The Jordan Curve Theorem

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Abstract

The Jordan Curve Theorem (JCT) is arguably best-known for stating an apparently obvious fact while reportedly being quite hard to prove. The goal of this article is to outline some of the difficulties that arise when attempting to prove the theorem, as well as to present a very accessible proof by Carsten Thomassen (1992) which relies on a well-known result from graph theory.

1 Definitions and statement

Before giving a precise formulation of the theorem, it is best to review the few basic notions which appear in the statement.

Let X be a topological space. A *path* in X is a continuous map $\gamma : [0,1] \to X$; one often says γ is a path *from* $\gamma(0)$ *to* $\gamma(1)$. The image $\mathrm{im}(\gamma) \subset X$ of a path γ is variously called a *curve*, an *arc* or again a path.

A path γ is called *simple* if the restriction $\gamma|_{[0,1)}$ is injective, which amounts to saying that $\mathrm{im}(\gamma)$ does not self-intersect. In the special case where $X = \mathbb{R}^2$ is the Euclidean plane, the image $\mathrm{im}(\gamma)$ of a simple path $\gamma : [0,1] \to \mathbb{R}^2$ is called a *Jordan curve* if γ is *closed*, i.e. if $\gamma(0) = \gamma(1)$, and a *Jordan arc* otherwise.

Example. Let $\mathcal{C} \subset \mathbb{R}^2$ be the ellipse with equation $b^2(x - x_0)^2 + a^2(y - y_0)^2 = a^2b^2$ for $a, b \in \mathbb{R}_{>0}$. Then $\mathcal{C}$ is a Jordan curve, as it is the image of $\gamma : [0,1] \to \mathbb{R}^2$, $t \mapsto (x_0 + a\cos 2\pi t, y_0 + b \sin 2\pi t)$. ◇

Next, recall that a topological space X is called *path-connected* if for any two points $x, y \in X$ there exists a path from x to y. The maximal (w.r.t. inclusion) path-connected subsets of a space X are called the *path-components* of X. For instance, a line in the plane is path-connected, while a hyperbola has precisely[1] two path-components.

We are now finally ready to state the Jordan Curve Theorem.

Theorem 1. *Let $\mathcal{C} \subset \mathbb{R}^2$ be a Jordan curve. Then $\mathbb{R}^2 \setminus \mathcal{C}$ has precisely two path-components (, and both have $\mathcal{C}$ as their boundary).*

[1] Unless it degenerates to a pair of intersecting lines.

2 The need for a proof

The reader will now probably concur that the JCT merely states an obvious geometrical fact, namely that a non-self-intersecting curve divides the plane in an "inner" and an "outer" region, and that the proof is bound to be straightforward. Indeed, there *is* an elementary proof available whenever the Jordan curve at hand is "nice". To illustrate this, let us sketch the proof of the theorem in the case where $\mathcal{C}$ is a polygon without self-intersections.

We first show that $\mathbb{R}^2 \setminus \mathcal{C}$ has *at least* two path-components. For each point $P = (x_0, y_0)$ of $\mathbb{R}^2 \setminus \mathcal{C}$ consider the half-line h_P given by $y = y_0$, $x \geq x_0$. We say that a half-line *crosses* one of the sides of $\mathcal{C}$ if they intersect in precisely one point and this point is not a vertex of $\mathcal{C}$. We now call P of $\mathbb{R}^2 \setminus \mathcal{C}$ *even* (*odd*) if the number of sides h_P crosses is even (odd).

Intuition, possibly coupled with a drawing, now suggests that the odd (even) points of $\mathbb{R}^2 \setminus \mathcal{C}$ are precisely those in the inner (outer) region of $\mathcal{C}$. While this will turn out to be true, figures 1 and 2 of [3] are hopefully enough to dissuade the reader from trusting his or her intuition, even when the curve is smooth.

What can surely be said is the following: if P is an even (odd) point and $\widetilde{\gamma} : [0, 1] \to \mathbb{R}^2 \setminus \mathcal{C}$ is a path *in* $\mathbb{R}^2 \setminus \mathcal{C}$ with $P = \widetilde{\gamma}(0)$, then $Q = \widetilde{\gamma}(1)$ is again even (odd). The argument is familiar: consider the largest $t \in [0, 1]$ such that $\widetilde{\gamma}(s)$ has the same parity as P for all $s < t$, and show that if $t < 1$ then $\widetilde{\gamma}(t + \varepsilon)$ still has the same parity as $\widetilde{\gamma}(t)$ for a small enough ε, so necessarily $t = 1$. It now follows that each path-component of $\mathbb{R}^2 \setminus \mathcal{C}$ consists either entirely of odd points or entirely of even points, and it is not hard to see that there must be at least one of each, thus $\mathbb{R}^2 \setminus \mathcal{C}$ indeed has at least two path-components.

In order to show that the number of path-components is also at most two, consider a disc $\mathcal{D}$ which only intersects one side of $\mathcal{C}$, so that $\mathcal{D} \cap \mathcal{C}$ is a chord of $\mathcal{D}$. Then clearly $\mathcal{D} \setminus (\mathcal{C} \cap \mathcal{D})$ has exactly two path-components; this follows from classical Euclidean geometry. Now suppose there were three points P_1, P_2, P_3 lying in distinct path-connected components of $\mathbb{R}^2 \setminus \mathcal{C}$. For each point P_i we construct a $\gamma'_i : [0, 1] \to \mathbb{R}^2 \setminus \mathcal{C}$ with the following properties:

(i) $\gamma'_i(0) = P_i$;
(ii) $\gamma'_i(1) \in \mathcal{D} \setminus (\mathcal{C} \cap \mathcal{D})$;
(iii) $\gamma'_i(t)$ is "very close" to $\mathcal{C}$ (the distance is smaller than a fixed $\varepsilon > 0$) for all t sufficiently large.

(A way to obtain γ'_i is to first consider a half-line starting in P_i which intersects $\mathcal{C}$, then modify it so that the resulting path "runs parallel" to $\mathcal{C}$ until it lands inside $\mathcal{D}$.)

Now we see by the pigeonhole principle that two of the three paths necessarily land in the same region of $\mathcal{D} \setminus (\mathcal{C} \cap \mathcal{D})$. But this region is path-connected, so two of the P_i's can be joined by a path and hence lie in the same path-component of $\mathbb{R}^2 \setminus \mathcal{C}$, which contradicts our assumption. The proof is thus completed.

It is easy to see that the above approach also works if the path $\gamma : [0,1] \to \mathbb{R}^2$ which describes the given Jordan curve $\mathcal{C} = \mathrm{im}(\gamma)$ is piecewise C^1. However, the assumptions of the JCT only require the path γ to be *continuous*, and calculus tells us that continuous functions need not be differentiable at *any* point. Therefore, to conclude this section we would like to offer an example of just how "nasty" Jordan curves can be, or more precisely of how spectacularly our intuitive understanding of them fails.

Consider the square $Q \subset \mathbb{R}^2$ of side length $\sqrt{2}$ whose vertices lie on the coordinate axes, and let $T = \mathcal{F}_0$ denote the "upper triangle"; the area of T equals 1. If we cut out a triangle with the same vertex and same height as T but with a smaller base, the resulting figure $\mathcal{F}_1$ consists of two triangles T_0, T_1 joined at the vertex; its area equals $1 - r_1$ for some $r_1 \in \,]0,1[$. We can now apply a similar process to each of T_0, T_1, thus obtaining a figure $\mathcal{F}_2$ consisting of four triangles $T_{00}, T_{01}, T_{10}, T_{11}$ so that each two have at most one vertex in common and whose combined area can be written as $(1 - r_1)(1 - r_2)$ for some $r_2 \in \,]0,1[$. Iterating yields

$$\mathcal{A} := \bigcap_{i \geq 0} \mathcal{F}_i = \bigcup_{\omega \in \{0,1\}^{\mathbb{N}}} T_\omega.$$

I claim that $\mathcal{A}$ is a Jordan arc. Indeed, consider the map γ which sends $t \in [0,1]$ with binary expansion[2] $0.t_0 t_1 t_2 ...$ to the unique point of $\mathcal{A}$ which lies in $T_{t_1} \cap T_{t_1 t_2} \cap ...$; it is easily checked to be injective and continuous. For instance, if the triangles we cut out are chosen in an appropriate way with $r_j = 1/3 \; \forall j$, then $\mathcal{A}$ is the so-called Koch curve, see [5] for the rigorous description and [4] for an online simulation.

Let us now set $r_j = r^2/j^2$ for all j, where $r \in \,]0,1[$. If we apply the so-called Weierstrass factorization theorem to the function $z \mapsto \sin \pi z$ and rearrange terms, we obtain

$$\prod_{j=1}^{\infty} \left(1 - \frac{r^2}{j^2}\right) = \frac{\sin \pi r}{\pi r}.$$

Observe that the left-hand side equals the area of $\mathcal{A}$ for our particular choice of the r_j's, while the map $]0,1[\,\to\,]0,1[$, $r \mapsto \sin \pi r / \pi r$ is bijective, therefore upon appropriate choice of r *one can obtain a Jordan arc of any given (positive) area* < 1. Accordingly we can also obtain a Jordan *curve* of any given positive area, e.g. by joining $\mathcal{A}$ with its reflection across the x-axis. This clearly contradicts the intuitive notion of a curve as "a line without breadth" and hopefully persuades the reader of the need for a rigorous proof for the Jordan Curve Theorem.

[2]It is well-known that "some" rational numbers in $[0,1]$ possess two expansions; one terminates after finitely many steps while the other does not. Here we always take the infinite expansion.

3 Historical overview

Before we dive into the actual proof of the theorem, we briefly go over the history of the Jordan Curve Theorem and its proofs.

It appears that the JCT was considered trivial for a long time. Only in the 19^{th} century did Bernhard Bolzano (1781-1848) point out that the following statement needed a proof:

Wenn eine in sich zurückkehrende Linie in einer Ebene liegt, und man verbindet einen Punct derselben Ebene der von ihr eingeschlossen wird mit einem anderen Puncte dieser Ebene, der aber von ihr nicht eingeschlossen wird, durch eine zusammenhängende Linie, so muss diese die zurückkehrende Linie schneiden.

If a closed curve lies in a plane, and one joins a point of this plane which is enclosed by it with another point of this plane which is *not* enclosed by it by a connected arc, then this arc must cross the closed curve.

The JCT in its modern form was first formulated and proved by Camille Jordan (1838-1922) in its *Cours d'Analyse* (1887). Jordan claimed (without proof) that the theorem holds for polygons without self-intersections, then showed that every closed curve can be approximated by polygons in an appropriate sense and that the result for polygons can be carried over to the general case.

Jordan's proof was regarded as incomplete at his time; indeed, Jordan often mentions *quantités infiniment petites*, "infinitely small quantities", which certainly must have sounded non-rigorous to his contemporaries. It is often claimed that the first complete proof is found in the paper *Theory on plane curves in nonmetrical Analysis Situs* by Oswald Veblen (1880-1960). In the preamble, Veblen wrote:

JORDAN*'s explicit formulation of the fundamental theorem ... is justly regarded as a most important step in the direction of a perfectly rigorous mathematics. This can be confidently asserted whether we believe that perfect rigor is attainable or not. His proof, however, is unsatisfactory to many mathematicians.*

Veblen's proof is interesting in that it does not use methods from analysis; instead, it fits into his axiomatic approach to classical geometry, and the very concept of a Jordan curve is axiomatised in that framework.

Since Veblen's proof, many mathematicians have provided proofs for the JCT; here we only mention a few of those that are considered elementary. One such proof is Edwin E. Moise's, published in 1977; it was shortened by Ryuji Maehara in 1984 with an application of Brouwer's fixed-point theorem. Another short and elementary proof is the one by Helge Tverberg (1980), which incidentally yields a value $r = r(\mathcal{C})$ so that the inner region of $\mathcal{C}$ contains a circle of radius r. In 1992, Carsten Thomassen published a proof of the JCT which relies on graph theory; his is the approach we will follow in the next section. Thomassen's proof has also been rewritten into a so-called "formal proof", see [12].

Finally, it is worth mentioning that some results from the field of algebraic topology provide a way to prove the Jordan Curve Theorem and the following generalisation to higher dimensions: if a subset S of $\mathbb{R}^d$, $d \geq 2$ is homoeomorphic to S^{d-1}, then the complement $\mathbb{R}^d \setminus S$ has precisely two path-components. A proof which uses these methods can be found e.g. in the textbook *Algebraic Topology* (2002) by Allen Hatcher.

4 Thomassen's proof

In this section we present an elementary proof of the Jordan Curve Theorem, following the paper *The Jordan-Schoenflies Theorem and the Classification of Surfaces* (1992) by Carsten Thomassen.

A few preliminary observations are in order. In many steps of the proof we will use the fact that a Jordan curve $\mathcal{C}$ is always a *compact* subset of $\mathbb{R}^2$; this is because $\mathcal{C}$ is the continuous image of the compact interval $[0, 1]$. In particular, $\mathcal{C}$ is *bounded*, i.e., one can find a ball $\mathcal{B}$ of finite radius with $\mathcal{C} \subset \mathcal{B}$.

We now proceed to show that $\mathbb{R}^2 \setminus \mathcal{C}$ has *at least* two path-components. We begin with a geometric construction. Since $\mathcal{C}$ is compact, there exist a *leftmost* and a *rightmost* vertical line in $\mathbb{R}^2$ which have non-empty intersection with $\mathcal{C}$; we denote these by h_1 and h_2 respectively. Again by compactness, for $i = 1, 2$ there exists a point u_i in the set $h_i \cap \mathcal{C}$ with largest y-coordinate. The curve $\mathcal{C}$ now consists of two Jordan arcs from u_1 to u_2, an "upper" one $\mathcal{U}$ and a "lower" one $\mathcal{L}$, which intersect only at the endpoints. Let h be a line "between" h_1 and h_2, and let v_1 (v_2) be the point of $h \cap \mathcal{U}$ ($h \cap \mathcal{L}$) with smallest (largest) y-coordinate. Since $\mathcal{C}$ does not self-intersect, these points are distinct; let u_3 denote the midpoint of the segment of h joining v_1 to v_2.

Now let g be some horizontal line "above" $\mathcal{C}$ (in particular, disjoint from $\mathcal{C}$) and v_3 be a point on g "between" h_1 and h_2. We shall now show that $v_3, u_3 \in \mathbb{R}^2 \setminus \mathcal{C}$ cannot be joined by a path in $\mathbb{R}^2 \setminus \mathcal{C}$; it will follow that this set has at least two path-components. So suppose there were a path joining u_3 and v_3 which does not intersect $\mathcal{C}$. By going along h, we can clearly join u_3 to each of v_1 and v_2; on the other hand, by going along the upper (or lower) arc of $\mathcal{C}$ we can join each of u_1 and u_2 to each of v_1 and v_2. Finally, we join u_1 (or u_2) to v_3 by going along h_1 (or h_2) and g.

We have thus joined each of the u_i's to each of the v_j's; thus, the points u_1, u_2, u_3, v_1, v_2, v_3 together with the paths joining them form a plane realisation of the so-called *complete bipartite graph* $K_{3,3}$. Since no two of the edges intersect except at most at the endpoints, this graph is an *embedding of $K_{3,3}$ into the plane* $\mathbb{R}^2$. But it is a well-known result of graph theory that $K_{3,3}$ cannot be embedded in the plane. We have thus reached the contradiction we were looking for and proved the claim that $\mathbb{R}^2 \setminus \mathcal{C}$ has at least two path-components.

In order to show that $\mathbb{R}^2 \setminus \mathcal{C}$ has at most two path-components, we are going to use the fact that, if $\mathcal{A}$ is a Jordan arc, then $\mathbb{R}^2 \setminus \mathcal{A}$ is path-connected. This, in turn, can be shown as follows: given $P, Q \notin \mathcal{A}$, one can construct a graph Γ so that every point of $\mathcal{A}$ lies in some inner face of Γ while P and Q lie in the outer face. We shall not go into details here; suffice it to say that Γ is obtained as the union of squares of sufficiently small side length centred in points of $\mathcal{A}$ (cf. the similar concept of *tubular neighbourhoods* in differential geometry).

Suppose for the sake of contradiction that $u_1, u_2, u_3 \in \mathbb{R}^2 \setminus \mathcal{C}$ lie in distinct path-components of $\mathbb{R}^2 \setminus \mathcal{C}$, and let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ be three disjoint arcs on $\mathcal{C}$. By our previous remark, $\mathbb{R}^2 \setminus (\mathcal{C} \setminus \mathcal{A}_j)$ is path-connected for each j, so for each i we can find a path (in $\mathbb{R}^2 \setminus (\mathcal{C} \setminus \mathcal{A}_j)$!) from u_i to some other path-component of $\mathbb{R}^2 \setminus \mathcal{C}$. This path necessarily intersects $\mathcal{C}$ in a point p_{ij} on $\mathcal{A}_j$; we denote the path joining u_i to p_{ij} by γ_{ij}. Note that by modifying some of γ_{ij}'s we can assume that no two of them intersect except at most at the endpoints. If we now choose a point v_j on each $\mathcal{A}_j$, we see that it is possible to join each u_i to each v_j by going along γ_{ij} and $\mathcal{A}_j$, so again we obtain a plane graph isomorphic to $K_{3,3}$, which is absurd. The proof of the Jordan Curve Theorem is thus completed.

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