = (1, 0)$ and $B = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$; these are points on the unit circle around the origin O, and the central angle $\measuredangle AOB$ is clearly $\frac{\pi}{3}$. Finally I resolved to use the letter Δ to denote the triangle OAB and S to denote the whole sector delimited by the radii $\overline{OA}$, $\overline{OB}$ and the (shorter) arc connecting A to B.

So, to recap, I would be asking my students to rewrite the path integral

$$\frac{1}{2}\int_{\partial\Omega} \begin{pmatrix} x^3 \\ x^2 - y^2 \end{pmatrix} \cdot \mathrm{d}\vec{s}$$

as a double integral

$$\iint_{\Omega} x\, \mathrm{d}\mu\,,$$

then compute the latter as a difference

$$\iint_{S} x\, \mathrm{d}\mu - \iint_{\Delta} x\, \mathrm{d}\mu \tag{$*$}$$

where S and Δ are as above.

Obviously I would have to compute the integrals myself to make sure that they didn't become abstrusely complicated halfway through or require some weird trick that I couldn't expect my students to know. But even then, I couldn't think of any way of knowing if my computations *or* my end results were correct... except for computing the path integral directly and checking if it matched the value I obtained with Stokes. So I sat down and undertook the absurd task of evaluating the path integral "as is".

I made very standard choices for the parametrization of $\partial\Omega$: I used

$$\begin{aligned} \gamma_1 : \left[0, \frac{\pi}{3}\right] &\to \mathbb{R}^2, \\ t &\mapsto \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \end{aligned}$$

for the arc connecting A to B and

$$\begin{aligned} \gamma_2 : [0,1] &\to \mathbb{R}^2, \\ t &\mapsto \begin{pmatrix} \frac{1}{2}(1+t) \\ \frac{\sqrt{3}}{2}(1-t) \end{pmatrix} \end{aligned}$$

for the chord connecting B back to A. But even with such a simple parametrization, I got two of the ugliest integrals I had ever seen in my life:

$$\begin{aligned} \frac{1}{2}\int_{\partial\Omega} \begin{pmatrix} x^3 \\ x^2 - y^2 \end{pmatrix} \cdot \mathrm{d}\vec{s} = \frac{1}{2}\Bigg[\int_0^{\frac{\pi}{3}} &\begin{pmatrix} \cos^3 t \\ \cos^2 t - \sin^2 t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \mathrm{d}t \\ &+ \int_0^1 \frac{1}{16}\begin{pmatrix} (1+t)^3 \\ 2\left((1+t)^2 - 3(1-t)^2\right) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \mathrm{d}t\Bigg]. \end{aligned}$$

Actually, the second one's probably not bad, but the first one utterly baffled me. I can see that I can rewrite the integrand as $\cos(t)\left(\cos 2t - \frac{\sin 2t}{2}\right)$ but then my gut tells me that this can only end in either repeatedly applying a bunch of half-forgotten trigonometric formulas or repeatedly integrating by parts. So I chose to do neither of those things and instead typed the integral in Wolfram Alpha, who graciously complied and delivered

$$\int_0^{\frac{\pi}{3}} \cos(t)\left(\cos 2t - \frac{\sin 2t}{2}\right) \mathrm{d}t = \frac{\sqrt{3}}{4} - \frac{15}{64}.$$

Thrilled by the power of technology, I let Wolfram Alpha compute my second integral, as well, which turned out to have the value

$$\frac{15}{64} - \frac{\sqrt{3}}{6}$$

— and how convenient that those $\frac{15}{64}$'s would go ahead and cancel each other out, leaving me with the very neat result

$$\frac{1}{2}\int_{\partial\Omega}\begin{pmatrix} x^3 \\ x^2-y^2 \end{pmatrix}\cdot \mathrm{d}\vec{s} = \frac{\sqrt{3}}{24}.$$

Now the time has finally come to substantiate my earlier claim that *Stokes' Theorem is cool*. With Stokes' Theorem, evaluating our path integral is only a matter of computing the two integrals in $(*)$. Obviously, the first integral (the "minuend", if you please) should be computed using polar coordinates:

$$\begin{aligned}\iint_S x\,\mathrm{d}\mu &= \int_0^1\int_0^{\frac{\pi}{3}} r^2\cos\varphi\,\mathrm{d}r\,\mathrm{d}\varphi \\ &= \int_0^1 r^2\,\mathrm{d}r\cdot\int_0^{\frac{\pi}{3}}\cos\varphi\,\mathrm{d}\varphi \\ &= \frac{1}{3}\cdot\sin\frac{\pi}{3} = \frac{\sqrt{3}}{6}.\end{aligned}$$

As for the subtrahend

$$\iint_\Delta x\,\mathrm{d}\mu\,,$$

I thought of asking my students to compute it as a double integral

$$\iint x\,\mathrm{d}x\,\mathrm{d}y\,,$$

where of course they would have to find the limits of integration themselves. One has

$$\begin{aligned}\iint_\Delta x\,\mathrm{d}\mu &= \int_0^{\frac{\sqrt{3}}{2}}\int_{\frac{\sqrt{3}}{3}y}^{1-\frac{\sqrt{3}}{3}y} x\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}}\left(1+\frac{1}{3}y^2-\frac{2\sqrt{3}}{3}y-\frac{1}{3}y^2\right)\mathrm{d}y \\ &= \frac{1}{2}\left(\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{4}\right) = \frac{\sqrt{3}}{8}.\end{aligned}$$

So, to summarize one last time:

$$\begin{aligned}
\int_{\partial\Omega} \begin{pmatrix} x^3 \\ x^2 - y^2 \end{pmatrix} \cdot \mathrm{d}\vec{s} &= \iint_{\Omega} x \, \mathrm{d}\mu \\
&= \iint_{S} x \, \mathrm{d}\mu - \iint_{\Delta} x \, \mathrm{d}\mu \\
&= \frac{\sqrt{3}}{6} - \frac{\sqrt{3}}{8} \\
&= \frac{\sqrt{3}}{24},
\end{aligned}$$

as expected, but this time around it took us much less computational effort to obtain the correct value. *Ergo*, Stokes' Theorem is cool, as claimed.