Locally Compact Groups (v0.45)

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Note to the Reader

This document originates with a lecture course I taught with Harald Grobner at the University of Vienna in the spring term of 2021. During the course, my co-lecturer and I shared informal notes with our students, while also planning to use these as a basis for much more detailed lecture notes which would cover *and* expand on the full contents of the course. Carrying out this plan in the desired level of detail proved tremendously ambitious given my time constraints and other engagements as a PhD student. The current version (which carries the somewhat arbitrary version number v0.45) only fully covers the first of the three parts that made up the course. I hope to come back to this project in quieter times and conclude it; for now, consider it my *Dune: Part One*.

On a more factual note, clearly the current version is to be considered a preliminary one in many regards. Accordingly,

> *"Those who read beyond this page do so at their own peril."*
>
> Oscar Wilde ... sort of

Acknowledgements

Overall, I am very proud of how these notes have turned out. I couldn't have done it without my co-lecturer and PhD supervisor Harald Grobner, who believed in the course concept I pitched to him, helped me with the choice of contents and presentation, and gave input and feedback on the notes as they grew from early, incredibly sketchy versions to their current form. Without him, the course itself probably wouldn't even have taken place.

The same goes for my audience: a lecture course is nothing without students. Having motivated and interested people attending week after week motivated me in turn and spurred me to do better. I am grateful to them for pointing out mistakes, for their engagement outside of the (virtual) classroom and for their kind words about my lectures. A special thanks goes out to special audience member Sonja Žunar, who always has the kindest and most encouraging feedback.

Contents

Motivation and overview

Locally compact groups represent a joint cornerstone of algebra and analysis—a trait which will be apparent throughout these notes. As a consequence, the study of these objects may be approached from several directions, each providing its own set of motivating examples and considerations.

In this introduction we shall first focus on three "strands" of motivation, corresponding roughly to the three chapters of which this document is composed. Next, we shall give an overview of the contents, and outline the make-up of each chapter in the process. Finally, to conclude the introduction there will be a brief discussion of the relevant literature.

A first way to motivate the study of locally compact groups is the following consideration coming from the field of analysis. Traditionally and in most modern uses, the very term 'analysis' is intrinsically tied to the real field $\mathbb{R}$: the various flavours of analysis (ranging from 'real' or 'complex' analysis to functional analysis, harmonic analysis and so on) are all concerned with spaces that are obtained *from* $\mathbb{R}$ in some way ($\mathbb{R}^n$, $\mathbb{C}$, $\mathbb{C}^n$, smooth and analytic manifolds, topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$, ...), and the very properties of $\mathbb{R}$ seem almost indispensable in many key points, e.g.,

- to meaningfully discuss convergence of sequences and of infinite sums (or products) in $\mathbb{R}$ or $\mathbb{C}$;
- to prove that any finite-dimensional vector space over $\mathbb{R}$ is canonically a topological vector space over $\mathbb{R}$, which is then automatically a Banach space (with all norms being equivalent);
- to define the Lebesgue integral for functions on $\mathbb{R}^n$;

- to introduce concepts of differentiability and smoothness, test functions, distributions, etc. on $\mathbb{R}^n$ and hence on real manifolds;
- to talk about Fourier transforms and Schwartz functions on $\mathbb{R}^n$.

One could wonder if there even exist spaces different from $\mathbb{R}$, $\mathbb{R}^n$, etc. where some of these notions and results carry over. Clearly to answer this one needs to take into account the "least amount of structure" that is needed for the definitions or statements to make sense.

Consider as a simple case the existence of Lebesgue measure on $\mathbb{R}$. The result can be stated as follows:

. THEOREM. *There exists a unique (positive) measure λ on $\mathbb{R}$ which satisfies the following:*

(i) the λ-measurable sets are precisely[1] *the Borel subsets of $\mathbb{R}$, and every compact set has finite measure;*

(ii) λ is translation-invariant; and

(iii) $\lambda([0, 1]) = 1$.

Moreover, any nontrivial (positive) measure λ' satisfying (i) and (ii) is a positive scalar multiple of λ.

Suppose one wanted to generalize this result in this form to more general spaces. Such spaces would have to be equipped with both a topological structure and a group structure, which are needed to make sense of (i) and (ii), respectively. The rest of the statement then (intuitively) suggests that, in order to pin down the measure uniquely (not just up to scalars), we need at least one subset of nonzero finite measure. Refining these observations, we arrive at the notion of a *locally compact group*. One can then prove that for these groups, there exists a nonzero translation-invariant Radon measure[2], unique up to positive real multiples, known as *Haar measure*. This fact is a major step towards developing a robust theory of analysis on arbitrary locally compact groups, and as it turns out, virtually all of the notions and results mentioned

[1] There is also an analogous existence-and-uniqueness result for the strictly larger σ-algebra of *Lebesgue*-measurable sets; however, we do not need to get into such details here. We will come back to this in Chapter I.

[2] A precise definition for our purposes is given in §4 of Chapter I. .

above generalize to at least some class of locally compact groups (possibly with additional structure).

A second approach that will lead us to consider locally compact groups comes from the representation theory of groups. We shall return to this topic more systematically in Chapter III; for the moment being, we shall content ourselves with a partial account.

Consider, then, the following problem: given an abstract group G, one wants to find a concrete "realization" in which the group operation is as easy as possible to describe.

If G is finite, then one option is to realize the elements of G as permutations of some suitable (sufficiently large) set, i.e., to embed G into some permutation group $\mathfrak{S}_n$. (It is not hard to see that this is always possible.) In turn, elements of $\mathfrak{S}_n$ can be realized as permutation matrices, i.e. as linear transformations on $\mathbb{R}^n$ (in fact, on $\mathbb{K}^n$ for any field $\mathbb{K}$). This shows that for every finite group G and every field $\mathbb{K}$ there exists a finite-dimensional vector space V over $\mathbb{K}$ such that G affords a nontrivial group homomorphism $\rho\colon G \to \mathrm{GL}(V)$; in particular, the group operation on G can be "translated" to composition of linear maps on a vector space, i.e. to multiplication of matrices.

A pair (V, ρ) as above (minus the finiteness assumption on G and $\dim V$) is called a *($\mathbb{K}$-)linear representation* of the group G. For finite G and, say, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, one can prove:

. THEOREM (Maschke's Theorem). *Every $\mathbb{K}$-linear representation (V, ρ) decomposes as a direct sum of "building blocks" (*irreducible representations*).*

(Furthermore, although this is not part of Maschke's theorem: under the same assumptions, all irreducible $\mathbb{K}$-linear representations of G are finite-dimensional.)

One way to prove the above theorem relies on "averaging" functions f on G over the group G (by summing the finitely many values $f(g)$, $g \in G$ and dividing by $|G|$.) This suggests that the result could be generalized to any group G on which one can "average", e.g. by integrating and then dividing by the "size" of G. Thus, one would need a group equipped with a nonzero (positive) measure μ for which $\mu(G) < \infty$. But we saw earlier that locally compact groups are naturally equipped with an invariant measure, called Haar measure. As we shall see, such a group has finite Haar measure if and only if it is compact, so compact groups are the natural candidates for this generalization.

Of course, if one wants to bring topology into the game, then sooner or later one also needs the representation space V to have some “sensible” topology, and the representation needs to be continuous in a suitable sense. Once this is all made precise, it turns out that, indeed, Maschke’s theorem, as well as finite-dimensionality of irreducibles, generalize to (certain) continuous linear representations of *compact* groups on (certain) topological vector spaces. The representation theory of general locally compact groups is a vast and exciting subject and we will touch upon it towards the end of these notes.

A third aspect that motivates the study of locally compact groups is their ubiquity in modern number theory, of which we shall now give a “taste”.

As is well-known, prime numbers are at the very center of many questions in number theory. The problem of solving simultaneous congruences modulo powers of a fixed prime p leads one to consider so-called *p-adic numbers.* It turns out that, for each prime number p, the *field of p-adic numbers*, denoted by $\mathbb{Q}_p$, is a *locally compact field* and that, precisely by virtue of being locally compact, each $\mathbb{Q}_p$ shares many of the desirable properties of $\mathbb{R}$ which make analysis possible (cf. above).

This is the most fundamental case of a much more general phenomenon. Indeed, in number theory one is often led to consider so-called *algebraic number fields*, which are, per definition, finite-degree extensions of the rational field $\mathbb{Q}$. For these fields, there is a notion which naturally generalizes that of prime integers, namely that of *prime ideals.* It turns out that, again, the problem of solving congruences “yields”, broadly speaking, locally compact fields.

The more surprising result is that we also have a converse. To wit, if we start with a general (nondiscrete) locally compact field that is not the reals $\mathbb{R}$ or the complex numbers $\mathbb{C}$, then it can be of two types: either it contains a finite subfield (meaning that the elements $1,\ 1+1,\ 1+1+1,\ \dots$ are not all distinct), or, if it doesn’t, then *it is obtained from some algebraic number field by the process above.* This shows that studying algebraic number fields and studying locally compact fields are two intimately related tasks, and in fact, several fundamental results in algebraic number theory can be recast as topological statements involving locally compact fields or the locally compact *adèle ring* of a number field.

We are now ready to outline the topics that will be discussed in these notes. Cf. also the Table of Contents following the introduction.

The first chapter is devoted to locally compact groups in general, and more specifically to proving the fundamental result on existence and essential uniqueness of the Haar integral. The reader is not assumed to have any prior familiarity with locally compact groups; instead, the first section serves to introduce the larger framework of topological groups and discuss some general results—to the extent that they might be of use later on. It will then be easy to zero in on locally compact groups, which is the category we are really interested in. Since we want to talk about integration on such groups, we will then give a brief review of measure theory to recall some fundamental notions. The next natural step will then be to discuss invariant measures and specifically Haar measures. Finally, and crucially, the proofs of existence and essential uniqueness of Haar measure will be discussed. To conclude the chapter we will give an overview of some additional results and topics that sadly lie outside the scope of these notes; this section will be called "Vista", which is an idea borrowed from William Waterhouse's book [Water].

For one paradigmatic example of a simple but powerful application of Haar measure, we have chosen the topic of locally compact fields, which makes up the second chapter. Again, we shall start by discussing in great generality the concept of topological rings (fields being special rings); this will be easy, having already built the foundations of topological groups. Next we will move on to locally compact fields, which, simply by virtue of being locally compact, can be shown to have remarkably powerful properties. It will also be instructive to study vector spaces over such fields, which is where we'll see results analogous to the well-known ones for $\mathbb{R}$ or $\mathbb{C}$ that were mentioned above. At this point we will introduce the field of p-adic numbers as the paradigmatic example of what locally compact fields other than the reals or complex numbers "look like". Again, we will conclude the chapter with a "Vista".

The third and largest chapter will be devoted to representation theory, the study of realizations of abstract groups as groups of linear transformations on vector spaces, as discussed earlier. We shall revisit the motivation and the special case of finite groups as given earlier, and then explore the possible avenues of generalization. For a certain class of locally compact groups, namely *locally profinite groups*, and a certain class of their representations, most of the theory familiar from the finite-group case can be set up again without the need to dramatically redefine many of the main concepts. This is no longer true of general locally compact groups, which is why we will only be able to treat some special cases in the level of detail they deserve: the representation theory of compact groups, and that of locally compact abelian groups, which is, essentially, and at this point perhaps surprisingly, an abstract approach to classical Fourier analysis. Again, we will conclude with an overview of more advanced topics and suggestions for further reading in the "Vista" section.

To conclude this introduction, it is perhaps appropriate to briefly discuss relevant literature. The general structure of the account in these notes is not unlike that of the bulk of [Folland], with additional input from [GWarner] for the representation-theoretic part. There are two notable exceptions, namely the two "case studies" that were most inspired by the authors' interest in number theory: locally compact fields on the one hand, and the representation theory of locally profinite groups on the other. The main references used by the authors for these topics were [FANF] and [BushHen], respectively. However, many other sources have provided some of the material and inspired the exposition. For this reason, our policy will be to mention useful reference works at the beginning of the section to which they are relevant, and to suggest literature for further reading at the end of each chapter, in the "Vista" section. The eager reader is therefore invited to check these parts of the notes especially carefully. In the Preliminaries preceding Chapter I we have additionally included a compendium of the notions that are going to be used in the document.

Preliminaries

Foundations of mathematics We shall not need to be greatly concerned with foundations of mathematics in these notes. Instead, we shall follow in the honoured tradition of taking for granted that everything we say works in Zermelo–Fraenkel set theory with the axiom of choice (ZFC) and pointing out each instance in which we use the axiom of choice in an argument.

Set theory Familiarity with elementary set-theoretic notation, operations, and terminology is assumed. Occasionally, terms such as *family*, *collection* or *class* might be used as synonyms of "set".

Convention. We use $\subseteq$ as a general symbol for inclusion (not necessarily strict); if we *require* an inclusion to be strict, then we shall use $\subsetneq$, while $\subset$ will sometimes be used for inclusions which are evidently strict, e.g.: $\mathbb{Q} \subset \mathbb{R}$ (s. below).

A (set-theoretic) *function* $f\colon X \to Y$ has Y as its *codomain* (or *target space*), and $f(X) \subseteq Y$ as its *range* or *image*. A function might be alternatively called a *map* or a *mapping* in certain set phrases such as 'continuous map' or 'linear map'. This should not create confusion with the use of these same terms for the morphisms of category theory, nor is familiarity with category theory necessary to read these notes.

Convention. For sets X and Y, the collection of all set-theoretic functions $X \to Y$ will be denoted by Y^X whenever necessary. Thus, no distinction is made on the formal level between tuples $(y_x)_{x\in X}$ of elements of Y indexed by the elements of X, on the one hand, and functions $X \to Y$ on the other.

Pointwise operations with functions Let X and Y be sets, and suppose that Y affords a *binary operation*, i.e. a set-theoretic function

$$\begin{aligned} Y \times Y &\to Y \\ (y_1, y_2) &\mapsto y_1 * y_2. \end{aligned}$$

Then the set Y^X of functions $X \to Y$ affords a pointwise version of $*$: for elements f, $g \in Y^X$, one may define $f * g \in Y^X$ by $(f * g)(x) := f(x) * g(x)$ for all $x \in X$. Analogously, if Y affords an *external binary operation*, i.e. a function

$$\begin{aligned} Z \times Y &\to Y \\ (z, y) &\mapsto z \odot y \end{aligned}$$

for some set Z, then we obtain an external binary operation $Z \times Y^X \to Y^X$ via the formula $(z \odot f)(x) = z \odot f(x)$ (where $z \in Z$, $f \in Y^X$, and x ranges over the elements of X).

It is easily checked in either case that properties of the (original) binary operation such as associativity, commutativity, etc. continue to hold true for its pointwise version. In conclusion, if a set Y affords one or more (possibly external) binary operation(s) which make it into a group [or a ring, or a vector space over some field $\mathbb{K}$], then, for any set X, the set Y^X of functions $X \to Y$ becomes a group [ring; $\mathbb{K}$-linear space] with the pointwise operations induced by those on Y. (Cf. also the paragraphs below for more on algebraic structures.)

Number systems Throughout these notes, the following standard notations will be used freely:

$\mathbb{N}$ ...... the set of natural numbers (including 0),

$\mathbb{Z}$ ...... the integers,

$\mathbb{Q}$ ...... the rational numbers,

$\mathbb{R}$ ...... the reals,

$\mathbb{C}$ ...... the complex numbers.

Each of the first four sets will always be identified with a subset of the following one(s) in the standard way. Moreover, all of these sets will always be tacitly equipped with the usual operations $+$ and $\cdot$, and all except $\mathbb{C}$ will always be considered with their natural order structure. Additionally, the reals and the complex numbers will generally be equipped with their standard topological (metric) structure, and on either space, the notation $|\cdot|$ will stand for the standard Euclidean absolute value.

Notations such as $\mathbb{Z}_{\geq 0}$ or $\mathbb{R}_{>0}$ should be self-explanatory. We shall also use the common notations $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

We again make the reader aware of the following

Convention. $\mathbb{N} = \{n \in \mathbb{Z} : n \geq 0\}$.

Other blackboard bold symbols Often symbols in a blackboard bold typeface (such as $\mathbb{N}$, $\mathbb{Z}$, etc. from the previous paragraph) have special meanings for certain authors and should therefore be used sparingly. If need be, we shall use the following:

$\mathbb{F}_q$ ...... the (essentially unique) finite field with cardinality q (q being a prime power);

in a slight extension, we shall use $\mathbb{F}$ for a finite field whose cardinality is not known, not specified, and/or not relevant.

Convention. We will occasionally use $\mathbb{K}$ to denote a field (such as a field of coefficients, or a "ground field" in the sense of linear algebra) which is initially *completely arbitrary* and which might later be specialized to some concrete "blackboard bold field" such as $\mathbb{F}$, $\mathbb{R}$ or $\mathbb{C}$.

In the text (§1 of Chapter I), we shall introduce the notation $\mathbb{T}$ for the set of complex numbers of modulus 1.

Cardinality A set is *countably infinite* if it admits a bijection to the natural numbers. We shall use the phrase '*at most countable*' for sets which are either finite or countably infinite. It might then be ambiguous to simply write '*countable*', but hopefully this will not lead to any confusion in these notes. (For instance, the statement "A countable union of countable sets is again countable" is true whether 'countable' is taken to mean 'countably infinite' or 'at most countable'.)

Groups The definition of a *group* is standard. Most commonly, a group is defined as a non-empty set together with an associative binary operation on that set (sometimes called the *group law*) and a distinguished element of that set (the *neutral element*) subject to certain axioms, most notably the existence of an *inverse* for each element of the group.

Two main conventions are used when denoting groups: in *multiplicative* notation, the group law is written as $\cdot$ or even simply as juxtaposition, the neutral element is denoted 1, and the inverse of an element g is denoted g^{-1}; whereas in *additive* notation, one uses $+$, 0, and $-g$, respectively. Additive notation is reserved for *abelian* (i.e., commutative) groups.

CONVENTION. When denoting a *general* group, we will denote the group law multiplicatively, the neutral element by e and the inverse of g by g^{-1}. Thus, a group is a triple $(G, \cdot, e)$. Often we will simply write $(G, \cdot)$ or even G in lieu of $(G, \cdot, e)$. The neutral element will also be called *the identity.*

Familiarity with the following notions is assumed: subgroups, left (right) cosets of a subgroup, the index of a subgroup, normal subgroups, quotient groups, group homomorphisms. The *trivial subgroup* is $\{e\}$.

Rings and fields In the absence of any explicit mention to the contrary, *all rings in these notes will be commutative with unity.* Accordingly, a *subring* is an additive subgroup which contains 1 and is closed under multiplication, and an *ideal*—an additive subgroup which is closed under multiplication by elements of the whole ring—is always two-sided.

By a related convention, *all fields are commutative.* A field is *finite* if and only if it has finite cardinality. A *subfield* (of some ring) is a subset which is a field with the operations inherited from the ambient ring.

Topology A *topology* on a set X is a family of subsets of X, called the *open sets* of the topology, subject to certain axioms. Familiarity with the following notions is assumed: base for a topology; subbase for a topology; the topology *generated* by a given base (or subbase).

A *topological space* is a set together with a topology on it (the latter is customarily omitted from the notation). A *point* of a topological space is an element of the underlying set.

Familiarity with the following notions is assumed: closed subsets; adherent points of a subset; the closure of a subset; the interior of a subset. The closure of a subset A shall be regularly denoted by $\overline{A}$ or $\mathrm{cl}(A)$, and the interior of A will be A° or $\mathrm{int}(A)$.

We shall also compare pairs of topologies on the same set using the terms '*coarser*' and '*finer*'. The *trivial topology* on a set is the coarsest possible topology on that set; the *discrete topology* is the finest one. A *discrete space* is a topological space whose topology is discrete.

The terms *subspace topology* and *trace topology* are synonymous.

For a point x of a topological space X, a *neighbourhood* of x is a subset of X whose interior contains x. (Observe that neighbourhoods are not necessarily open.) The family of all neighbourhoods of x is the *neighbourhood system* or *neighbourhood filter* at x. We assume familiarity with the notion of a *neighbourhood basis* at a point x. Moreover, for us, a *local base* is the same as a neighbourhood basis (the elements of a local base *need not be open*).

Familiarity with the notion of *continuity* (at a point) of a map between topological spaces is assumed. We shall also speak of the *initial* and *final* topology on a set with respect to a given family of maps, the most important examples being the *product topology* and the *quotient topology*, respectively. We also expect knowledge of the terms '*open map*', '*closed map*', and '*homoeomorphism*'. A *bicontinuous* map is a homoeomorphism.

Familiarity with the notions of *first-countable* and *secound-countable* topological spaces is assumed.

Finally, we assume familiarity with the notion of *topological distinguishability* of two points. The reader is also expected to have encountered the first few separation axioms: T_0, T_1, and most importantly T_2, better known as the *Hausdorff condition*.

Metric spaces Familiarity with metric spaces is assumed, including the following notions: the natural (induced) topology on a metric space; the meaning of such terms as 'open ball', 'closed ball', 'radius', etc.; Cauchy sequences; (Cauchy) completeness and the completion of a metric space. Recall that in a complete metric space X (with metric d), a subset $Y \subseteq X$ is closed if and only if it is complete with respect to the restricted metric $d|_{Y\times Y}$.

A topological space is *metrizable* if there exists a metric on its underlying set which induces the given topology on said set. Since metric spaces satisfy all separation axioms, a metrizable topology must necessarily be Hausdorff (T_2).

Compact topological spaces Familiarity with the notion of an *open cover* of a topological space is assumed. Throughout these notes, a topological space X is *compact* if every open cover of X has a finite subcover. (*Caveat*: some authors, such as [Bourbaki-top], call these spaces *quasicompact* and reserve the term 'compact' for quasicompact spaces which are additionally Hausdorff; but *for us, compact spaces are not necessarily Hausdorff*.) The following basic standard facts concerning compact spaces might be used without reference to expedite several proofs throughout these notes.

- A space is compact if and only if every family of closed subsets having the finite intersection property has non-empty intersection.
- A finite (subspace of a) topological space is always compact, and a (subspace of a) Hausdorff topological space is finite if and only if it is both compact and (when equipped with the subspace topology) discrete.
- Continuous images of compact (sub-)spaces are again compact.
- A closed subspace of a compact space is itself compact; and
- a compact subspace of a Hausdorff space is necessarily closed. Hence:
- let $\mathcal{F}$ be a non-empty family of compact subsets of a Hausdorff space X. Then the intersection of all elements of $\mathcal{F}$ is compact.
- A metric [metrizable] space is compact if and only if it is *sequentially compact*, meaning that every sequence has a convergent subsequence.[3] Hence:
- a compact metric space is (Cauchy) complete.

[3]For general (meaning: not necessarily metrizable) topological spaces, sequences are not enough to check for compactness. Instead, we have that a topological space is compact if and only if every net has a cluster point (or, equivalently, if and only if every net has a convergent subnet).

- A (subspace of a) metric space is compact if and only if it is (Cauchy) complete and *totally bounded*, which means that for every $\varepsilon > 0$ there exists a finite cover of that (sub-)space by open balls of radius ε.
- (*Tychonoff's theorem*) An arbitrary product of compact spaces is again compact.

(Note that the last two items are proved using the axiom of choice.)

Formal power series Let A be a commutative ring with unity. The *ring of formal power series (in one variable) with coefficients in A*, variously denoted by $A[\![x]\!]$, $A[\![t]\!]$ or similar, is, by definition, the set $A^{\mathbb{N}}$ of (set-theoretic) functions $\mathbb{N} \to A$ equipped with pointwise addition (cf. earlier paragraphs) and multiplication given as follows: for f, $g\colon \mathbb{N} \to A$, the product fg is determined by the formula

$$(fg)(n) = \sum_{k=0}^{n} f(k)g(n-k), \qquad n \in \mathbb{N}.$$

(It is routine to check that, with these operations, $A[\![x]\!]$ is indeed a commutative ring with 1, containing a copy of A as a subring.)

The symbol x (t, ...) appearing in the notation $A[\![x]\!]$ ($A[\![t]\!]$, ...) stands for the unique element of $A^{\mathbb{N}}$ defined by

$$n \mapsto \begin{cases} 1, & n = 1, \\ 0, & \text{else.} \end{cases}$$

This notational convention has the following advantages.

(a) It is easily checked that, if a function $\boldsymbol{a}\colon \mathbb{N} \to A$, $n \mapsto a(n)$ has *finite support*, i.e. if the set $\{n \in \mathbb{N} : a(n) \neq 0\}$ is finite, then

$$\boldsymbol{a} = \sum_{n\,:\,a(n)\neq 0} a(n)x^n.$$

(This is, of course, to be understood as an equality of elements of $A[\![x]\!]$.)

(b) For two elements $\boldsymbol{a}$, $\boldsymbol{b}$ as in the previous item, the product $\boldsymbol{ab}$ can be computed simply by "collecting like terms".

These observations inform the following general notational convention for arbitrary elements of $A[\![x]\!]$.

Convention. When a function $\mathbb{N} \to A$, $n \mapsto a(n) = a_n$ is regarded as an element of $A[\![x]\!]$, it is customary to denote it by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

and to call such an element $f(x)$ a *formal*[4] *power series (in the variable x with coefficients in A).*

With this convention, the product of two elements of $A[\![x]\!]$ can again be computed simply by "collecting like terms".

[4]The word "formal" serves as a reminder that these infinite sums are not to be understood as limits of their sequences of partial sums; however, they *can* be regarded in this way for a suitable metrizable topology on $A[\![x]\!]$, called the (x)-*adic topology*, cf. Chapter II.

Chapter I

Locally compact groups and the Haar integral

The goal of this chapter is to introduce the eponymous protagonists of these notes, as well as to establish the fundamental properties of Haar measure. The first section serves as a primer on topological groups, while the specialization to locally compact groups is deferred to §2. Next, a preparatory review of measure theory (§3) will pave the way for a discussion of invariant measures, and Haar measures in particular, in §4. Existence and essential uniqueness of Haar measure are the subject of sections 5 and 7, respectively, with the intervening §6 reviewing the necessary background in integration theory. The final section is devoted to showing how to determine left and right Haar measure on several well-known examples of locally compact groups. To conclude the chapter we will give an overview of some additional results and topics that sadly lie outside the scope of these notes; this (unnumbered) section will be called "Vista", which is an idea borrowed from William Waterhouse's book [Water].

1 Generalities on topological groups

From a pragmatic point of view, this section has the main aim to lay down some of the foundations on which the remainder of the notes will build up: it introduces the fundamental notion of a topological group and contains key observations which will be used many times, often implicitly, in the sequel.

However, it was also written with the secondary intention to make the reader acquainted with some of the more unique aspects of the theory, be it unexpected results or ingenious proof ideas—to omit these would be to not do the theory justice. Still our treatment here is, by necessity, far from comprehensive. More can be found in the the Vista at the end of this chapter, but for a truly comprehensive account the reader is referred to [Bourbaki-top, Chapter III] or [SWarner, Chapter I].

1.1 The central notion

1.1. We delve right in with the pivotal definition.[1] A *topological group* is a group $(G, \cdot, e)$ together with a topology on its underlying set G such that the two maps

$$\begin{aligned} G \times G &\to G, \\ (g, h) &\mapsto g \cdot h, \end{aligned} \tag{G.LAW}$$

and

$$\begin{aligned} G &\to G, \\ g &\mapsto g^{-1}, \end{aligned} \tag{INV}$$

are continuous. (Here, it is understood that the topology on the domain $G \times G$ of the former map is the product topology.)

1.2. *Examples.*

(1) It follows from standard arguments to be found in any introductory analysis book that the additive group of the reals, together with the standard Euclidean topology on $\mathbb{R}$, is a topological group. The same arguments apply of course to the additive groups of the spaces $\mathbb{C}$, $\mathbb{R}^n$, $\mathbb{C}^n$ (for any $n \in \mathbb{N}$) with their respective standard topologies.

(2) Similarly, the multiplicative group of the reals, meaning the set $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ together with multiplication, is a topological group when endowed with its standard topology. Plainly the same is true of such groups as

[1] The reader is reminded to consult the relevant paragraphs in the preliminary considerations on p. page numbers!ff. should questions of notation or terminology arise.

$\mathbb{C}^*$ and $\mathbb{R}_{>0}$. Another example of a multiplicative group which will feature prominently later on is the so-called *circle group* $\mathbb{T}$, which can be described most simply as a multiplicative subgroup of $\mathbb{C}^*$:

$$\mathbb{T} = \{z \in \mathbb{C}^* : |z| = 1\}$$

with the topology inherited from $\mathbb{C}$. The name "circle group" is easily explained by keeping in mind that, upon identifying $\mathbb{C}$ with Euclidean plane $\mathbb{R}^2$, the subspace $\mathbb{T}$ is none other than the unit circle S^1 in the plane $\mathbb{R}^2$. One may also think of $\mathbb{T}$ as the group $U(1)$ of unitary 1×1-matrices (with complex entries). The notation $\mathbb{T}$ comes from interpreting the unit circle as a (one-dimensional) torus.

(3) It follows immediately from comparing the respective definitions[2] that every (real or complex) *Lie group* is automatically a topological group. Thus, matrix groups such as the general linear group $\mathrm{GL}(n, \mathbb{R})$ or the orthogonal group $\mathrm{O}(n, \mathbb{R})$ are topological groups in a natural way (for any $n \geq 1$).

In fact, the knowing reader might have realized that *all* examples discussed so far, including those from items (1) and (2) above, are Lie groups. However, the notion of a topological group is clearly much more general, as the next examples show.

(4) Consider the additive group of $\mathbb{Q}$ together with its natural topology as a subspace of $\mathbb{R}$. It is easily seen to be a topological group, but it is not a Lie group.[3]

(5) Any topological vector space over $\mathbb{R}$ or $\mathbb{C}$ is a topological group with addition.

(6) Any group, when endowed with the discrete topology, becomes a topological group.

(For contrast, observe that a topological vector space over $\mathbb{R}$ or $\mathbb{C}$ is a Lie group if and only if it is finite-dimensional, and that, by the most widespread

[2] We have included the definition of a Lie group in the Vista.

[3] The latter claim will become apparent from the discussion in §2 below; more precisely, we shall see that all Lie groups are locally compact groups whereas $\mathbb{Q}$ isn't.

definition of a Lie group, a discrete group is a Lie group if and only if it is *at most countable*; cf. the Vista.)

Observe that all examples discussed so far (with the possible exception of (5)) are even Hausdorff. We shall see later (see 1.19) that forgoing the Hausdorff property means having to deal with points that are "lumped together" (*topologically indistinguishable*), so this is not only undesirable, it also hardly ever happens in practice. For this reason, some authors reserve the term 'topological group' for the Hausdorff ones. By contrast, our definition in these notes does *not* explicitly rule out non-Hausdorff groups, so that, for us, the following *does* constitute an example of a topological group:

(7) The trivial topology (the coarsest-possible topology) makes every group into a topological group. $\diamond$

1.3. *Remarks on the definition.*

(1) One might wonder why the definition of a topological group features a condition on (G.LAW) (as well as (INV)) but no condition on, say, the maps

$$\begin{aligned} G &\to G, \\ x &\mapsto g\cdot x \end{aligned} \qquad \text{and/or} \qquad \begin{aligned} G &\to G, \\ x &\mapsto x\cdot g \end{aligned} \tag{$*$}$$

as g ranges over the elements of G. To answer this, recall the following result from general topology first:

1.4. Lemma. *Let X, Y and Z be topological spaces, and let $f : X \times Y \to Z$ be a continuous map, where the domain is equipped with the product topology. Then, for any fixed $x \in X$, the map $f_x\colon Y \to Z$, $y \mapsto f(x, y)$ is continuous; analogously, the map $f_y\colon X \to Z$, $x \mapsto f(x, y)$ is continuous for every $y \in Y$.*

(The proof is a straightforward verification and is left to the interested reader.) It follows that continuity of (G.LAW) implies continuity of the maps from ($*$), so that, when the former is in force, it is not necessary to expressly require the latter. By contrast, there is only a partial converse to this statement, as shown by [SWarner, Chapter 1, Exercise 1.10]; in other words, imposing continuity of the maps from ($*$) *instead of* continuity of (G.LAW) might lead to a strictly larger class of objects. It is then not entirely easy to explain *a priori* why one choice should be more convenient or natural than the other, except perhaps through a "categorical" lens (s. the Vista for more on this, but cf. also E.4.(c)); at this point in time, the reader might simply have to trust that the definition from 1.1 is the "correct" one until the theory itself has had a chance to (hopefully) provide enough evidence for this.

(2) It is also natural to wonder whether the definition can be simplified by omitting, say, the condition on the inversion map (INV). (I.e., whether it automatically follows from the other condition, as is the case for Lie groups.) In general, this is not the case: Exercise E.1 gives an example of a group G and a topology on G such that the group law (G.LAW) is continuous but the inversion map is not. (Amusingly, there also exist a group G and a topology on G such that (INV) is continuous but (G.LAW) isn't, see [SWarner, Chapter 1, Exercise 1.3].)

$\diamond$

1.5. *Remark.* On the surface, it might appear that the study of topological groups is, loosely speaking, a "proper subset" of the study of abstract groups. In actuality, the opposite is true: as was mentioned above, every abstract group can be viewed as a Hausdorff topological group by endowing it with the discrete topology, and so, in a sense, any result that holds for (Hausdorff) topological groups groups also holds for all abstract groups.[4] An added bonus in considering groups with their natural topologies (when available) rather than simply as abstract groups is, of course, that the added structure makes it easier to "tell apart" pairs of groups that are indistinguishable from a merely algebraic viewpoint: a particularly illuminating example is explored in Exercise E.3. $\diamond$

1.2 Basic properties

1.6. Having seen how many objects fall into the scope of our central definition, it is time to see some of its more direct consequences. The first result is as simple as its ramifications are powerful, and is concerned primarily with

[4] Although, of course, many results might turn out to be uninteresting for discrete groups.

the maps

$$l_g \colon G \to G, \quad x \mapsto g \cdot x \qquad \text{and} \qquad r^g \colon G \to G, \quad x \mapsto x \cdot g \tag{L\&R}$$

(where g ranges over the elements of the group $(G, \cdot)$) that were already discussed in an earlier remark. For fixed g, these maps are called *left* and *right translation by* g, respectively. There seem to be no standard notations in the literature for these maps—and in fact special notation is hardly ever needed at all: if $A \subseteq G$ and $g \in G$, it is far more common to denote $l_g(A)$ and $r^g(A)$ by gA and Ag, respectively. Our own choice of notation borrows somewhat from the idea of so-called covariant and contravariant indices and should serve as a reminder that $l_{gh} = l_g \circ l_h$ whereas $r^{gh} = r^h \circ r^g$ (with g, $h \in G$ arbitrary).

We can now state the result:

1.7. PROPOSITION. *Let G be a topological group. Then:*

(i) for any $g \in G$, both left and right translation by g define homoeomorphisms from G into G; and

(ii) the inversion map (s. (INV)) is a homoeomorphism from G into itself.

Proof. The second claim is immediate, since the map is clearly self-inverse and is continuous by assumption.

As for the first claim, it was already argued above that l_g and r^g are continuous for any $g \in G$ (the notations being as in (L&R)). But each l_g is a bijective map with inverse $l_g^{-1} = l_{g^{-1}}$, which is again continuous; so every l_g is a homoeomorphism, as claimed. The argument for right translations is entirely analogous (or, alternatively, one may use the fact that $r^g = \nu \circ l_{g^{-1}} \circ \nu$, where ν denotes the inversion map $g \mapsto g^{-1}$.) □

There are a number of immediate corollaries, which are nevertheless well worth writing down explicitly. The first two are direct consequences of parts (i) and (ii) of the proposition, respectively:

1.8. COROLLARY. *Let G be a topological group, A be a subset of G, and g a point in G. Then the following are equivalent:*

(i) A is open;

(ii) gA is open;

(iii) Ag is open.

Moreover, the statement remains true if the word 'open' is replaced by 'closed' or 'compact' throughout.

(For the meaning of the notations in (ii) and (iii), see 1.6 above.)

1.9. Corollary. *Let G be a topological group, A be a subset of G. Then A is open [closed, compact] if and only if*

$$A^{-1} := \{a^{-1} : a \in A\} \subseteq G$$

is open [closed, compact].

To state the next corollary, we introduce the following notation: for subsets A, B of a group G, one writes

$$AB := \{a \cdot b : a \in A, b \in B\} \subseteq G.$$

1.10. Corollary. *Let G be a topological group. If $A \subseteq G$ is open and $B \subseteq G$ is arbitrary, then AB and BA are open.*

Proof. Write $AB = \bigcup_{g \in B} Ag$ and analogously for BA. $\square$

Perhaps the most striking consequence of the proposition is that a topological group looks "locally the same" around each of its points (in more rigorous terms, it is a *homogeneous space*): indeed, if g, h are arbitrary elements of the topological group G, then there exists a homoeomorphism from G to G which sends g to h, namely (say) $l_h \circ l_g^{-1}$, or $r^h \circ (r^g)^{-1}$. In this connection, note the following result.

1.11. Corollary. *Let G be a topological group, U be a subset of G and g, h be points in G. Then the following are equivalent:*

(i) U is a neighbourhood of g;

(ii) $hg^{-1}U$ is a neighbourhood of h;

(iii) $Ug^{-1}h$ is a neighbourhood of h.

Moreover, the statement remains true if the word 'neighbourhood' is replaced by 'open neighbourhood', 'closed neighbourhood', 'compact neighbourhood', 'compact open neighbourhood', ... throughout.

In particular (by setting $h = e$), we obtain that, as U runs over the neighbourhoods of e, both gU and Ug run over the neighbourhoods of g, and this exhausts the neighbourhoods of g. This can be used to prove that a group homomorphism between topological groups is continuous if and only if it is continuous at the identity (this is Exercise E.5). At the same time, it points to another major property of topological groups: to completely describe their topology it suffices, at least morally, to describe a neighbourhood base around the neutral element. For a precise statement and proof, see e.g. [SWarner, Chapter 1, Corollary 1.5]; for an important special case which will feature in our discussion of locally compact fields in Chapter II, see Exercise E.6.

1.3 Local bases at the identity

1.12. As a natural continuation to the preceding discussion, our next task will be to investigate the system of neighbourhoods of the identity in a topological group. To that purpose, it is convenient to introduce the following definition: a subset A of a group G is said to be *symmetric* if it has the property that $A^{-1} = A$ (the notation being as in 1.9). The reason for the terminology is apparent if one considers the familiar group $(\mathbb{R}, +)$: plainly, a subset of this group is symmetric in the above sense if and only if it is symmetric around 0 in the usual geometric sense.

1.13. *Remark.* Clearly all subgroups are examples of symmetric subsets, but the converse is not true: think e.g. of $(-1, 1) \subset \mathbb{R}$. In fact, it is easy to see that, in $\mathbb{R}$, no neighbourhood U of the neutral element can be a subgroup except for $\mathbb{R}$ itself. This is true in any connected group; see the Vista for more on this. $\diamond$

The importance of symmetric subsets and particularly symmetric neighbourhoods is made apparent by the following result.

1.14. Proposition. *Let G be a topological group. Then:*

(i) The symmetric open neighbourhoods of the identity e form a local base at e. In other words, for every neighbourhood U of e there exists a symmetric open neighbourhood V such that $V \subseteq U$.

(ii) For every neighbourhood U of e there exists a symmetric open neighbourhood W such that $WW \subseteq U$.

Proof. To show the first claim, let U be a neighbourhood of e. Without loss of generality, U can be taken to be open in G. But then $V = U \cap U^{-1}$ has the sought-after property (cf. 1.9).

As for the second claim, let U be again an arbitrary neighbourhood of the identity. Then continuity of the group operation (the map from (G.LAW)) at (e, e) guarantees the existence of a neighbourhood $\tilde{W}$ such that $\tilde{W}\tilde{W} \subseteq U$. By (i) we may then take a symmetric open neighbourhood $W \subseteq \tilde{W}$, and we have $WW \subseteq U$ *a fortiori*. □

1.4 Constructions

The previous result concludes our study of the neighbourhood system at the identity for the moment. In the next result, we focus on *constructions* with topological groups instead.

1.15. Proposition.

(i) A subgroup of a topological group, when equipped with the subset topology, is again a topological group.

(ii) An arbitrary product of topological groups is a topological group with the product topology.

(iii) A quotient of a topological group G by a normal subgroup H is again a topological group with the quotient topology. The canonical map $\pi_H : G \twoheadrightarrow G/H$, $g \mapsto gH$ is continuous and open. If the subgroup is open, then the quotient is discrete.

For the duration of the proof, we will resort to the following notations: for any group G, the inversion map (cf. (INV)) will be denoted by ν_G, and the map describing the group law (cf. (G.LAW)) will be denoted by μ_G. We shall also write q_G for the map $\mu_G \circ (\mathrm{id}_G \times \nu_G)$, i.e.,

$$\begin{aligned} q_G \colon G \times G &\to G, \\ (g, h) &\mapsto g \cdot h^{-1}. \end{aligned}$$

With these notations, we have the following auxiliary result which will simplify the proof of the proposition:

1.16. LEMMA. *Let G be a group and let a topology be given on G. In order for both μ_G and ν_G to be continuous, it is necessary and sufficient that $q_G = \mu_G \circ (\mathrm{id}_G \times \nu_G)$ be continuous.*

Proof of the lemma. This is Exercise E.2. □

Proof of the proposition. For the first claim, let H be a subgroup of the topological group G. Then μ_H is none other[5] than the restriction of μ_G to $H \times H$, and analogously $\nu_H = \nu_G|_H$. Thus, their continuity follows from the continuity of the "unrestricted" maps μ_G and ν_G (which holds by assumption) together with the definition of the subspace topology.

For the second claim, let $(G_\alpha)_{\alpha \in A}$ be a (non-empty) family of topological groups, and set $G \coloneqq \prod_{\alpha \in A} G_\alpha$. By 1.16, it suffices to show that $q = q_G$ is continuous; by that same reason, we may use that for each α the map $q_\alpha \coloneqq q_{G_\alpha}$ is continuous.

To prove the claim, recall that the product topology on G is, by definition, the initial topology with respect to the various projections $G \to G_\alpha$; in other words, q is continuous if and only if $\pi_\alpha \circ q$ is continuous for all α, where π_α is the projection to the "α-th" factor. But π_α is a group homomorphism, so $\pi_\alpha \circ q = q_\alpha \circ (\pi_\alpha \times \pi_\alpha)$; we already argued that the first map on the right-hand side is continuous, and it is not hard to check (by standard point-set topology) that the same holds for the second map.

Finally, consider the third claim. First of all, the quotient $G/H = \{gH : g \in G\}$ is indeed a group with the operation $gH \cdot g'H \coloneqq (gg')H$, because H was assumed normal.

Next, consider the map π_H. The quotient topology is the final topology with respect to this map, which makes π_H automatically continuous. The definition of the quotient topology also entails that $A \subseteq G/H$ is open if and only if $\pi_H^{-1}(A)$ is open in G; it follows that π_H is an open map, because if U is open in G, then $\pi_H^{-1}(\pi_H(U)) = UH$ is also open in G by 1.10, hence $\pi_H(U)$ itself is open.

[5] Actually, the restrictions of μ_G and ν_G a priori map into G, but since H is a subgroup, their range is actually H, so their codomain can taken to be H.

Next, one proves that G/H is indeed a topological group. As was the case for products, we may use 1.16 to reduce to the following: use continuity of q_G to infer continuity of $q_{G/H}$. The decisive step is now to realize that $q_{G/H} \circ (\pi_H \times \pi_H) = \pi_H \circ q$ and use the readily-seen continuity of $\pi_H \times \pi_H$.

As for the very last assertion, this follows easily from the openness of π_H. Indeed, if H is open in G, then $\pi_H(H)$ is open in G/H. But $\pi_H(H)$ is none other than the trivial subgroup of G/H, so by 1.7 all singletons in G/H are open, which was to be shown. □

1.17. *Examples.*

(1) Clearly, for any $n \geq 1$, the topological group $(\mathbb{R}^n, +)$ is none other than the n-fold product of the topological group $(\mathbb{R}, +)$ with itself. The same holds true if one replaces $\mathbb{R}$ by $\mathbb{C}$.

(2) In 1.2, we already considered $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{T} \subset \mathbb{C}^*$ as subgroups with the subspace topology.
Now consider the inclusion $\mathbb{Z} \subset \mathbb{R}$. Then the subspace topology on $\mathbb{Z}$ is precisely the discrete topology.

(3) The above proposition implies in particular that the quotient $\mathbb{R}/\mathbb{Z}$ is a topological group. Indeed, this group is isomorphic (as a topological group!) to the circle group $\mathbb{T}$ introduced in 1.2.(2); in other words, there exists a map $\mathbb{R}/\mathbb{Z} \to \mathbb{T}$ which is both a group homomorphism and a homoeomorphism (namely, the map which sends the coset $x + \mathbb{Z}$ to $\mathrm{e}^{2\pi \mathrm{i} x}$). ◇

1.5 Open and closed subgroups

The next result lists some of the most striking products of the interaction between the group structure and the topological structure on a topological group.

1.18. Proposition.

(i) Let H be a [normal] subgroup of a topological group G. Then its closure is again a [normal] subgroup.

(ii) Open subgroups of a topological group G are automatically closed.

(iii) A closed subgroup of a topological group G which is of finite index is automatically open.

(iv) An open subgroup inside a compact group is of finite index and is itself compact.

Proof. Fill in proof of first claim, cf. [SWarner, Chapter I, Thm. 2.1].

For the second and third claims, recall that, if H is a subgroup of a group G, then G is the union of the left cosets gH as g ranges over the elements of G. On the other hand, $H = \bigcup_{g \in H} gH$, and so the complement of H in G is simply the union of the remaining cosets, i.e., $G \setminus H = \bigcup_{g \notin H} gH$. Thus, if G is a topological group and H is an open subgroup, then the complement of H is the union of sets of the form gH and hence itself open (cf. 1.8). Similarly, if H is closed and of finite index then its complement can be written as a finite union of translates of the closed set H, and is hence again closed.

Finally, suppose U is an open subgroup of a compact group K. Then one may write $K = \bigcup_{k \in K} kU$. This is an open cover of K, so, by compactness, there is a finite subcover: $K = \bigcup_{i=1}^{n} k_i U$. Thus $\{k_1, \ldots, k_n\}$ contains a system of representatives for K modulo U, hence $[K : U] \leq n < \infty$. The last assertion follows from (ii) together with the well-known fact that a closed subset of a compact space is again compact. □

Some of the most noteworthy implications of these easy statements are on connectedness, as discussed in the Vista.

1.6 Hausdorff groups

To conclude this section, we want to focus on the Hausdorff property for topological groups. Earlier, we remarked that all groups encountered in practice are Hausdorff, and in fact, starting in the next section we will impose the Hausdorff condition on all groups we want to study. So it is worthwhile asking how much of a restriction this actually is. The relevant result is:

1.19. Proposition.

(i) A topological group is Hausdorff if and only if it is T_0, and this is the case if and only if the trivial subgroup is closed.

(ii) For every topological group G there exist a Hausdorff group G' and a surjective group homomorphism $\pi\colon G \twoheadrightarrow G'$ such that every continuous map $G \to X$ to a T_0 topological space X factors uniquely through π.

Observe that part (ii) reads like a universal property, and indeed, it quickly implies that G' is unique up to isomorphism of topological groups.

To prove the proposition, we start with some preparations which are relevant to both claims.

1.20. LEMMA. *In every topological group G, there exists a closed normal subgroup N such that:*

(i) $\overline{\{g\}} = gN$ for every $g \in G$; and

(ii) in order for two points g, $h \in G$ to be topologically indistinguishable, it is necessary and sufficient that they lie in the same N-coset.

Proof of the lemma. Plugging $g = e$ into the first claim, we see that the only possible choice for N is $N = \overline{\{e\}}$. This is indeed a closed normal subgroup of G by 1.18.(i) (because $\{e\}$ is normal in G). Upon invoking 1.8, we see that

$$\overline{\{g\}} = \bigcap_{\substack{F \text{ closed} \\ g\in F}} F = \bigcap_{\substack{C \text{ closed} \\ e\in C}} gC = g \cdot \bigcap_{\substack{C \text{ closed} \\ e\in C}} C = g \cdot \overline{\{e\}}$$

for any $g \in G$, whence the claim.

As for (ii), this now follows immediately from (i) if we recall that two points x and y of a topological space X are topologically indistinguishable if and only if the singletons $\{x\}$ and $\{y\}$ have the same closure in X. $\square$

Proof of the proposition. Clearly in a Hausdorff group all points are closed, so in particular the trivial subgroup is closed. On the other hand, if $\{e\}$ is closed then *all* singletons are closed (by 1.8), which in turn implies the T_0 condition. Hence, the nontrivial part of the first claim lies in showing that a T_0 group is automatically Hausdorff.

So suppose that G is T_0, let $g, h \in G$ be distinct, and set $x = g^{-1}h$. Then x and e are distinct, hence topologically distinguishable. We shall now use the lemma to infer that $e \notin \overline{\{x\}}$.

For suppose that $e \in \overline{\{x\}}$. Then $\overline{\{e\}} \subseteq \overline{\{x\}}$, i.e., by the above lemma, $eN \subseteq xN$. But N-cosets are either disjoint or identical, so it would follow that

$\overline{\{e\}} = eN = xN = \overline{\{x\}}$, i.e., e and x are topologically indistinguishable, a contradiction.

Thus, $e \notin \overline{\{x\}}$. This means that e is not an adherent point of the singleton $\{x\}$; accordingly, there exists some neighbourhood U of e which does not contain x. By 1.14, there exists a symmetric open neighbourhood V of e such that $VV \subseteq U$. Then gV and hV are disjoint, because otherwise there would exist elements $v_1, v_2 \in V$ such that $gv_1 = hv_2$, but then $x = g^{-1}h = v_1v_2^{-1} \in VV^{-1} = VV \subseteq U$, a contradiction. Since g and h were arbitrary, this proves that G is Hausdorff.

To prove (ii), let X be an arbitrary T_0 space and φ be a continuous map from G to X. Then $\varphi(g) = \varphi(h)$ whenever g and h are topologically indistinguishable; in other words, φ is constant on N-cosets, where N is as in the previous lemma. But then φ factors uniquely through the quotient G/N, which is Hausdorff by Exercise E.7. Hence, we can take G' and π to be the quotient G/N and the quotient map $G \twoheadrightarrow G/N$, respectively. □

The final result of this section has the purpose of showing that common constructions do not lead outside of the category of Hausdorff groups. Its proof is easy and left as an exercise (E.7).

1.21. Proposition.

(i) Subgroups of Hausdorff groups are again Hausdorff.

(ii) Arbitrary products of Hausdorff groups are again Hausdorff.

(iii) A quotient of a Hausdorff group by a closed normal subgroup is again Hausdorff.

1.22. *Remark.* Recall that the Hausdorff property and the T_0 property fit into the larger range of so-called separation axioms. In the Vista, the reader will find further discussion concerning separation axioms for topological groups as well as another property that separation axioms are famously related to, namely metrizability. ◇

2 Locally compact groups

In this section, we first discuss the notion of local compactness for general topological spaces, addressing some of the different conventions that exist in the literature. Next, we specialize to locally compact groups, showing that this category encompasses all Lie groups—and thus many of the examples from the previous section—as well as other, as-yet-unseen types of groups. In this connection, we shift our focus to those locally compact groups which are, in a sense, "furthest away" from being Lie groups, called *locally profinite groups*—we will see more and more examples of these throughout these notes, and their distinctive qualities (even among locally compact groups) will ultimately earn them a dedicated section in the final chapter.

2.1 Local compactness in general

2.1. As was hinted at above, usage of the phrase 'locally compact' can vary in the literature. Firstly, just as with the term 'compact', some authors require locally compact spaces to be Hausdorff whereas others do not. The qualifier 'locally' complicates matters further: often, 'locally P' (where P denotes some property of topological spaces) means, by definition, that each point of the space at hand has a local base consisting of subsets with the property P, but sometimes, e.g. with 'locally Euclidean', one may more concisely (and yet equivalently) require that each point have *at least one* open neighbourhood with the property P. Thus, we see that either of the following conditions—possibly in conjunction with the requirement that X be Hausdorff—could be a natural candidate for defining 'local compactness' for a topological space X:

(LC1) Every point of X has a compact neighbourhood.

(LC2) Every point of X has a local base of compact neighbourhoods.

Both of these conditions can indeed be found in the literature, and in fact, they are not the only ones. Some authors work with either of the following, which are obtained by swapping the word 'compact' out for 'relatively compact'.[6]

(LC3) Every point of X has a relatively compact neighbourhood.

(LC4) Every point of X has a local base of relatively compact neighbourhoods.

Clearly, (LC1) is the least restrictive of these as well as the easiest to verify in practice; for this reason, in these notes, a topological space X will be called

[6] Recall that a subset of a topological space X is *relatively compact* if its closure in X is compact.

locally compact if and only if it satisfies (LC1) from above. However, this condition shall rarely appear on its own; our focus will be almost exclusively on spaces which are *locally compact Hausdorff* (or *LCH* for short), i.e., spaces that are both Hausdorff and satisfy (LC1). Not only are such spaces clearly "nicer" to work with; adding the assumption of "Hausdorfness" has the pleasant side effect that it resolves the ambiguity in the definition of local compactness, meaning that there is no disagreement across the literature around the meaning of the phrase 'locally compact Hausdorff space'. This is the content of the following result:

2.2. PROPOSITION. *For a Hausdorff space X, (LC1)–(LC4) are equivalent.*

Proof. First, it is apparent that for *any* X (not necessarily Hausdorff!), we have the implications (LC4) $\Longrightarrow$ (LC3) $\Longrightarrow$ (LC1) and (LC2) $\Longrightarrow$ (LC1). (It is in fact also true for any X that (LC3) implies (LC4), but we shall not need this.)

From now on, let X be Hausdorff. Then compact subsets of X are closed and in particular relatively compact, so that now (LC1) implies (LC3) and (LC2) implies (LC4). In summary, (LC2) is now the strongest of all four, with (LC1) being the weakest. We shall now close the circle by showing that (LC1) in turn implies (LC2). To do this, we use the fact that, in a Hausdorff space X satisfying (LC1), every point has a local base consisting of closed neighbourhoods; this will be addressed in the Vista below.

Thus, let X be a Hausdorff space satisfying (LC1), and let x be any point in X. Then x has both a compact neighbourhood K (directly by (LC1)) and a local base $\mathcal{B}$ of closed neighbourhoods (by the aforementioned result). To prove that X satisfies (LC2), we need to exhibit a neighbourhood basis around x consisting of compact subsets. To that end, consider $\{C \cap K : C \in \mathcal{B}\}$. Clearly this is again a neighbourhood basis around x. But it is also easy to see that each $C \cap K$ is compact in X: first, since X is Hausdorff, K is closed, so $C \cap K$ is also closed (in X) if C is; but then $C \cap K$ is a closed subspace of the compact space K, hence again compact, as claimed. □

2.3. Locally compact Hausdorff spaces form a well-behaved and well-studied class of topological spaces (cf. the Vista for some results of topological interest), which is however also comfortably "large"—it is easily checked to contain:

(1) all discrete spaces;

(2) compact Hausdorff spaces (since compact spaces trivially satisfy (LC1));

(3) Euclidean space of any (finite) dimension; and hence

(4) any locally Euclidean Hausdorff space, and so in particular all (real or complex) manifolds.

We shall later find other examples of such spaces by means of the following general result:

2.4. Proposition. *Let X be a locally compact Hausdorff space. Then all closed subsets of X and all open subsets of X are again locally compact in the subspace topology.*

Proof. Fill in this proof! Cf. e.g. SE1887556. □

2.2 Local compactness for groups

2.5. The entirety of the above discussion applies to general topological spaces, so it is now time to bring the focus back onto topological groups. Clearly it is meaningful to talk about locally compact Hausdorff topological groups—meaning, of course, topological groups whose underlying topological spaces are LCH. However, it is important to note that the predominant convention in the literature is to drop the word 'Hausdorff' as though it were implied by either 'locally compact' or 'topological group' (even when the author's definitions would *not* imply this). In other words, a topological group is usually said to be *locally compact* if (and only if) its topology is locally compact *as well as* Hausdorff. We shall adhere to this seemingly universal convention in these notes too; in this section and all sections to come, 'locally compact group' is *always* to be understood as a shorthand for 'locally compact Hausdorff topological group'.

2.6. *Examples.* The following are examples of locally compact groups:

(1) the additive groups of $\mathbb{R}$ and $\mathbb{C}$, and in fact of $\mathbb{R}^n$ and $\mathbb{C}^n$ (for any $n \in \mathbb{N}$);

(2) the multiplicative groups $\mathbb{R}^*$, $\mathbb{R}_{>0}$, $\mathbb{C}^*$ and $\mathbb{T}$ (the circle group) considered in 1.2.(2);

(3) all Lie groups;

(4) all discrete groups;

(5) all compact Hausdorff topological groups.

In fact, most of these claims follow directly from the discussion in 2.3. (For (2), use 1.18, or view (2) as a subcase of (3).) Alternatively, they can be inferred immediately (with the possible exception of (3)) from the following result:

2.7. LEMMA. *For a Hausdorff topological group G, the following are equivalent:*

(i) G is locally compact;

(ii) G has at least one compact subset with non-empty interior;

(iii) the neutral element of G has a compact neighbourhood.

Proof. Use 1.8 and 1.11. □

Further examples of locally compact groups will be discussed in the next subsection, and with the introduction of locally compact fields in Chapter II, we will be able to produce even more. For the moment, let us simply mention (expanding on the case (3) of Lie groups above) that groups defined by polynomial equations over a nondiscrete locally compact field can be topologized in a canonical way, turning them into locally compact groups. (See §3 of Chapter II for a particularly important class of examples.) ◇

2.8. *Non-examples.*

(1) The additive group of $\mathbb{Q}$ (topologized as in 1.2) is *not* locally compact. We argue by contradiction: suppose that $\mathbb{Q}$ were locally compact. Then in particular $0 \in \mathbb{Q}$ would have a compact neighbourhood K. Because the topology on $\mathbb{Q}$ (and hence on K) is induced by the standard Euclidean metric on $\mathbb{R}$, it follows that K is sequentially compact. We shall obtain a contradiction by exhibiting a sequence in K that has no convergent subsequence in K.
First, pick an open neighbourhood U of 0 contained in K. Then U can be written as $V \cap \mathbb{Q}$ for some open neighbourhood V of 0 in $\mathbb{R}$. Now pick an irrational number $\alpha \in V$ and a sequence $x_1, x_2, \dots$ such that:

- each term x_n lies in V;
- each term x_n is rational; and
- the sequence converges to α.

Then $(x_n)_{n\geq 1}$ is a sequence in U, but no subsequence of U has a limit in U, and this is the desired contradiction. In fact, this argument proves that a subset of $\mathbb{Q}$ with non-empty interior cannot be compact (this would, however, also follow from 2.7).

(2) A topological vector space over $\mathbb{R}$ or $\mathbb{C}$ is neither locally compact nor Hausdorff in general; more concretely, such a space can never be locally compact if it is infinite-dimensional over the ground field (in this regard, cf. also §3 of Chapter II.) Thus, readers acquainted with some functional analysis may take their favourite infinite-dimensional Banach (or Hilbert, or Fréchet) space as an additional example of a Hausdorff group which is not locally compact. $\diamond$

Up until this point, it might appear that the split between locally compact groups and all other topological groups runs along the exact same line as the split between *Lie* groups and all other topological groups. In other words, we have so far failed to provide a single concrete example of a locally compact group—or indeed of a compact Hausdorff group—which is not already a Lie group (except for, say, "an uncountable group with the discrete topology", which is artificial and ultimately uninteresting). The examples presented in the next subsection will fill this gap. Before devoting ourselves to these, however, it is appropriate to provide one easy general result on how locally compact groups behave under fundamental topological constructions.

2.9. Proposition.

(i) A closed subgroup of a locally compact group is LCH with the subspace topology.

(ii) A quotient of a locally compact group by a closed normal subgroup is again locally compact.

(iii) Finite products of locally compact groups are again locally compact.

2.10. *Remark.*

(1) The first claim of the proposition fails if the subgroup is not closed (as we saw with $\mathbb{Q} \subset \mathbb{R}$). In fact, [SWarner, Chapter I, Corollary 2.4] shows that, in a Hausdorff group, a locally compact subgroup is necessarily closed.

(2) Compare the last claim of the proposition with *Tychonoff's theorem*, which states that *arbitrary* products of *compact* spaces are again compact. Tychonoff's theorem implies that arbitrary products of locally compact spaces, *when equipped with the box topology*, are again locally compact; however, the more useful topology on a product is generally the product topology, and with this latter topology, arbitrary products of locally compact spaces are *not* themselves locally compact in general. $\diamond$

Proof of 2.9. (a) follows from 2.4.

(b) is e.g. [SWarner, Chapter I, Thm. 3.22.(1)].

As for (c), cf. (2) in the above remark. $\square$

2.3 Locally profinite groups

We have remarked earlier that, so far, all our concrete examples of locally compact groups also happen to be Lie groups, and that this subsection will fill this gap.

2.11. We warm up by working out an example that should be understandable without much background and will pop up several times in the future. Let A be an abelian group, denoted additively, and consider

$$\begin{aligned} G &= A^{\mathbb{N}} \\ &= \{(a_n)_{n\in\mathbb{N}} : a_n \in A \text{ for all } n \in \mathbb{N}\} \\ &= \{f : f \text{ is a set-theoretic function } \mathbb{N} \to A\}. \end{aligned}$$

(The equality of the three sets on the right-hand side is essentially a matter of definition and does not hide any deep claims; hopefully having several possible realizations on the page will facilitate understanding rather than hinder it.)

Naturally G admits a binary operation, namely termwise/pointwise addition, which turns it into an abelian group. On the other hand, G also admits a fairly natural topology, which can be described equivalently in any of the following ways (the verification of the equivalence is left to the interested reader):

(i) the topology induced by the metric

$$d((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) = 2^{-\inf\{n\,:\,a_n\neq b_n\}}$$

(with the conventions $\inf\emptyset = \infty$ and $2^{-\infty} = 0$);

(ii) the topology in which, for any $x = (a_n)_{n\in\mathbb{N}} \in G$, a subset $U \subseteq G$ is a neighbourhood of x if and only if U contains a set of the form

$$\{y = (b_n)_{n\in\mathbb{N}} \in G : a_n = b_n \text{ for all } n \leq N\}$$

for some natural number N;

(iii) the topology induced on G as in Exercise E.6 by the following family of subgroups:

$$\Sigma = \{H_n : n \in \mathbb{N}\},$$

where, for $n \in \mathbb{N}$,

$$H_n \coloneqq \{(a_n)_{n\in\mathbb{N}} : a_0 = a_1 = \cdots = a_n = 0\};$$

(iv) the coarsest topology with respect to which all of the "truncation maps"

$$G \to A^n,$$
$$(a_n)_{n\in\mathbb{N}} \mapsto (a_0, a_1, \ldots, a_{n-1})$$

(where n ranges over the positive integers) are continuous, each A^n being equipped with the discrete topology;

(v) the product topology, where each factor A is, again, equipped with the the discrete topology.

It is easy to see that this topology makes G into a Hausdorff topological group. It should be about as easy (see Exercises E.8 and E.9) to check that G has the following properties:

(ZD) every point in G has a neighbourhood basis consisting of neighbourhoods which are *both* open *and* closed in G;

(TD) the only non-empty connected subsets of G are the singletons $\{x\}$, $x \in G$.

(See 2.12 below for a more detailed discussion of these.) Either property demonstrates that G is "very far" from connectedness: it should be easy to see that a topological space satisfying (ZD) cannot at the same time be locally connected unless its topology is discrete (and our G is clearly not discrete). In particular, *G cannot possibly be a Lie group*. On the other hand, *if A is a finite group, then G is locally compact* and in fact even compact (s. Exercise E.10). Thus, our goal of showing that the class of locally compact groups is strictly larger than that of Lie groups has been attained.

2.12. *Remark*. The labels (ZD) and (TD) used above are suggestive of the names that are sometimes given to these properties in the more general context of point-set topology, namely *zero-dimensional* and *totally disconnected*, respectively. (Unfortunately, there is no complete agreement in the literature over the terminology.) It is the point of the already-referenced Exercise E.9 that a T_1 space which satisfies (ZD) also automatically satisfies (TD). The converse is not true in general, but we are in luck:

2.13. PROPOSITION. *A locally compact Hausdorff space which satisfies (TD) also automatically satisfies (ZD).*

(The proof is not very involved, see MO37392; also, in the context of locally compact groups there is a refinement, see 2.20 below.) This puts us in a position to use the phrase '*totally disconnected*' to mean either of the properties (ZD) and (TD) *so long as* we are discussing LCH spaces (and hence in particular when we are discussing locally compact groups). ◇

2.14. A topological group which is compact, Hausdorff and is totally disconnected (see the preceding remark) is called a *profinite group*. The deeper reason for this terminology lies in the following result:

2.15. PROPOSITION. *For a topological group G, the following are equivalent:*

(i) G is compact, Hausdorff and totally disconnected;

(ii) G is, algebraically and topologically, the projective limit *of an inverse system of finite (discrete) groups.*

We shall not prove this result in full, nor will we need to understand projective limits in full generality—readers who are already acquainted with them will however have no difficulties in verifying that, if A is discrete, then $G = A^{\mathbb{N}}$ is indeed the projective limit of the groups $G_i = A^i$, $i \geq 1$, with the "projections" given by truncation maps. (Indeed, this is precisely the content of (iv) from 2.11 above.)

2.16. *Remark.* Some readers might be familiar with projective limits from one of the common definitions of the ring $\mathbb{Z}_p$ of p-adic integers. We will define this object in a slightly different fashion in Chapter II.

Readers might also be aware that the Galois group of a Galois extension E/F (possibly of infinite degree) can be viewed as the projective limit of the Galois groups $\mathrm{Gal}(F'/F)$ as F' runs over those intermediate fields $F \subseteq F' \subseteq E$ which are of finite degree and Galois over the ground field F. Having equipped each of the finite groups $\mathrm{Gal}(F'/F)$ with the discrete topology, one obtains a corresponding projective limit topology on $G = \mathrm{Gal}(E/F)$, which is then called the *Krull topology* and clearly (either by definition or by the preceding proposition) turns G into a profinite group. Curiously, it has been shown ([Water2]) that every profinite group arises as the Galois group of some Galois extension. $\diamond$

2.17. By analogy with the preceding definition, a topological group which is locally compact, Hausdorff and totally disconnected will be called a *locally profinite group*. (For some authors, such groups are called *t.d. groups* instead, where clearly "t.d." is short for "totally disconnected".) The remainder of this subsection will be devoted to them, in preparation for the larger and larger role they will play later on.

2.18. *Observation.* On an imaginary scale ranking all locally compact groups by their connectedness properties, locally profinite groups are intuitively[7] at one extreme, with (connected) Lie groups at the other end of the spectrum. $\diamond$

2.19. *Remark.* The choice of wording 'locally profinite' might seem at odds with the usual meaning of the qualifier 'locally' in point-set topology (cf. also the discussion at the beginning of this section), but is also somewhat justified in retrospect by the following powerful result. $\diamond$

[7] Technically, a countable discrete group would be both locally profinite and a (zero-dimensional) Lie group, ruining the analogy.

2.20. THEOREM (van Dantzig). *For a Hausdorff topological group G, the following are equivalent:*

(i) G is a locally profinite group;

(ii) the identity element e in G has a neighbourhood basis consisting of compact open subgroups.

Sketch of proof. Fill in this proof! □

2.21. *Examples.*

(1) Clearly, all discrete groups and all profinite groups are locally profinite.

(2) More importantly, in Chapter II we will make the acquaintance of two infinite families of nondiscrete totally disconnected locally compact fields.

(3) A host of other examples will be available once we have looked at locally compact fields more systematically. In fact, we have mentioned that a group defined by analytic or polynomial equations over a nondiscrete locally compact field is canonically a locally compact group. We will see that, if the base field is totally disconnected (cf. 2.12), then so is the group, making it locally profinite. ◇

Finally, we conclude the section with a basic result concerning operations with locally profinite groups. Its proof is easy and is left as an exercise (E.11); in fact, the bulk is proved already in 2.9.

2.22. PROPOSITION.

(i) A closed subgroup of a locally profinite group is again locally profinite.

(ii) A finite product of locally profinite groups is again locally profinite.

(iii) The quotient of a locally profinite group by a closed normal subgroup is again locally profinite.

3 Review of measure theory

As was hinted at before, the main result of this chapter will be the existence and essential uniqueness, on every locally compact group, of a certain measure known as Haar measure. Thus, the next step is to recall some generalities on measures. We use the term "recall" because we expect our typical reader to have a nonnegligible background in mathematical analysis, to the extent that they have seen a construction of Lebesgue measure and the definitions of the Lebesgue integral and of Lebesgue spaces (so-called L^p spaces) fleshed out in previous courses; however, we have resolved to write this section as well as §6 in such a way as to benefit readers who might not fit this description, by giving what is essentially a soft introduction to measure and integration theory. (Readers who feel confident about their measure-theoretic background might therefore consider skipping ahead to §4.) Regrettably, a truly self-contained treatment is not feasible for the scope of this course: in some instances, it will be inevitable to refer to works such as [Rudin], [Folland2], [Cohn], [Halmos] or (for readers fluent in German) [Elst], which feature a far more systematic and comprehensive treatment.

3.1 Conceptual introduction and outer measures

3.1. From a conceptual viewpoint, measure theory is a way of formalizing a notion of "size" for subsets of a certain set. (More accurately, "size" ought rather to be understood in a geometric sense—such as quantifying an area or a volume—than in the already-familiar sense of cardinality.) The formalism of measure theory is especially fruitful insofar as it provides a rigorous basis both for a general theory of integration (as we will see below) and for the modern mathematical (axiomatic) approach to probability theory; in this light, one can perhaps start to appreciate why the automatic existence of Haar measure on any locally compact group is a noteworthy fact.

The intuitive concept of "size" is made rigorous by the mathematical notion of a *measure*. As a first step towards a formal definition of this notion, it is helpful to isolate some properties of "size" (of subsets of a fixed set) that its rigorous counterpart is expected to fulfill.

For instance, measures of sets should be given by nonnegative real numbers (or infinity), with the empty set being assigned measure 0. (We might express this by saying that such a measure is "*positive*".) Moreover, if we have an inclusion $A \subseteq B$ of (sub-)sets then naturally we expect the "smaller" set A to have measure less than or equal to that of B—i.e., the measure should be *monotonic* (or *monotone*). Finally, if a set can be written as a union (finite or at most countably infinite[8]) of a family of its subsets (possibly overlapping with each other), then clearly its measure should be less than or equal to the total sum of the measures of the sets which cover it; moreover, if the union is disjoint, then we should expect actual equality to hold. These last two conditions are known as *countable subadditivity* and *countable additivity*, respectively.

The first important realization at this point is that a positive, monotonic, countably subadditive function taking subsets of a fixed set X to nonnegative real numbers (possibly extended to include $+\infty$) *need not* be countably additive in general. In order to obtain a function with all four properties, one often has to give up the ambition of being able to measure *all* subsets of X and instead specify *which* subsets should be "*measurable*" and restrict to those. Thus, it could be said that the notion of "size" can be formalized in two different ways; however, in actuality, only one notion—the one with countable additivity built into the definition—is graced with the "true" title of a measure, whereas the other notion is considered ancillary. Nevertheless, since it is this latter notion which can be defined with less preparations, this is the one that we shall consider first.

3.2. By definition, an *outer measure* on a set X is a function $\mu^*\colon \mathcal{P}(X) \to [0,\infty]$, where $\mathcal{P}(X)$ denotes the power set of X, which satisfies the following properties:

(a) $\mu^*(\emptyset) = 0$;

(b) if A, B_1, B_2, ... are subsets of X such that $A \subseteq \bigcup_{i=1}^\infty B_i$, then $\mu^*(A) \leq \sum_{i=1}^\infty \mu^*(B_i)$.

Thus, in the language of the previous paragraph, outer measures are precisely those functions that are positive, monotonic and countably subadditive (but not necessarily countably additive).

3.3. *Examples.* As will become clear from the discussion in §3.3, examples of measures and of outer measures are, in a sense, interchangeable. (More precisely, any measure yields an outer measure by extension and any outer mea-

[8] Allowing for countable set operations turns out to be key, if perhaps not intuitive at first glance; for instance, it makes it possible to consider (increasing or decreasing) sequences of sets and to interpret countable (sub-)additivity as a certain form of continuity, cf. e.g. [Halmos, p. 38, Theorems D and E].

sure yields a measure by restriction.) Nonetheless, it is instructive to provide a concrete nontrivial example at this point.

To this end, consider the set $X = \mathbb{R}$, and let $\mathcal{H} \subset \mathcal{P}(X)$ denote the collection of bounded half-open intervals in $\mathbb{R}$ of the form $(a, b]$. For any $I = (a, b] \in \mathcal{H}$, we shall denote the *length* $b - a$ of the interval I by $\ell(I)$. Now, for $A \subseteq \mathbb{R}$ arbitrary, set

$$\lambda^*(A) \coloneqq \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : \begin{array}{c} (I_n)_{n\geq 1} \text{ is a countable} \\ \mathcal{H}\text{-cover of } A \end{array} \right\}. \tag{OUT.MEAS.}$$

(Of course, $(I_n)_{n\geq 1}$ being a countable $\mathcal{H}$-cover of A means that each I_n is in $\mathcal{H}$ and that $A \subseteq \bigcup_{n\geq 1} I_n$. Moreover, as usual, $\inf \emptyset = \inf \{\infty\} = \infty$.) Then it is readily checked (cf. also the remark below) that λ^* is an outer measure on $\mathbb{R}$ and that λ^* *extends* ℓ in the sense that $\lambda^*(I) = \ell(I)$ for all $I \in \mathcal{H}$. For reasons that will become clear soon, λ^* is known as *Lebesgue outer measure* on $\mathbb{R}$. $\diamond$

3.4. *Remark.*

(1) Observe that, in the above example, $\lambda^*(A)$ is determined by approximating A "from the outside" (or "from above") by countable covers of intervals. This might give some retroactive insight into the nomenclature "outer measure".

(2) The above example is merely a concrete instance of a much more general phenomenon: for any set X, *any* collection $\mathcal{H} \subseteq \mathcal{P}(X)$ containing $\emptyset$ and *any* function $\ell\colon \mathcal{H} \to [0, \infty]$ such that $\ell(\emptyset) = 0$, (OUT.MEAS.) defines an outer measure on X (s. [Folland2, Prop. 1.10]). A sufficient condition for the resulting outer measure to extend the original function ℓ is that the collection $\mathcal{H}$ be a so-called *semiring*, which means that

   (a) $\emptyset \in \mathcal{H}$,

   (b) $\mathcal{H}$ is closed under finite intersections, and

   (c) for A, $B \in \mathcal{H}$, the set-theoretic difference $A \setminus B$ can be written as a finite disjoint union of sets in $\mathcal{H}$,

   and that the function ℓ be positive and countably additive on $\mathcal{H}$.[9] It is

[9] For a proof of this statement, see e.g. [Elst, Kap. II, Satz 4.5a)–b)], or alternatively [Folland2, Prop. 1.13a]—in the latter source, the assertion is proved for an *algebra* rather than for a semiring, but the necessary modification is straightforward.

easily seen that both halves of this condition are satisfied in the above example. $\diamond$

3.2 Measurable spaces

3.5. As was hinted at in the introductory paragraph 3.1, countable additivity typically comes at the price of giving up measurability of *all* sets. In practice, this means that, unlike outer measures, a “proper” (countably additive) measure on a set X is not *a priori* defined on the entirety of the power set $\mathcal{P}(X)$, but only on a certain subset (which needs to be specified together with the measure). Accordingly, the question arises as to which collections of subsets of X—i.e., which subsets of $\mathcal{P}(X)$—qualify as potential domains for a measure.

The answer is encapsulated in the notion of a *measurable space*. By this, we mean a pair $(X, \mathcal{A})$ consisting of a set X and a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ of subsets of X with the following properties:

(a) $\emptyset$, $X \in \mathcal{A}$;

(b) $\mathcal{A}$ is closed under complementation: if $A \in \mathcal{A}$, then $X \setminus A \in \mathcal{A}$;

(c) $\mathcal{A}$ is closed under countable unions: if $A_1, A_2, \ldots \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

(That is, the collection $\mathcal{A}$ is a *σ-algebra*—please note that some authors use the term ‘σ-field’ instead.) The elements of $\mathcal{A}$ are called the *$\mathcal{A}$-measurable* (or simply *measurable*, if $\mathcal{A}$ is understood) subsets of X.

3.6. *Remark.* By combining (b) and (c) from the previous paragraph, we see that a σ-algebra $\mathcal{A}$ is automatically closed under countable intersections; taking into account (a) then allows us to conclude that $\mathcal{A}$ is also closed under finite unions and finite intersections, and hence (again by (b)) also set differences. In particular, every σ-algebra is a semiring as defined in 3.4. $\diamond$

3.7. *Examples.* Clearly, for any set X, the following are σ-algebras:

(a) the whole power set $\mathcal{P}(X)$;

(b) the trivial σ-algebra $\{\emptyset, X\}$;

(c) $\{\emptyset, A, X \setminus A, X\}$ whenever A is any (proper non-empty) subset of X.

Moreover, arbitrary intersections of σ-algebras are again σ-algebras. It then easily follows that, for any set X and any collection $\mathcal{E} \subseteq \mathcal{P}(X)$ of subsets of X, there exists a smallest σ-algebra which contains $\mathcal{E}$, denoted $\sigma(\mathcal{E})$ and called the σ-algebra *generated* by $\mathcal{E}$.

The most important application of this construction, at least for our purposes, is the following. If X is a topological space, then the natural choice of a σ-algebra on X is the one generated by the collection $\mathcal{O}$ of open sets, in symbols $\sigma(\mathcal{O})$. This σ-algebra is called the *Borel-σ-algebra of* X and will be denoted $\mathcal{B}_X$; its sets are called *Borel sets*. Note that, because $\mathcal{B}_X$ is closed under complementation, it also contains the collection of closed subsets of X (i.e.: closed sets are Borel) and is in fact generated by it. $\diamond$

3.8. *Caveat.* Apparently, some authors define the Borel-σ-algebra of a topological space as the σ-algebra generated by the collection of *compact* (rather than open) sets. In general, neither σ-algebra need be contained in the other, see E.12. On the other hand, it is important to note that, in a *Hausdorff* topological space, compact subsets are automatically closed and hence (by the previous example) Borel. $\diamond$

3.9. *Remark.* In the special case $X = \mathbb{R}$ (with the usual Euclidean topology), several systems of generators for the Borel-σ-algebra are known. In fact, it is easy to see that $\mathcal{B}_{\mathbb{R}}$ is generated by any of the following:

(1) the collection of open bounded intervals;

(2) the collection of closed bounded intervals;

(3) the collection $\mathcal{H}$ of bounded intervals of the form $(a, b]$, as in 3.3;

(4) the collection of bounded intervals of the form $[a, b)$;

(5) the collection of intervals of the form $(-\infty, b)$;

(6) the collection of intervals of the form $(-\infty, b]$;

and the list goes on (see e.g. [Folland2, Prop. 1.2]). This abundance of equivalent descriptions for $\mathcal{B}_{\mathbb{R}}$ is especially useful when checking measurability of functions (s. 6.24.(6) below). $\diamond$

3.3 Measures

3.10. We are now in a position to give the fundamental definition of this section. A *(positive) measure*[10] on a measurable space $(X, \mathcal{A})$ is a function $\mu \colon \mathcal{A} \to [0, \infty]$ which satisfies the following properties:

(a) $\mu(\emptyset) = 0$;

(b) if A_1, A_2, ... are elements of $\mathcal{A}$ such that $A_i \cap A_j = \emptyset$ whenever $i \neq j$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

(I.e., a measure on $(X, \mathcal{A})$ is a positive, countably additive function on $\mathcal{A}$.) A *measure space* is a triple $(X, \mathcal{A}, \mu)$ where $(X, \mathcal{A})$ is a measurable space and μ is a measure on $(X, \mathcal{A})$.

3.11. *Remarks on the definition.*

(1) A measure is clearly automatically *finitely additive* in the sense that the measure of a finite disjoint union of measurable sets is the sum of the individual measures.

(2) Because (by the arguments in 3.6) a σ-algebra $\mathcal{A}$ contains the set difference $B \setminus A$ whenever A, $B \in \mathcal{A}$, it readily follows that a measure on $\mathcal{A}$ is automatically monotonic on $\mathcal{A}$.
It should also be easy to see that every countable union $\bigcup_{i=1}^{\infty} A_i$ of $\mathcal{A}$-measurable sets can be rewritten as a countable *disjoint* union $\bigcup_{i=1}^{\infty} A_i'$, where $\mu(A_i') \leq \mu(A_i)$ for all $i \geq 1$. From this we infer that a measure on $\mathcal{A}$ is also automatically countably subadditive on $\mathcal{A}$.

(3) The above statements apply more generally to semirings; more precisely, a positive, countably additive function on a semiring (cf. 3.4) is automatically monotonic and countably subadditive on its domain. The arguments for this are easy modifications of the ones given above for σ-algebras.

(4) The definition of a measure given *supra* is at odds with the one used by some authors, most notably [Bourbaki]. For a discussion of how to reconcile these two perspectives, see 7.23.(3). ◇

[10] There also exist so-called *signed measures*, *complex measures*, ...; however, in this note, the word "measure", when used without qualifiers, will always stand for "positive measure".

3.12. *Examples.* On any measurable space $(X, \mathcal{A})$, we have the following trivial examples of measures:

(1) the *zero measure* (also known as the *trivial* measure), which sends every $A \in \mathcal{A}$ to 0.

(2) the function which sends $\emptyset$ to 0 and every non-empty $A \in \mathcal{A}$ to ∞.

A more interesting example is *counting measure*, which is obtained by setting $\mu(A)$ equal to the cardinality of A when A is finite and $\mu(A) = \infty$ else.

Another example of a very different flavour is given by *Dirac measures*. For every $x \in X$, the Dirac measure in x is the function $\mathcal{A} \to [0, \infty]$ defined as follows:

$$A \mapsto \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

(The reader eager to get some more practice with the newly-introduced notions is of course welcome to check that each of the above does indeed define a measure.)

Finally, most readers are probably aware of the existence of *Lebesgue measure*. One possible route to its construction will be reviewed at the end of this subsection, but the contents of the following sections will also allow us to construct Lebesgue measure in a different and independent way as Haar measure on the locally compact group $(\mathbb{R}, +)$. $\diamond$

3.13. Admittedly, the examples discussed in the previous paragraph might not be enough to convince a skeptic encountering measure theory for the first time of its usefulness. This has to do with the fact that many truly interesting examples of measures (including Haar measure) are obtained through a non-obvious procedure which involves outer measures. We will now shed some light on the core step of this procedure.

Let μ^* be an outer measure on a set X. A subset $A \subseteq X$ is called μ^*-*measurable (in the sense of Carathéodory)* if

$$\forall Q \subseteq X \colon \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \setminus A).$$

While the definition has been said to be "not in the least intuitive"[11] (but cf.

[11] This is taken from an often-quoted passage in E. Hewitt and K. Stromberg's book *Real and Abstract Analysis*.

also e.g. [Halmos, §11, p. 44] or [Folland2, p. 29] for some tentative motivation), it provides us with the following powerful result, which we state without proof (see e.g. [Folland2, Carathéodory's Theorem 1.11, p. 29]).

3.14. THEOREM. *Given an outer measure μ^* on a set X, the μ^*-measurable sets form a σ-algebra $\mathcal{M}(\mu^*) \subseteq \mathcal{P}(X)$, and the restriction of μ^* to this σ-algebra is countably additive, i.e., a measure on $(X, \mathcal{M}(\mu^*))$.*

3.15. One major application of this result is the "extension" of positive, countably additive functions to suitably large collection of sets starting from smaller ones. More verbosely, let X be a set, $\mathcal{H} \subseteq \mathcal{P}(X)$ be a semiring (cf. 3.4), and $\ell \colon \mathcal{H} \to [0, \infty]$ be positive and countably additive on $\mathcal{H}$—then, as was discussed in 3.4, the formula (OUT.MEAS.) yields an outer measure μ^* on X, and $\mu^*(A) = \ell(A)$ for all $A \in \mathcal{H}$. We now have the following assertion[12], which can be seen as a "coordinate result" to the preceding theorem: every set $A \in \mathcal{H}$ is μ^*-measurable in the sense of Carathéodory. (I.e., in the notation of the theorem, $\mathcal{H} \subseteq \mathcal{M}(\mu^*)$.) It follows that $\mathcal{M}(\mu^*)$ already contains the σ-algebra $\sigma(\mathcal{H})$ generated by $\mathcal{H}$. (This is simply by definition of the latter, s. 3.7.) But then the theorem—together with the fact that μ^* extends ℓ—guarantees that the restriction of μ^* to $\sigma(\mathcal{H})$ is an extension of ℓ to $\sigma(\mathcal{H})$. In other words, by this procedure we can obtain measures in the proper sense of the word (i.e., whose domain is a σ-algebra) even when the initial function ℓ was not given on enough sets to meet this requirement.

3.16. *Parenthetical.* In the preceding discussion, it is natural to ask "what happens" when $\mathcal{H}$ *is already a σ-algebra $\mathcal{A}$ to begin with* (and hence ℓ is already a measure μ). In this case, there is clearly no gain in extending $\ell = \mu$ to $\sigma(\mathcal{H}) = \sigma(\mathcal{A})$, since the latter is simply $\mathcal{A}$. Nevertheless $\mathcal{M}(\mu^*)$ is still a σ-algebra which contains $\mathcal{A}$ and $\mu^*|_{\mathcal{M}(\mu^*)}$ is still a measure which extends μ. So the domain of *any* given measure can potentially be enlarged. (However, [Folland2, Chapter 1, Exercise 20b, p. 32] shows that repeating this procedure will not result in an even larger extension; in fact, it can be seen that $\mathcal{M}(\mu^*)$ is already the largest σ-algebra which contains $\mathcal{A}$ and on which μ^* is a measure.) The extension $\mu^*|_{\mathcal{M}(\mu^*)}$ incidentally has a more concrete description as the

[12] Cf. e.g. [Elst, Kap. II, Satz 4.5a)], or alternatively [Folland2, Prop. 1.13b], with the same caveat as in footnote 9.

saturation of the *completion* of μ, see [Folland2, Chapter 1, Exercise 22, p. 32]. $\diamond$

3.17. *Examples.* To conclude this section, we put together the insights of the last few paragraphs and come back to Lebesgue measure, arguably the most important example of a measure and yet the only one that, in accordance with our decision to "start from scratch", we have not been able to say much about so far.

Thus, let $X = \mathbb{R}$ and resume the notations ℓ, $\mathcal{H}$ and λ^* (Lebesgue outer measure) from 3.3. Applying 3.14 to λ^* yields a measure space $(X, \mathcal{M}(\lambda^*), \lambda^*|_{\mathcal{M}(\lambda^*)})$, and by the preceding remark, together with 3.9.(3), the σ-algebra $\mathcal{M}(\lambda^*)$ contains the Borel-σ-algebra $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{H})$ of $\mathbb{R}$. One calls the elements of $\mathcal{M}(\lambda^*)$ the *Lebesgue-measurable subsets* of $\mathbb{R}$ and the restriction $\lambda^*|_{\mathcal{M}(\lambda^*)}$ *Lebesgue measure* on $\mathbb{R}$. There are several helpful characterizations of Lebesgue measurability for a subset $E \subseteq \mathbb{R}$, see e.g. [Elst, Kap. II, Satz 7.4] and [Folland2, Thm. 1.19].

Lebesgue measure is also the *unique* extension of our initial length function ℓ to a measure on $\sigma(\mathcal{H}) = \mathcal{B}_{\mathbb{R}}$. This follows from the following celebrated theorem of Carathéodory ([Elst, Kap. II, Kor. 5.7], or [Folland2, Theorem 1.14] with the same caveat as in footnote 9)—together with the simple observation that the real line can be written as a countable union of intervals of finite length.

3.18. Proposition. *Let $(X, \mathcal{A})$ be a measurable space, $\mathcal{A} = \sigma(\mathcal{H})$ for a semiring $\mathcal{H}$, and $\ell\colon \mathcal{H} \to [0, \infty]$ be a positive, countably additive function on $\mathcal{H}$. Suppose that*

($*$) *X can be written as a countable union $\bigcup_{n \geq 1} E_n$ where each E_n belongs to $\mathcal{H}$ and $\ell(E_n) < \infty$ for all $n \geq 1$.*

Then ℓ admits a unique *extension to a measure on $\sigma(\mathcal{H}) = \mathcal{A}$.*

(One often says that ℓ is *σ-finite*, or that X is *σ-finite for ℓ*, if condition (*) is met.)

Finally, the following properties of Lebesgue measure (denoted λ in what follows for the sake of conciseness) are easily established.

(1) λ is *translation-invariant* in the sense that, if $E \subseteq \mathbb{R}$ is Lebesgue-measurable, x is a real number and $x + E$ denotes the (left) translate of

E by x (see 1.6 for the notation and recall that $\mathbb{R}$ is denoted additively), then $x + E$ is again Lebesgue-measurable with $\lambda(x + E) = \lambda(E)$.
(In fact, this is immediate from the fact that ℓ has the same property on $\mathcal{H}$ and from (OUT.MEAS.).)

(2) (As was already noted above,) Every Borel set is Lebesgue-measurable; moreover, any bounded Borel set (and in particular every compact set) has finite measure, i.e., it holds true that $\lambda(B) < \infty$ whenever $B \in \mathcal{B}_{\mathbb{R}}$ is bounded.

These observations provide us with a perfect segue into the next section. $\diamond$

4 Haar measure

In the last section, we devoted a lot of attention to Lebesgue measure on the real line, recalling how it can be constructed and some of its fundamental properties. At the end of the section, we made two easy observations which will become especially important to us in the immediate future: we noted that Lebesgue measure is translation-invariant and that it is finite on bounded measurable sets and hence in particular on compact sets.

Clearly these are properties of measures which can be investigated, and indeed defined, much more generally than just for the real line. As a matter of fact, given a measure space $(X, \mathcal{A}, \mu)$, translation-invariance as formulated in 3.17 can be made sense of whenever the underlying set X is equipped with a group structure[13], and the condition of finiteness on compact sets only requires a topological structure on X with the property that compact sets are measurable. It might then be natural to want to look into these properties, both separately and jointly, i.e. when the underlying set is, say, a topological group. In the course of §4.2 it will become apparent that there is much to be gained from restricting to locally compact Hausdorff topologies, which we introduced in §2; accordingly, the entire section will ultimately "converge" to the pivotal definition of Haar measure on a locally compact group, which will be the main focus of the rest of the chapter.

[13] Or, more generally, when X is *acted on* by a group, cf. footnote 14 below.

For bibliographical remarks, we refer to the individual subsections, esp. remarks 4.2.(1), 4.9, 4.14 and the introductory paragraph of §4.3.

4.1 Invariant measures

4.1. Let G be a group, $\mathcal{A} \subseteq \mathcal{P}(G)$ be a σ-algebra of subsets of G. Suppose that $\mathcal{A}$ is *left-invariant* in the sense that, for every $\mathcal{A}$-measurable subset $E \subseteq G$ and every $g \in G$, the left translate gE of E by g is again $\mathcal{A}$-measurable. A measure μ on the measurable space $(G, \mathcal{A})$ will then be said to be *left-invariant* if $\mu(gE) = \mu(E)$ for all $E \in \mathcal{A}$ and all $g \in G$. (Naturally, there is an entirely analogous definition for the notion of *right-invariance*, obtained by swapping out gE for Eg throughout.)

4.2. *Remarks on the definition.*

(1) It is perhaps appropriate to warn the reader that the above definition cannot be found word for word in any of the major references for these notes; more precisely, the assumption it features on the σ-algebra $\mathcal{A}$ (the domain of the prospective invariant measure μ) is not commonly found elsewhere, even though it is clearly necessary to ensure that the expression $\mu(gE)$ [or $\mu(Eg)$ for right-invariance] is always meaningful (or "well-defined") when E is $\mathcal{A}$-measurable and g is an element of G.

This discrepancy can easily be explained as follows. (We shall only address left-invariance in the following paragraphs, but everything applies *mutatis mutandis* to the right-invariant counterparts as well.) In most[14] of the sources for this note, the definition of a left-invariant measure appears only in connection with Haar measure; in this particular context, G is given as a *topological* group and the σ-algebra $\mathcal{A}$ is typically taken to be the Borel-σ-algebra $\mathcal{B}_G$ of G (as introduced in 3.7). But then the assumption on $\mathcal{A}$ is automatically verified (this is E.15) and so there is no need to introduce a general notion of a "left-invariant σ-algebra".

[14]Bourbaki's definition [Bourbaki, Chapter VII, §1, no. 1, Definition 1.a)] is an exception to this statement; in fact, it is, in some respects, even more general than ours, since it applies to measures on spaces which are *acted on* by a group, rather than just measures on groups. On the other hand, it was already mentioned in 3.11 that measures are defined differently in Bourbaki's books (in particular, they are only defined on locally compact Hausdorff topological spaces), cf. also 7.23.(3).

(To our knowledge, this terminology is only used in one of the sources, namely in the last few paragraphs of [Joys, Chapter 7].)

(2) The assumption of left-invariance [right-invariance] of a σ-algebra $\mathcal{A}$ can also be equivalently phrased as the requirement that the left translation map l_g [the right translation map r^g] be a *measurable map* from $(G, \mathcal{A})$ to $(G, \mathcal{A})$ for every $g \in G$. (For more on left and right translation maps, see 1.6; for a definition of the notion of a measurable map, see e.g. 6.19 below.) $\diamond$

Before moving on to examples, we make two easy observations about passing from left-invariant σ-algebras and measures to right-invariant ones.

4.3. Let G be an abelian group. Then

(i) a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(G)$ is left-invariant if and only if it is right-invariant,

and if either holds, then

(ii) a measure μ on $(G, \mathcal{A})$ is left-invariant if and only if it is right-invariant.

(This is simply because, if G is abelian, then $gE = Eg$ for any $E \in \mathcal{A}$ and any $g \in G$.)

4.4. Now let G be *any* group (not necessarily abelian). We introduce two *ad hoc* items of notation: for a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(G)$, let $\mathcal{A}^\vee \subseteq \mathcal{P}(G)$ denote the set

$$\mathcal{A}^\vee \coloneqq \{E^{-1} : E \in \mathcal{A}\}$$

(s. 1.9 for the notation), and for a measure μ on $(G, \mathcal{A})$, let $\mu^\vee$ denote the function $\mathcal{A}^\vee \to [0, \infty]$ given by

$$\mu^\vee(E) \coloneqq \mu(E^{-1}).$$

Then the following are easily checked (E.16):

(i) For every σ-algebra $\mathcal{A} \subseteq \mathcal{P}(G)$, the set $\mathcal{A}^\vee$ is again a σ-algebra, and $(\mathcal{A}^\vee)^\vee = \mathcal{A}$.

(ii) A σ-algebra $\mathcal{A} \subseteq \mathcal{P}(G)$ is left-invariant if and only if $\mathcal{A}^\vee$ is right-invariant, and viceversa.

(iii) If $\mathcal{A} \subseteq \mathcal{P}(G)$ is a σ-algebra and μ is a measure on $(G, \mathcal{A})$, then $\mu^\vee$ is a measure on $(G, \mathcal{A}^\vee)$, and $(\mu^\vee)^\vee = \mu$.

(iv) If $\mathcal{A} \subseteq \mathcal{P}(G)$ is a left-invariant [right-invariant] σ-algebra, then a measure μ on $(G, \mathcal{A})$ is left-invariant [right-invariant] if and only if the corresponding measure $\mu^\vee$ on $(G, \mathcal{A}^\vee)$ is right-invariant [left-invariant].

4.5. *Examples.*

(1) Examples of left- or right-invariant σ-algebras are easily found: for instance, for any group G, both the largest and the smallest σ-algebra on G are left- and right-invariant. A nontrivial example is given by the Borel-σ-algebra on any topological group G, cf. E.15, or by the Lebesgue-σ-algebra on $\mathbb{R}$ as discussed in 3.17.

(2) Let G be any group and let $\mathcal{A}$ denote the power set $\mathcal{P}(G)$ of G. Then the following examples of measures we encountered in 3.12 are both left- and right-invariant on $(G, \mathcal{A})$:
   (a) the zero measure;
   (b) the measure μ defined by $\mu(\emptyset) = 0$ and $\mu(E) = \infty$ for every non-empty $E \in \mathcal{A}$;
   (c) counting measure.

(3) Clearly any positive scalar multiple of a (left- or right-)invariant measure is again invariant. In more detail, whenever μ is a left-invariant measure on $(G, \mathcal{A})$ and c is a positive real number, the map $\mathcal{A} \to [0, \infty]$ given by $A \mapsto c\mu(A)$ is again a left-invariant measure on $\mathcal{A}$. (And of course there's an entirely analogous statement for right-invariant measures.) $\diamond$

4.6. The above examples show that invariance is not as restrictive a condition as one might think, and that on any group there exist several invariant measures. It is then difficult to argue, on a purely algebraic basis, why Lebesgue measure is more suited to the real line than, say, counting measure; clearly geometrical/topological considerations play a more important role in picking out the "proper" invariant measure in this context. These considerations will be explored in the next subsection.

4.2 Radon measures

This subsection is devoted to investigating measures on topological spaces, especially those which can be considered to be “well-behaved” with respect to the given topological structure. The ultimate product of this discussion will be the introduction of *Radon measures*, a notion of such importance that we have taken plenty of care in giving a precise definition (4.13) and contrasting it with other definitions found in the literature (4.14). In an attempt to provide motivation and justification for our particular choice of definition, we shall present a general result (4.8) which will also find important applications in later sections.

4.7. Drawing inspiration, once again, from Lebesgue measure on the real line (see 3.17 for more details), there are two very natural conditions to impose on measures on a topological space X, namely:

(a) that it should be defined “at least” on all open and all closed subsets[15] of X, and

(b) that the measure of compact subsets be finite—assuming, of course, that compact subsets of X are indeed measurable.

While these conditions might immediately appear reasonable, it is by far not immediate, given a (Hausdorff) topological space X, to produce a measure on it which satisfies both properties—except of course for the zero measure. In fact, putting aside our motivating example of Lebesgue measure on the real line (whose very existence is, as we know, anything but trivial), there is only one other nontrivial example we can offer right away: if the space X is discrete, then it is easily checked (this is E.17) that counting measure has the required properties. (In contrast, counting measure clearly fails to satisfy (b) on, say, $X = \mathbb{R}$.)

In light of our discussion of extensions of measures in §3, the following tentative strategy might seem appealing: first assign, to each compact subset of the given space X, a nonnegative (finite!) real number; then, extend the assignment to the Borel-σ-algebra $\mathcal{B}_X$ by the standard technique explored in 3.15. The resulting measure would then indeed have both of the desired properties (a) and

[15] Equivalently, this can be phrased as the condition that the σ-algebra of measurable sets should contain the *Borel-σ-algebra* $\mathcal{B}_X$ of X, cf. 3.7.

(b). There are, however, some obvious issues: firstly, the compact subsets of X do not necessarily generate the Borel-σ-algebra (cf. 3.8), and secondly, they do not even form a semiring in general! (This is checked immediately even for $X = \mathbb{R}$.) From this we deduce that obtaining a measure on all Borel sets from just the compact ones, if at all possible, would need a different kind of technique, and that the resulting measure might not be a true *extension* in the sense of 3.15.

Fortunately for us, there is a partial result which will be sufficient for our purposes. Loosely speaking, it says that, if the space X has "sufficiently many" compact subsets, then there is indeed a way to obtain a measure on all Borel sets from just the compact ones, and that the thus-obtained measure will even have some additional compatibility conditions with the given topology. More precisely, the condition of 'having sufficiently many compact subsets' is made rigorous by the property of being a locally compact Hausdorff space—which will thus get its first real chance to shine since its introduction in §2—and the result reads as follows.

4.8. Proposition. *Let X be a locally compact Hausdorff space, and let $\mathcal{K} = \mathcal{K}(X)$ denote the collection of compact subsets of X. Further, let an assignment $h\colon \mathcal{K} \to \mathbb{R}_{\geq 0}$ be given with the following properties:*

- *(h1) h is* monotonic *on $\mathcal{K}$, i.e., $h(K) \leq h(K')$ whenever K, $K' \in \mathcal{K}$ satisfy $K \subseteq K'$;*
- *(h2) h is* finitely subadditive *on $\mathcal{K}$, i.e., $h(K \cup K') \leq h(K) + h(K')$ for any K, $K' \in \mathcal{K}$;*
- *(h3) h is* finitely additive *on $\mathcal{K}$, i.e., $h(K \cup K') = h(K) + h(K')$ whenever K, $K' \in \mathcal{K}$ are disjoint.*

Now define $\mu^*\colon \mathcal{P}(X) \to [0, \infty]$ as follows: for any $U \subseteq X$ open, set

$$\mu^*(U) := \sup\{h(K) : K \subseteq U, K \text{ compact}\}, \tag{1}$$

and for $E \subseteq X$ arbitrary, set

$$\mu^*(E) := \inf\{\mu^*(U) : U \supseteq E, U \text{ open}\}. \tag{2}$$

Then:

(i) μ^* *is an outer measure.*

Now let $\mathcal{A}$ denote the σ-algebra of μ^*-measurable sets of X. (See 3.13 and 3.14.) By 3.14 (where $\mathcal{A}$ was denoted by $\mathcal{M}(\mu^*)$), the restriction $\mu := \mu^*|_{\mathcal{A}}$ is a measure on $(X, \mathcal{A})$. We have:

(ii) *$\mathcal{A}$ contains the Borel-σ-algebra $\mathcal{B}_X$ of X.*

(iii) *$\mu(K) < \infty$ whenever $K \subseteq X$ is compact.*

Moreover,

(iv) *$\mu(K) \geq h(K)$ for every $K \in \mathcal{K}$.*

(v) *μ is* inner regular on open subsets, *i.e.,*

$$\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ compact}\}, \qquad U \subseteq X \text{ open}.$$

(vi) *μ is* outer regular *(on all measurable subsets), i.e.,*

$$\mu(E) = \inf\{\mu(U) : U \supseteq E, U \text{ open}\}, \qquad E \in \mathcal{A}.$$

The proof will be given immediately below, but first, a few observations are in order.

4.9. *Remarks on the proposition.*

(1) Note that the proposition does *not* state that μ^* agrees with h on $\mathcal{K}(X)$; indeed, μ is *not* necessarily an *extension* of h to $\mathcal{A} \supseteq \mathcal{B}_X \supseteq \mathcal{K}(X)$ in general. (Cf. 3.8 for the last inclusion.) See [Elst, Kapitel VIII, Fortsetzungssatz 2.4] for an amended version of this result in which the measure μ is indeed an extension of h, provided h satisfies a certain "tightness" property. Another important case in which μ^* is an extension of h is addressed in E.19 (using terminology to be introduced in 4.13 below).

(2) As a bibliographical side note, it should be noted that the above result is not contained in any of the references as such; instead, its formulation and proof were obtained by taking arguments which are common to the proofs (as given e.g. in [Cohn] or [Folland2]) of two fundamental results: existence of Haar measure on the one hand and the Riesz Representation Theorem on the other. The same arguments can be found in [Gleason], which largely follows [Cohn]. (However, *beware* that in [Gleason], it is claimed that μ^* is an extension of h, which, as we have argued above, is *not true* in general!)

(3) Observe that, since $\mathcal{K}(X)$ is not a semiring in general, one cannot infer finite subadditivity from finite additivity and has to assume both properties separately. $\diamond$

Proof of 4.8. First of all, we invite the reader to verify the details of why the assignment $\mu^*\colon \mathcal{P}(X) \to [0,\infty]$ is well-defined.

To prove the first claim (i), we have to show that $\mu^*(\emptyset) = 0$ and that μ^* is monotonic and countably subadditive (cf. 3.2). The first half is straightforward since the empty set is both compact and open and $h(\emptyset) < \infty$ can only be zero by finite additivity. Monotonicity is also readily checked using monotonicity of h, the equations (1)–(2) and basic properties of suprema and infima.

As for countable subadditivity, let $\{E_n\}_{n\geq 1}$ be an arbitrary countable collection of subsets of X. We have to show that $\mu^*(\bigcup_{n\geq 1} E_n) \leq \sum_{n\geq 1} \mu^*(E_n)$. As a stepping stone, we shall first assume that *all E_n are open in X*, and accordingly denote them by U_n instead. Then $\bigcup_{n\geq 1} U_n =: U$ is again open in X, so $\mu^*(U)$—the left-hand side of the inequality we want to prove—is determined by (1). Accordingly, it suffices to prove that, whenever $K \subseteq U$ is compact, $h(K)$ does not exceed $\sum_{n\geq 1} \mu^*(U_n)$. To this end, we shall use the following purely topological lemma ([Cohn, Lemma 7.1.9]):

4.10. Lemma. *Let X be a Hausdorff space and let K be a compact subset of X. Suppose that K is contained in the union $\bigcup_{n=1}^N U_n$ of finitely many open subsets $U_n \subseteq X$. Then there exist compact subsets $K_n \subseteq U_n$, $n = 1,\ldots,N$, such that $K = \bigcup_{n=1}^N K_n$.*

Going back to the proof of the proposition, let $\{U_n\}_{n\geq 1}$ be as above and K be a compact subset of X contained in the union $U = \bigcup_{n\geq 1} U_n$. Because K is compact and $\{U_n\}_{n\geq 1}$ is an open cover of K, we deduce that there exists an index N such that $K \subseteq \bigcup_{n=1}^N U_n$. We can then immediately apply the lemma; keeping the notations introduced in its statement, we obtain

$$h(K) = h\left(\bigcup_{n=1}^N K_n\right) \leq \sum_{n=1}^N h(K_n) \leq \sum_{n=1}^N \mu^*(U_n) \leq \sum_{n=1}^\infty \mu^*(U_n)$$

(where, in the first, second, and third inequality, we used: finite subadditivity of h, (1), and nonnegativity of μ^*, respectively).

Now we move on to the general case of (i), i.e., the countable collection $\{E_n\}_{n\geq 1}$ is arbitrary. Recall that we have to show that $\mu^*(\bigcup E_n) \leq \sum \mu^*(E_n)$; since the inequality is trivially true if the right-hand side is ∞, we can restrict to the case that it is finite. It is then sufficient to show that, for every $\varepsilon > 0$, the inequality $\mu^*(\bigcup E_n) \leq \sum \mu^*(E_n) + \varepsilon$ holds true.

Thus, let $\varepsilon > 0$ be fixed but arbitrary. By (2), we can pick, for each $n \geq 1$, an open set $U_n \supseteq E_n$ such that $\mu^*(U_n) \leq \mu^*(E_n) + \frac{\varepsilon}{2^n}$. Then, using the special case we have already proved above, we find that

$$\mu^*\left(\bigcup_{n\geq 1} E_n\right) \leq \mu^*\left(\bigcup_{n\geq 1} U_n\right) \leq \sum_{n\geq 1} \mu^*(U_n) \leq \sum_{n\geq 1} \mu^*(E_n) + \varepsilon,$$

as desired.

To prove the second claim (ii), it suffices to prove that all open subsets of X are μ^*-measurable; in other words, we need to show that, if $U \subseteq X$ is open and $Q \subseteq X$ is arbitrary, then $\mu^*(Q) = \mu^*(Q \cap U) + \mu^*(Q \setminus U)$. Observe that, in fact, the inequality "$\leq$" is automatically true by subadditivity of μ^* (because Q is covered by $Q \cap U$ and $Q \setminus U$), so we only need to show "$\geq$". Hence, we can restrict to the case $\mu^*(Q) < \infty$, because the inequality is trivially satisfied if $\mu^*(Q) = \infty$. Under the assumption of finiteness of $\mu^*(Q)$ we can then further reduce the problem to showing that

$$\mu^*(Q) + \varepsilon \geq \mu^*(Q \cap U) + \mu^*(Q \setminus U) \tag{$\star$}$$

for every $\varepsilon > 0$.

Thus, let U and Q be as above, and let $\varepsilon > 0$ be fixed but arbitrary. By (2), we can pick an open $V \supseteq Q$ such that $\mu^*(V) \leq \mu^*(Q) + \frac{\varepsilon}{3}$; we can then ignore Q from now on and focus on showing that

$$\mu^*(V) + \frac{2}{3}\varepsilon \geq \mu^*(V \cap U) + \mu^*(V \setminus U), \tag{$\dagger$}$$

from which ($\star$) will immediately follow.

The advantage of replacing Q by an open set is that now $V \cap U$ is open in X, so we can express $\mu^*(V \cap U)$ via (1) and accordingly find a compact set $K \subseteq V \cap U$ such that $h(K) \geq \mu^*(V \cap U) - \frac{\varepsilon}{3}$. There's more: because $K \subseteq U$, and K is closed in X, we have that $V \setminus K$ is an open superset of

$V \setminus U$, therefore, again by (1), we can find a compact set $L \subseteq V \setminus K$ such that $h(L) \geq \mu^*(V \setminus K) - \frac{\varepsilon}{3}$. By construction, K and L are even disjoint, hence (by finite additivity of h)

$$\begin{aligned} h(K \cup L) = h(K) + h(L) &\geq \mu^*(V \cap U) + \mu^*(V \setminus K) - \frac{2}{3}\varepsilon \\ &\geq \mu^*(V \cap U) + \mu^*(V \setminus U) - \frac{2}{3}\varepsilon \end{aligned} \tag{$\spadesuit$}$$

(the last "$\geq$" uses monotonicity of μ^*). On the other hand, $K \cup L$ is clearly a compact subset of $(V \cap U) \cup (V \setminus K) \subseteq V$, so (1) yields $\mu^*(V) \geq h(K \cup L)$. This latter inequality together with ($\spadesuit$) implies (†) and hence concludes the proof of (ii).

As a consequence of (ii), *we can henceforth write* $\mu(B)$ *instead of* $\mu^*(B)$ *whenever* B *is Borel* and hence in particular whenever B is, say, open, closed, or compact (cf. 3.8); clearly the two quantities are equal (as extended real numbers) since μ is simply the restriction of μ^*. In particular this observation retroactively justifies the use of the expression $\mu(K)$ in the statement of (iii).

We now tackle the third claim (iii). For this we need an auxiliary topological result ([Cohn, Prop. 7.1.3]):

4.11. Lemma. *In a locally compact Hausdorff space, every compact subset is contained in a relatively compact open subset.*

(Recall that a *relatively compact* subspace is one whose closure is compact.)

Thus, let K be an arbitrary compact subset of X; we want to show that $\mu(K) < \infty$. To that end, pick an open, relatively compact set $U \supseteq K$ (this is possible by the above lemma). Because $\overline{U}$ is compact, monotonicity of h implies that $h(K') \leq h(\overline{U})$ for any compact set $K' \subseteq U$; by (1), it follows that $\mu(U) \leq h(\overline{U})$. On the other hand, $K \subseteq U$, so $\mu(K) \leq \mu(U)$ by monotonicity of the measure μ. In conclusion, we have shown that $\mu(K) \leq \mu(U) \leq h(\overline{U}) < \infty$, as desired.

We can now devote our attention to (iv)–(vi), but these are straightforward in comparison to the previous claims. To see why (iv) holds, let K be a compact subset of X; then $h(K) \leq \mu(U)$ for any $U \supseteq K$, by (1), and hence also

$\inf\{\mu(U) : U \supseteq K \text{ open}\} \geq h(K)$. But, by (2), the left-hand side is precisely $\mu(K)$, thus proving the claim.

As for (v), fix an open subset U and observe that, by monotonicity of μ joint with (iv), $\mu(K)$ is "squeezed" between $h(K)$ and $\mu(U)$ whenever $K \subseteq U$ is compact. The claim is then immediate by (1).

Finally, (vi) is simply a reformulation of (2) since μ agrees with μ^* on all sets appearing in the equality. □

4.12. The regularity conditions appearing in claims (v)–(vi) of the above proposition are extremely useful and can be seen as "natural" requirements of compatibility with the topology at hand just as much as (ii) or (iii) (which are just (a)–(b) from 4.7) can.[16] It is a standard fact (s. e.g. [Folland2, Theorem 1.18]) that Lebesgue measure λ on $X = \mathbb{R}$ has both properties—in fact, λ is even inner regular on *all* measurable sets, i.e., the equality from 4.8.(v) holds with the open set U replaced by any Lebesgue-measurable set $E \subseteq \mathbb{R}$ (and μ replaced by λ). Moreover, the same can easily be checked to hold true for counting measure on any discrete space (this is E.18).

It is then natural to ask whether inner regularity on open sets, possibly combined with some of the other properties listed in the proposition, is already enough to imply inner regularity on all measurable (or at least Borel) sets. As it turns out ([Rudin, Chapter 2, Def. 2.15]), the answer is 'no' in general, and "this flaw is in the nature of things". It follows that, if some form of inner regularity is desired in future discussions then it might be better to stick to the less restrictive condition, and in fact this is precisely what we do in the following central definition.

4.13. Let X be a LCH space. In this note, a measure μ on $(X, \mathcal{B}_X)$ will be called a *Radon measure on* X if it is finite on compact subsets, inner regular on open sets and outer regular on Borel sets. (I.e., if parts (iii), (v) and (vi) from 4.8 hold, with $\mathcal{A}$ replaced by $\mathcal{B}_X$ in (vi).)

4.14. *Remarks on terminology.*

[16] Interestingly, we shall see in 7.26 below that if the underlying space X is "sufficiently well-behaved", then the regularity properties (v)–(vi) are automatically satisfied for any measure on $(X, \mathcal{B}_X)$ which satisfies (iii).

(1) The definition given above is the same as in [Cohn] or [Folland2] and seems (to the authors) to be the one most suitable one to our ("Haar-measure-related") needs. In fact, it is precisely the one which guarantees a neat existence and uniqueness statement in the upcoming Riesz Representation Theorem (and hence in the proof of uniqueness of Haar measure). For an overview of different conventions on the meaning of the phrases 'Radon measure' and 'Borel measure' (the latter of which we will judiciously steer away from in this note), see e.g. MO109505.

(2) Observe that, in our definition, a Radon measure is defined *only* on the Borel sets. Again, this is to ensure "clean" uniqueness statements later on, but it is by no means a serious restriction: it is not difficult to enlarge the domain of a Radon measure whenever doing so is called for, s. also E.19.

(3) In this note, we will only consider Radon measures on spaces which are both locally compact and Hausdorff, but it is possible to give a similar definition for general Hausdorff spaces. This can be done e.g. by changing the condition of finiteness on compact subsets to *local finiteness*, which means that every point has a neighbourhood of finite measure. Clearly a Radon measure on a LCH space is locally finite, because compact neighbourhoods exist around every point and they have finite measure. Conversely, it is easily seen that, on *any* Hausdorff space, local finiteness implies finiteness on compact subsets (this is E.20). It is then understandable why some authors use the phrase 'local finiteness' to refer to the property we have called 'finiteness on compact subsets', but we shall not adhere to this convention. $\diamond$

We have already remarked in 4.12 above that a Radon measure (as we have just defined it) on a LCH space X need not be inner regular on all (measurable) subsets of X. Nevertheless, the defining properties are strong enough to yield inner regularity on all subsets of finite measure and more generally on all σ-*finite* subsets of X (cf. 3.18 for the definition of σ-finiteness). More explicitly,

4.15. Proposition. *Let μ be a Radon measure on a LCH space X. Then, whenever $B \in \mathcal{B}_X$ is a countable union of Borel sets B_n, $n \geq 1$, with $\mu(B_n) <$*

∞ *for all* $n \geq 1$, *it holds that*

$$\mu(B) = \inf\{\mu(K) : K \subseteq B \text{ compact}\}.$$

(This is [Folland2, Prop. 7.5].)

4.16. In particular, suppose that X is *σ-compact*, i.e. that X can be written as a countable union of compact subsets. Then, for any Radon measure μ on X, the measure space $(X, \mathcal{B}_X, \mu)$ is σ-finite. Since (measurable) subsets of σ-finite Borel sets are again σ-finite, we may conclude that a Radon measure μ on a σ-compact LCH space X is *always* inner regular on *all* Borel sets.

To conclude the subsection, we propose what is essentially a concise reformulation of the important (if technical) result 4.8 in light of our latest definition:

4.17. COROLLARY. *Let X be a LCH space. Then every h as in the statement of 4.8 yields a Radon measure μ on X via*

$$\mu := \mu^*|_{\mathcal{B}_X},$$

where: μ^* is as in the statement of 4.8, $\mathcal{B}_X$ denotes as usual the Borel-σ-algebra of X and the restriction is meaningful by 4.8.(ii).

4.3 Haar measures

If we agree that Radon measures are the "right" class of measures on (certain) topological spaces, then we are ready to turn our attention to measures on topological *groups* and synthesize the efforts of the previous two subsections into the following pivotal definition (cf. e.g. [Folland2, Chapter 11, p. 341]).

4.18. A *left Haar measure* on a locally compact group is a nonzero left-invariant Radon measure on that group. Similarly, a *right Haar measure* is a nonzero right-invariant Radon measure. A Radon measure which is simultaneously both a left and a right Haar measure may be termed a *bi-invariant Haar measure*.

4.19. *Examples.* It follows from what was said throughout this section so far that Lebesgue measure—more precisely, the restriction of Lebesgue measure to the Borel-σ-algebra $\mathcal{B}_{\mathbb{R}}$—is both a left and a right Haar measure on the locally compact abelian group $(\mathbb{R}, +)$. Moreover, putting together E.17, E.18 and 4.5 shows that, on any discrete group, counting measure is both a left and a right Haar measure. $\diamond$

4.20. *Remark.*

(1) By 4.3, any left Haar measure on a locally compact *abelian* group is also a right Haar measure, and viceversa. Moreover, on any (not necessarily abelian) locally compact group, any given left Haar measure μ can be used to obtain a right Haar measure $\mu^{\vee}$ and viceversa, as explained in 4.4 (cf. also E.21!); as a consequence, it is perfectly acceptable to focus on just, say, left Haar measures as far as existence and uniqueness statements are concerned.

(2) On the topic of uniqueness, it is important to make the (easy) observation that any positive scalar multiple (as in 4.5.(3)) of a left [right] Haar measure is again a left [right] Haar measure on that same group. Thus, *true* uniqueness of Haar measure is unattainable with our definition; the actual statement we may (and shall) aim for is that any two left [right] Haar measures must be *proportional*, i.e. a positive scalar multiple of one another. This will be proved in 7.1. $\diamond$

We defer further examples of Haar measures to later sections, and conclude the section by mentioning a few general properties of Haar measures instead.

4.21. Proposition. *Let G be a locally compact group.*

(i) If $U \subseteq G$ is a Borel set with non-empty interior, then $\mu(U)$ is strictly positive for any left [right] Haar measure μ on G.

(ii) G is discrete if and only if, for any (hence: every) left [right] Haar measure μ on G, the trivial subgroup has strictly positive measure, i.e., $\mu(\{e\}) > 0$.

(iii) G is compact if and only if, for any (hence: every) left [right] Haar measure μ on G, the whole group has finite measure, i.e., $\mu(G) < \infty$.

Proof. (We will only prove the claims for left Haar measures.)

To prove the first claim (i), suppose that G has a Borel subset U with non-empty interior such that $\mu(U) = 0$ for some left Haar measure μ on G. We shall show that μ must then be identically zero, contradicting the assumption that μ is a Haar measure.

First, upon replacing U by its interior U° we may—and shall—assume that U is actually open. (This is just to simplify the argument.) We then consider an arbitrary compact set $K \subseteq G$. By the results of §1, the union $\bigcup_{g\in G} gU$ of all left translates of U is the entire space G; in particular, the family of all these translates is a cover of K, and so, by compactness of K, there exist finitely many elements of G, say $g_1, \ldots, g_N$, such that $K \subseteq \bigcup_{n=1}^N g_n U$. (A similar argument was already seen in 1.18.(iv).) By the properties of the measure μ, we deduce that $\mu(K) \leq \sum_{n=1}^N \mu(g_n U) = \sum_{n=1}^N \mu(U) = 0$, hence equality must hold; since K was arbitrary, this means that μ is zero on all compact sets. But then, by inner regularity, we can only have $\mu(V) = 0$ for *all* open subsets $V \subseteq G$, hence, by outer regularity, $\mu(B) = 0$ for all Borel sets $B \subseteq G$. This is the desired contradiction.

We now set out to prove (ii). We follow the proof of [Bourbaki, Chapter VII, §1, no. 2, Prop. 2].

Clearly if G is discrete then $\{e\}$ is open in G, so the "only if" implication holds by (i). As for the converse, observe first of all that $\mu(\{g\}) = \mu(\{e\}) > 0$ for any $g \in G$ by left-invariance of μ. We conclude that, if K is a compact subset of G, then K is a finite set—because otherwise, K contains a countably infinite set C and $\mu(K) \geq \mu(C) = \sum_{k\in C} \mu(\{k\}) = \infty$, contradicting finiteness of μ on compact sets. The proof is then completed by the following observation.

4.22. Lemma. *Let X be a LCH topological space. Suppose that the only compact subsets of X are the finite ones. Then the topology of X is discrete.*

Proof of the lemma. Neglecting the trivial case $X = \emptyset$, it suffices to pick an arbitrary point $x \in X$ and show that the singleton $\{x\}$ is open in X.

To that end, let K be a compact neighbourhood of x; then, by assumption, K is a finite set. We conclude that the subspace topology on K is the discrete topology (because that is the only Hausdorff topology on a finite topological space.) In particular, $\{x\}$ is open in K in the subspace topology; in other words, $\{x\}$ can be written as $K \cap U$ for some open subset $U \subseteq X$. But then it also holds that $\{x\} = K^\circ \cap U$, so $\{x\}$ is open in X, as desired. $\square$

Fill in the proof of (iii)! (Cf. [Folland2, Prop. 11.4.(d)]) $\square$

5 Existence of Haar measure

The entirety of this section is devoted to showing existence of a left Haar measure (see 4.18) on any locally compact group, i.e., to proving the following statement:

5.1. THEOREM. *Let G be a locally compact group. Then there exists at least one left Haar measure μ on G.*

The proof we give is the one given in [Cohn, Theorem 9.2.1], which [Joys] attributes to André Weil. The same arguments are given in [Gleason] in an essentially self-contained fashion (but see 4.9.(2)). This is also essentially the same proof as in [Folland2, Theorem 11.8], [Joys, Chapter 7, §2], except that these sources do not construct a measure *per se* but rather a positive linear functional on a certain space of functions on the group at hand; cf. also the discussion in 7.27.

5.2. Before diving into the details of the proof, we would like to present a partial outline; more precisely, we shall give a preview of how some of the results encountered so far will factor into the argument, thus also indicating which steps remain to be carried out.

By 2.7, the assumption of local compactness on a topological group G means precisely that there exists a compact subset $K_0 \subseteq G$ with non-empty interior. Supposing that such a K_0 has been fixed, we shall define a set $\mathcal{I}(G; K_0)$ with two fundamental properties:

(1) $\mathcal{I}(G; K_0)$ is non-empty, and

(2) to each element of $\mathcal{I}(G; K_0)$ there corresponds to a function $h\colon \mathcal{K}(G) \to \mathbb{R}_{\geq 0}$ which satisfies the assumptions of 4.8, where $\mathcal{K}(G) \subseteq \mathcal{P}(G)$ will denote the collection of compact subsets of G.

The aforementioned proposition 4.8—or rather, its corollary 4.17—will then yield, for each such h, a Radon measure μ on G. The final, decisive step will be to prove that any μ obtained in this way is not identically zero and is left-invariant in the sense of 4.1; once this has been accomplished, we will have shown that the set of left Haar measures on G is non-empty, as desired.

5.1 The Haar covering number

5.3. The first step in the proof is an observation which draws on our discussion of topological groups in §1. Recall that we proved in 1.18 that, if an open subgroup sits inside a compact one, then it must be of finite index. (Because, loosely speaking, the translates of the smaller group will cover the larger one and then compactness will force finiteness of the covering.) Later, in the proof of 4.21.(i), we adapted the argument to work in a more general setting and established that, whenever K is a compact subset of a topological group G and $U \subseteq G$ is open and non-empty, finitely many translates of U will suffice to cover K. (In other words, we got rid of the assumption that U and K be subgroups.)

Because the natural numbers form a well-ordered set, there is then, in the above situation, even a *minimal* number of translates which can be taken as a cover. It should be readily clear (see also E.22) that this minimum is a rough estimate for the "relative size" of K with respect to U. This warrants the following definition.

Given an arbitrary topological group G, a compact set $K \subseteq G$ and a subset $V \subseteq G$ with non-empty interior, the nonnegative integer

$$(K : V) \coloneqq \min\{n \in \mathbb{N} : \exists g_1, \ldots, g_n \in G \text{ such that } K \subseteq \bigcup_{i=1}^{n} g_i V^\circ\}$$

will be called the *(left) Haar covering number* of K with respect to V.

5.4. *Remarks on the definition.*

(1) Observe that V is not required to be open: for the definition to be meaningful, it suffices that V have non-empty interior. (This is because we take translates of the *interior* of V, rather than V itself.) The added generality will be useful later.

(2) Recall that, in this note, $0 \in \mathbb{N}$. It is easily checked that, if K and V are as in the above definition, the quantity $(K : V)$ vanishes if and only if K is the empty set (with no condition on V). $\diamond$

5.5. A few properties of the Haar covering number are easily checked, e.g.:

(i) it is left-invariant in the sense that, if K and V are as in the above definition, then $(gK : hV) = (K : V)$ for any g, $h \in G$;

(ii) it is monotonic in the first argument, i.e., $(K_1 : V) \leq (K_2 : V)$ if $K_1 \subseteq K_2$ whenever everything is defined; and

(iii) it is finitely subadditive in the first argument, i.e., $(K_1 \cup K_2 : V) \leq (K_1 : V) + (K_2 : V)$ whenever K_1, K_2 are arbitrary compact sets and V is as above.

(See E.23.) It follows that, if we were to *fix* an *open non-empty* subset $V \subseteq G$ and set $c_V(K) \coloneqq (K : V)$ for every $K \subseteq G$ compact, then c_V would be a nonnegative real-valued function on $\mathcal{K}(G)$ satisfying two of the three properties required to apply 4.8—as well as a form of left-invariance. It is, however, easily seen that c_V fails to be finitely additive in general: if V is "too big" with respect to two given non-empty compact subsets K_1 and K_2—more precisely: if some translate of V contains the union $K_1 \cup K_2$—then clearly $c_V(K_1 \cup K_2) = 1 \neq 2 = c_V(K_1) + c_V(K_2)$. The version of finite additivity that we do have is the following.

5.6. Proposition. *Let G be a topological group, V be an open non-empty subset of G, and K_1, K_2 be compact subsets of G. Furthermore let c_V be defined as in 5.5. If $K_1V^{-1} \cap K_2V^{-1} = \emptyset$, then $c_V(K_1 \cup K_2) = c_V(K_1) + c_V(K_2)$.*

Proof. We know from 5.5.(ii) that $c_V(K_1 \cup K_2) \leq c_V(K_1) + c_V(K_2)$, so it remains to prove the inequality in the opposite direction. Unravelling the definition 5.3, this means that it suffices to show the following: if n is a natural number such that $K_1 \cup K_2$ can be covered by n translates of V, then there exist elements $g_1, \ldots, g_n \in G$ such that $K_1 \subseteq \bigcup_{i=1}^{k} g_iV$ and $K_2 \subseteq \bigcup_{i=k+1}^{n} g_iV$ (for some $k \in \{1, \ldots, n\}$).

Thus, let n be a natural number with the required property, and let $g'_1, \ldots, g'_n$ be elements of G such that $K_1 \cup K_2 \subseteq \bigcup_{i=1}^{n} g'_iV$. We claim that each g'_iV intersects *at most one* of the two sets K_1 and K_2. Indeed, if some g'_iV intersected both, then g'_i would lie in the intersection of K_1V^{-1} and K_2V^{-1}, contradicting the assumption of the proposition. Therefore, we may partition the index set $\{1,\ldots,n\}$ into a disjoint union $I_1\dot{\cup}I_2$ so that $K_j \subseteq \bigcup_{i\in I_j} g'_iV$ for $j=1,2$. (Unless n is minimal, there is more than one such partition.) But then clearly the elements $g_1,\ldots,g_n$ we have been seeking can be obtained by suitably permuting[17] the elements $g'_1,\ldots,g'_n$, and this establishes the result. □

5.7. The implications of the proposition are best understood as follows. Having fixed the ambient topological group G, let $\mathcal{U}$ denote the family of all open neighbourhoods of the identity in G. Then for every pair of non-empty disjoint compact sets K_1 and K_2 in G, and for every $U\in\mathcal{U}$ "sufficiently small" (i.e., small enough that the translates K_1U^{-1} and K_2U^{-1} are still disjoint, if at all possible), the function $c_U\colon \mathcal{K}(G)\to\mathbb{R}_{\geq 0}$ would have the desired property of finite additivity: $c_U(K_1\cup K_2)=c_U(K_1)+c_U(K_2)$. However, there is—in general[18]—no "universal" U which works for all pairs of compact sets. Thus, to get a truly finitely additive function on $\mathcal{K}(G)$, we will have to consider, say, a "limit" of the functions c_U as U shrinks more and more. Of course, this only makes sense if we regard the c_U as existing in a space with some topological structure that ensures the existence of (at least) one such "limit".

As a first naïve attempt to do this, observe that each c_U is a function $\mathcal{K}(G)\to\mathbb{R}_{\geq 0}$, and that the space of all such functions is, at least abstractly, none other than the product space $(\mathbb{R}_{\geq 0})^{\mathcal{K}(G)}=\prod_{K\in\mathcal{K}(G)}\mathbb{R}_{\geq 0}$. (Here, a function f is identified with the tuple $\big(f(K)\big)_{K\in\mathcal{K}(G)}$; cf. the equivalent descriptions of the space $A^{\mathbb{N}}$ from 2.11.) It is then meaningful to talk about convergence of sequences, or more generally of *nets*[19], in this product space.

Now, readers familiar with more advanced point-set topology will recognize that $\{c_U\}_{U\in\mathcal{U}}$ is indeed a net in this product space—with the index set $\mathcal{U}$ being,

[17] In more detail: set k equal to the cardinality of I_1, choose a bijection $\{1,\ldots,k\}\to I_1$, extend it to a bijection σ of the set $\{1,\ldots,n\}$ and set $g_i := g'_{\sigma(i)}$ for $i=1,\ldots,n$.

[18] Of course, if the group G is discrete, there exists a "smallest" neighbourhood of the identity (i.e. one that is included in all the others), namely $U=\{e\}$. In this case, $c_{\{e\}}$ will be finitely additive. On the other hand, one can show that, in a Hausdorff nondiscrete group, there is never a "smallest" neighbourhood of e!

[19] Familiarity with nets will not be strictly necessary to the understanding of the rest of the proof. Nonetheless, the language of nets is particularly suitable to explaining the reasoning behind the next step in the argument, so we shall use it in the rest of this paragraph and a few more times in this section.

of course, ordered by reverse inclusion. However, we cannot expect this net to converge or even to have any cluster points (i.e.: limits of subnets), because as $U \in \mathcal{U}$ shrinks more and more, the Haar covering number $(K : U) = c_U(K)$ of a fixed compact K will typically become arbitrarily large. This is where local compactness comes in in a crucial way.

5.2 The core of the argument

For the sake of simplicity, we shall now fix, for the entirety of this subsection, a *locally compact* topological group G and a distinguished compact subset $K_0 \subseteq G$ with non-empty interior. (Such a subset exists by 2.7.) We also retain the notation $\mathcal{U}$ for the family of all open neighbourhoods of the identity in G and abbreviate by $\mathcal{K}$ the notation $\mathcal{K}(G)$ introduced previously for the family of all compact subsets of G.

5.8. The properties of the distinguished set K_0 ensure that, if K and V are as in 5.3, then both $(K : K_0)$ and $(K_0 : V)$ are meaningful. It is then easy to see (E.24) that $(K : V) \leq (K : K_0)(K_0 : V)$ for any such K and V. Using this, we can draw an immediate conclusion from 5.6 above.

5.9. Corollary. *For every $U \in \mathcal{U}$, the assignment*

$$K \mapsto h_U(K) := \frac{(K : U)}{(K_0 : U)}$$

defines a monotonic, finitely subadditive function on $\mathcal{K}$ with values in $\mathbb{R}_{\geq 0}$, which moreover has the following properties.

(i) h_U is left-invariant in the sense that $h_U(gK) = h_U(K)$ for every $K \in \mathcal{K}$ and every $g \in G$.

(ii) Whenever K_1, $K_2 \in \mathcal{K}$ are such that $K_1U^{-1} \cap K_2U^{-1} = \emptyset$, one has that $h_U(K_1 \cup K_2) = h_U(K_1) + h_U(K_2)$.

(iii) For every $K \in \mathcal{K}$, the function value $h_U(K)$ is bounded above by the Haar covering number $(K : K_0)$.

5.10. Going back to the discussion of 5.7, we can again think of each h_U as a point in the product space $\prod_{K\in\mathcal{K}} \mathbb{R}_{\geq 0}$ (and of $\{h_U\}_{U\in\mathcal{U}}$ as a net in this space).

But unlike earlier, now each $h_U(K)$ has an upper bound independent of U, so h_U can actually be thought of as a point in the product space

$$\prod_{K\in\mathcal{K}} [0,(K:K_0)] =: X.$$

This space is, naturally, to be considered equipped with product topology; as a product of compact intervals, it is then itself a compact topological space by Tychonoff's theorem.

5.11. We are now finally ready to define the set $\mathcal{I}(G;K_0)$ alluded to in 5.2.

In the language of nets, $\mathcal{I}(G;K_0)$ can be described concisely as the set of *cluster points* of the net $(h_U)_{U\in\mathcal{U}}$ in the product space X introduced above. An equivalent, more elementary description is:

$$\mathcal{I}(G;K_0) := \bigcap_{V\in\mathcal{U}} \mathrm{cl}(\{h_U : U\in\mathcal{U}, U\subseteq V\}),$$

where cl denotes the closure operator in the topological space X. Observe that, since clearly $\mathcal{I}(G;K_0)\subset X$, elements of $\mathcal{I}(G;K_0)$ have a natural interpretation as functions $h\colon \mathcal{K}\to\mathbb{R}_{\geq 0}$ with the additional property that $h(K)\leq (K:K_0)$ for every $K\in\mathcal{K}$.

It remains, of course, to prove that $\mathcal{I}(G;K_0)$ indeed has the properties claimed in 5.2. This is the content of the next two results.

5.12. Proposition. *The set $\mathcal{I}(G;K_0)$ defined in the previous paragraph is non-empty.*

Proof. Since $\mathcal{I}(G;K_0)$ is the set of cluster points of the net $\{h_U\}_{U\in\mathcal{U}}$, the claim is a special case of the well-known fact that, in a compact topological space, every net has at least one cluster point. The proof given here is merely a specialization of the standard proof of this more general fact.

For $V\in\mathcal{U}$, let F_V be shorthand for $\mathrm{cl}(\{h_U : U\in\mathcal{U}, U\subseteq V\})$. (Cf. 5.11.) We claim that the family $\{F_V\}_{V\in\mathcal{U}}$ has the *finite intersection property*, i.e.: whenever $V_1,\ldots,V_n$ are elements of $\mathcal{U}$ (the index n being some positive integer), the intersection $\bigcap_{i=1}^n F_{V_i}$ is non-empty. But this is plain to see: the element h_U with $U := \bigcap_{i=1}^n V_i\in\mathcal{U}$ clearly lies in $\bigcap_{i=1}^n F_{V_i}$. It then follows by a well-known characterization of compactness that the family $\{F_V\}_{V\in\mathcal{U}}$ has itself non-empty intersection, i.e.: $\bigcap_{V\in\mathcal{U}} F_V\neq\emptyset$. But the left-hand side is precisely $\mathcal{I}(G;K_0)$, and the proof is complete. $\square$

5.13. Proposition. *Let $\mathcal{I}(G; K_0)$ be as above, and let $h \in \mathcal{I}(G; K_0)$, viewed as a function $h\colon \mathcal{K} \to \mathbb{R}_{\geq 0}$. (Cf. 5.11.) Then:*

(i) h is monotonic, finitely subadditive and finitely additive on $\mathcal{K}$;

(ii) h is left-invariant, i.e.: $h(gK) = h(K)$ for every $K \in \mathcal{K}$ and every $g \in G$;

(iii) $h(K_0) = 1$; in particular, h is not identically zero.

Proof. We know by 5.9 that, if h is replaced by h_U (for some $U \in \mathcal{U}$) in (i)–(iii), then all claims hold true except for finite additivity. In order to "transfer" these results to h, we may use the fact that h is a limit of the h_U together with the fact that the conditions of monotoniticy, invariance, etc. are continuous.

In more detail, recall that $\mathcal{I}(G; K_0)$ sits inside the space $X = \prod_{K \in \mathcal{K}} [0, (K : K_0)]$, equipped with its natural product topology. In particular, each of the projections $p_K\colon X \to [0, (K : K_0)]$ (where $K \in \mathcal{K}$) is continuous. (Note that, if we think of the elements of X as functions $f\colon \mathcal{K} \to \mathbb{R}_{\geq 0}$, then the projection p_K is given by sending $f \in X$ to $f(K) \in [0, (K : K_0)] \subset \mathbb{R}_{\geq 0}$.) This yields continuity of several important (families of) functions $X \to \mathbb{R}_{\geq 0}$, the easiest example being

$$f \mapsto f(K_0) - 1 = p_{K_0}(f) - 1.$$

Clearly this map is constant (more precisely: identically zero) on $\{h_U\}_{U \in \mathcal{U}}$, hence it is constant on $\{h_U : U \in \mathcal{U}, U \subseteq V\}$ for every $V \in \mathcal{U}$. Because it is continuous, we may then conclude that it is also identically zero on the *closure* $\mathrm{cl}(\{h_U : U \in \mathcal{U}, U \subseteq V\})$ for every $V \in \mathcal{U}$, and hence also on $\mathcal{I}(G; K_0)$. This proves that $h(K_0) = 1$ and thus establishes (iii).

The other claims (with the exception of finite additivity of h) are established by the same argument, applied to a different function. For instance, to show (ii), i.e. left-invariance, it suffices to note that, for each $K \in \mathcal{K}$ and each $g \in G$, the function $X \to \mathbb{R}_{\geq 0}$ given by

$$f \mapsto f(K) - f(gK) = p_K(f) - p_{gK}(f)$$

is continuous on X and identically zero on $\{h_U : U \in \mathcal{U}\}$. To show finite subadditivity, one observes that, for each pair $(K_1, K_2) \in \mathcal{K} \times \mathcal{K}$,

$$f \mapsto f(K_1) + f(K_2) - f(K_1 \cup K_2)$$

is continuous on X and takes only nonnegative values on $\{h_U : U \in \mathcal{U}\}$. Similarly, to prove monotonicity, i.e. $h(K_1) \leq h(K_2)$ for $K_1, K_2 \in \mathcal{K}$ with $K_1 \subseteq K_2$, the function to consider is

$$f \mapsto f(K_2) - f(K_1).$$

It remains to prove finite additivity of h. To that end, let $K_1, K_2 \in \mathcal{K}$ be disjoint. Clearly, if we can find a distinguished $V \in \mathcal{U}$ such that $K_1V^{-1} \cap K_2V^{-1} = \emptyset$, then it will also hold that $K_1U^{-1} \cap K_2U^{-1} = \emptyset$ for every $U \in \mathcal{U}$ such that $U \subseteq V$, hence: the function $X \to \mathbb{R}_{\geq 0}$ defined by

$$f \mapsto f(K_1) + f(K_2) - f(K_1 \cup K_2),$$

beside being continuous (as already established above), will be identically zero on $\{h_U : U \in \mathcal{U}, U \subseteq V\}$ by 5.9.(ii); the same argument as above will then yield that f is identically zero on $\mathcal{I}(G; K_0)$, i.e., that h is finitely additive.

It remains to find a suitable set V. This is accomplished essentially by applying the following general results ([Cohn, Prop. 7.1.1] and [Cohn, Prop. 9.1.3], respectively).

5.14. Lemma. *In a Hausdorff space X, disjoint compact sets K_1 and K_2 may be separated by disjoint open neighbourhoods $U_1 \supseteq K_1$ and $U_2 \supseteq K_2$.*

5.15. Lemma. *Let G be a topological group, U be an open subset of G and K be a compact subset of U. Then there exists an open neighbourhood V of the identity of G such that $KV \subseteq U$.*

We may apply the first lemma directly (to $X = G$) to obtain disjoint open sets $U_1 \supseteq K_1$ and $U_2 \supseteq K_2$. Applying the second lemma to each pair yields open neighbourhoods V_1, V_2 of the identity in G such that $K_1V_1 \subseteq U_1$ and $K_2V_2 \subseteq U_2$. Then $V = (V_1 \cap V_2)^{-1}$ has the property that $K_1V^{-1} \cap K_2V^{-1}$ is empty (because $K_1V^{-1} \cap K_2V^{-1} \subseteq K_1V_1 \cap K_2V_2 \subseteq U_1 \cap U_2 = \emptyset$) and the above arguments conclude the proof. $\square$

5.3 Conclusion of the proof

We can now complete the proof of the existence result 5.1.

Proof of 5.1. The outline of the proof was for the most part already given in 5.2: because G is locally compact, there exists a compact subset $K_0 \subseteq G$ with non-empty interior, which in turn gives rise to a set $\mathcal{I}(G; K_0)$ as defined in 5.11. By 5.12, it is possible to pick an element $h \in \mathcal{I}(G; K_0)$, which by 5.13 satisfies the assumptions (h1)–(h3) of 4.8. This latter result—or rather its corollary 4.17—then yields a Radon measure μ on G. It thus suffices to check that μ is left-invariant and not identically zero.

To prove that μ is left-invariant, let E be a Borel set in G and g be an element of G. Then, by outer regularity, $\mu(gE)$ is an infimum taken over all open sets which contain gE. But (essentially by 1.8), every such open set is of the form gU for some open $U \supseteq E$. Thus, to prove that $\mu(gE) = \mu(E)$ it will suffice to show that $\mu(gU) = \mu(U)$ for every open set $U \subseteq G$ and every $g \in G$.

Now, by the definition given in the statement of 4.8, $\mu(gU)$ is just the supremum (in $\mathbb{R}$) of the set $\{h(C)\}$ where C runs over the compact subsets of gU and h is as in the previous paragraph. But again by 1.8, every such C is of the form gK for some compact $K \subseteq U$, and it was proved in 5.13 that $h(gK) = h(K)$ for all $K \in \mathcal{K}(G)$ and all $g \in G$. Therefore we indeed find that μ is left-invariant on open sets and, by the previous argument, on all Borel sets.

Finally, we need to show that μ is not identically zero. By 4.8.(iv), we know that $\mu(K) \geq h(K)$ for any compact $K \subseteq G$. Applying this to our distinguished compact set K_0, we see that $\mu(K_0) \geq h(K_0) = 1$, and hence in particular $\mu(K_0) \neq 0$, which establishes the claim. □

Let us note an immediate corollary.

5.16. Corollary. *On every locally compact group, there exists at least one right Haar measure.*

Proof. Let G be a locally compact group. By 5.1, there exists a left Haar measure μ on G. Then, by 4.20.(1), $\mu^\vee$ is a right Haar measure on G. (See *loc. cit.* for the notation $^\vee$.) □

6 Review of integration theory

Having established that every locally compact group possesses at least one left Haar measure, we shall turn to the question of uniqueness next. To obtain our central result in this direction (7.1 below), we will rely on integration theory "*à la* Lebesgue" (i.e., resting in turn on measure-theoretic foundations). For this reason, we have opted to devote to this auxiliary topic the entirety of the present section, which thus takes on the shape of a parenthetical digression much like §3. (In fact, we may consider this section to serve as the continuation and conclusion of the discussion started in §3.) An additional reason to look at integration theory so extensively may be found in the fact that spaces of integrable functions will be among the main protagonists of Chapter III.

The interested reader is invited to consult the references given in §3 for more on integration theory.

6.1 Introduction

6.1. The origins of integration theory lie in the following ubiquitous problem: given a (say) continuous function f on a compact interval $[a, b] \subset \mathbb{R}$ with values in the nonnegative real numbers, determine the area of the region $A = A(f)$ enclosed in the plane $\mathbb{R}^2$ by the graph of the function, the x-axis and the two vertical lines $x = a$ and $x = b$. (This area is traditionally known as the *integral* of f from a to b and denoted by $\int_a^b f(x)\,\mathrm{d}x$.)

Clearly, if the function f is of a very simple form, then its integral can be computed easily by falling back on elementary plane geometry: for instance, if f is constant, then the region A described in the previous paragraph is simply a rectangle and its area is simply $y(b-a)$, with y being the unique value attained by f. Slightly more generally[20], suppose that f is a *step function*, i.e., suppose that there exist finitely many intermediate points $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ such that f is constant on each subinterval (x_{i-1}, x_i). (The values at the endpoints of the subintervals are irrelevant.) Then the area of the region A can again be determined with ease: it equals

$$\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}), \tag{$\int$STEP}$$

where t_i is any point in (x_{i-1}, x_i).

[20] Note that we are departing from the requirement that f should be continuous!

6.2. *Parenthetical.* It is important to observe that, while the requirement that f only assume nonnegative values is crucial to our interpretation of ($\int$STEP) as the area of the region A, the formula itself makes sense much more generally: if the nonnegativity assumption is lifted, ($\int$STEP) will simply describe a *signed* area in general (rather than an area in the classical sense, which would inescapably be nonnegative). In fact, one may define step functions with values in $\mathbb{C}$ (or indeed, any real or complex vector space) in precisely the same way as we did before, and the formula ($\int$STEP) remains meaningful in this more general setting.

6.3. Now let f be an arbitrary function on $[a, b]$ with values in the reals (or the complex numbers, cf. 6.2). Given a *tagged partition*

$$a = x_0 < t_1 < x_1 < \cdots < x_{n-1} < t_n < x_n = b$$

of the interval $[a, b]$, we may again look at the quantity ($\int$STEP), which we call the *Riemann sum* of f associated to the partition in question. As our readers presumably know, one says that f is *Riemann-integrable* if, upon taking finer and finer (tagged) partitions of $[a, b]$, the corresponding Riemann sums converge to a common value, which is then called the *(Riemann) integral* of f from a to b.

By making explicit reference to partitions of the domain $[a, b]$, the Riemann integral ultimately relies on the ordering of the reals, which does not bode well for generalization. However, we may take a different point of view: since Riemann sums are, by construction, simply integrals of suitable step functions, we may conclude that being Riemann-integrable simply means affording arbitrarily good approximations by step functions (in a suitable sense). This suggests the following strategy:

(1) find a suitable analogue or generalization of step functions, for which the integral is easily defined; and then

(2) extend the definition to functions which afford arbitrarily good approximations (in a suitable sense) by functions as in (1).

6.2 The Lebesgue integral

6.4. In accordance with the strategy laid out in the previous paragraph, let us take stock of what makes step functions especially suitable for the formula ($\int$STEP).

Two observations can be made right away. Firstly, a crucial property of step functions is that they only attain finitely many values. Additionally, if f is a step function on a compact interval $[a, b]$, the inverse image of each y in the range of f is typically a bounded interval in $\mathbb{R}$—more precisely, it will be a finite disjoint union of such intervals, each of which may or may not include its endpoints (and might in fact even consist of a single point). Regardless of the specifics, each of these sets $f^{-1}(y) \subseteq [a, b]$ certainly lies in the σ-algebra generated by the subintervals of $[a, b]$, i.e., it is a Borel-measurable subset of $[a, b]$.[21] In particular, the (Riemann-)integral of f can be rewritten without making reference to any partition of $[a, b]$, namely as follows:

$$\int_a^b f(x)\,\mathrm{d}x = \sum_y y \cdot \lambda\big(f^{-1}(y)\big)$$

(with λ being, of course, Lebesgue measure; the sum is indexed by the values in the range of f). While the appearance of Lebesgue measure is perhaps not surprising—we are, after all, trying to measure an area by falling back on the "lengths" of the sides of certain rectangles, and λ is merely a rigorous tool to measure lengths—the new viewpoint has extraordinary potential for generalization. Indeed, as a first step, we may immediately generalize the notion of a step function to much more general domains as follows.

6.5. Let $(X, \mathcal{A})$ be a measurable space, and let f be a function on X with values in the reals or the complex numbers.[22] Then f is called a *simple function* on $(X, \mathcal{A})$—or simply on X—if it satisfies the following two conditions:

(a) f only attains finitely many values; and

[21] Here we are implicitly using the following fact (cf. e.g. [Elst, Kapitel I, Korollar 4.6]): if X is a topological space and $Y \subseteq X$ is a Borel set in X, then the Borel-σ-algebra of Y (considered as a topological space in its own right with the subspace topology) is $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}_X\} = \{B \in \mathcal{B}_X : B \subseteq Y\}$. In particular, $\mathcal{B}_Y \subseteq \mathcal{B}_X$ if $Y \subseteq X$ is Borel.

[22] In accordance with 6.2, we can—and will—treat the real- and complex-valued cases simultaneously.

(b) for each value y in the range of f, the inverse image $f^{-1}(y) \subseteq X$ is $\mathcal{A}$-measurable in X.

6.6. *Remarks.*

(1) Observe that the above definition does not make reference to any measure on $(X, \mathcal{A})$!

(2) It will often be practical to identify the real-valued functions on a set X with those complex-valued functions on X whose range is contained in $\mathbb{R} \subset \mathbb{C}$. (Thus, the real-valued functions become a subset of the complex-valued ones.) One should convince oneself that this does not have any repercussions on the above definition: a function $f\colon X \to \mathbb{R}$ is simple if and only if it is simple when viewed as a complex-valued function. $\diamond$

6.7. *Examples.*

(1) On any measurable space $(X, \mathcal{A})$, all constant complex-valued functions on X (and hence in particular all constant real-valued functions) are simple.

(2) For a measurable space $(X, \mathcal{A})$ and a measurable subset $A \in \mathcal{A}$, the *indicator function* $1_A\colon X \to \{0, 1\} \subset \mathbb{R}$ is a simple function.

(3) Given finitely many simple functions on a common measurable space, every (pointwise) linear combination of them is again a simple function; in other words, the $\mathbb{K}$-valued simple functions on a measurable space $(X, \mathcal{A})$ form a $\mathbb{K}$-linear vector space with the obvious (pointwise) operations. (Here, we have used '$\mathbb{K}$' as a placeholder for either of the symbols '$\mathbb{R}$' and '$\mathbb{C}$'.) In fact, it is immediate that every simple function is a finite linear combination of indicator functions 1_A, $A \in \mathcal{A}$ as per the previous item.

(4) Let $(X, \mathcal{A})$ be a measurable space, and f be a $\mathbb{K}$-valued simple function on X, with $\mathbb{K}$ equal to either $\mathbb{R}$ or $\mathbb{C}$. It is readily seen that, for every function $g\colon \mathbb{K} \to \mathbb{K}$, the composition $g \circ f$ is again a simple function on X.

(5) The discussion carried out in 6.4 shows that every step function on a compact interval $[a, b] \subset \mathbb{R}$ is a simple function (the domain being, of course, equipped with its Borel-σ-algebra). On the other hand, there

clearly exist simple functions on $[a, b]$ which are not step functions, such as the indicator function of the set $\mathbb{Q} \cap [a, b]$. (Cf. (2).) $\diamond$

6.8. Defining the integral requires a little more care, even for such simple functions as the ones introduced above. In fact, let us, for the sake of simplicity, consider real-valued simple functions on the measurable space $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, equipped with Lebesgue measure λ. It is arguably clear from the discussion in 6.4 that the integral of such a function f should look like

$$\sum_{y \in f(X)} y \cdot \lambda\big(f^{-1}(y)\big). \tag{$\int$?}$$

(Observe that, because the function f is assumed simple, its range $f(X)$ is a finite set, so the sum is finite.) A major hurdle at this point is the fact that measures are (by design) explicitly allowed to attain the value ∞, while calculus teaches us that many operations with ∞ are inadmissible. Indeed, we can easily manufacture examples where problems appear.

(1) Suppose that f is such that $f^{-1}(0)$ has infinite measure (e.g.: f is the indicator function of a bounded interval.) Then ($\int$?) involves the operation $0 \cdot \infty$.

(2) Take f to be the *sign function*, which sends 0 to 0 and every $x \neq 0$ to $\frac{x}{|x|}$. Then f is a simple function for which ($\int$?) reads $\infty - \infty$.

These problems are not of the same nature. In fact, the former is essentially only a matter of convention: if we agree that sets on which f is identically zero are not supposed to contribute anything to the integral of f, regardless of how large they are (in terms of the measure λ), then the appropriate thing to do is to simply set $0 \cdot \infty := 0$ in this particular context. With this convention, the integral of 1_B, with $B \subseteq \mathbb{R}$ Borel, is always equal to $\lambda(B)$, which is indeed both intuitive and convenient.

The potential appearance of the undefined expression $\infty - \infty$ is a more serious problem, which cannot be solved conclusively as much as circumvented by restricting to functions for which it does not occur. Suppose, for instance, that the (simple) function at hand happens to only attain nonnegative real values. (In this case the function itself is said to be *nonnegative*.) Then surely no negative signs can occur in ($\int$?), so all operations with ∞ should be well-defined in the intuitive way—provided that one takes into account the convention $0 \cdot \infty := 0$ from the preceding paragraph. Accordingly, we posit the following definition.

6.9. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f a [complex-valued] simple function on $(X, \mathcal{A})$. Suppose that f is *nonnegative*, i.e. that $f(x) \in \mathbb{R}_{\geq 0}$ for all $x \in X$. Then the *integral of* f with respect to μ is defined to be

$$\int_X f \, \mathrm{d}\mu := \sum_{y \in f(X) \subset \mathbb{R}_{\geq 0}} y \cdot \mu\big(f^{-1}(y)\big) \in [0, \infty]. \qquad (\textstyle\int, \text{ PT. 1})$$

(The convention $0 \cdot \infty := 0$ is in place.)

6.10. The following properties of the integral are checked immediately.

(0) If $(X, \mathcal{A})$ is a measurable space and $f = 1_A$ is the indicator function of a measurable set $A \in \mathcal{A}$, then $\int_X 1_A \, \mathrm{d}\mu = \mu(A)$.

(1) If f is a nonnegative simple function on $(X, \mathcal{A})$ and $c \in \mathbb{R}_{\geq 0}$, then $\int_X cf \, \mathrm{d}\mu = c \int_X f \, \mathrm{d}\mu$.

(2) If f and g are nonnegative simple functions on $(X, \mathcal{A})$, then $\int_X (f + g) \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu + \int_X g \, \mathrm{d}\mu$.

(3) If f and g are nonnegative simple functions on $(X, \mathcal{A})$ with $f \geq g$, meaning $f(x) \geq g(x)$ for all $x \in X$, then $\int_X f \, \mathrm{d}\mu \geq \int_X g \, \mathrm{d}\mu$.

6.11. It is now time to turn to phase (2) of the strategy laid out in 6.3. Thus, let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a complex-valued function on X. Because, so far, we only know how to integrate nonnegative simple functions on X, it is only natural to start with the special case where f is itself nonnegative, i.e., with values in $\mathbb{R}_{\geq 0}$. It is also clear that, while f need not be simple, it should not be "too far removed" from simple functions—otherwise we would not be able to apply the special case we have already worked out. All things considered, we make the following assumption:

(SM) there exist nonnegative simple functions $f_1, f_2, \ldots$ such that $0 \leq f_1 \leq f_2 \leq \cdots$ and $f(x) = \lim_{n \to \infty} f_n(x)$ for all $x \in X$.

This assumption is clearly automatically satisfied for all nonnegative simple functions on $(X, \mathcal{A})$. Moreover, the set of nonnegative functions on X satisfying

(SM) is easily seen to be closed under (finite pointwise) sums and scaling by nonnegative real numbers.

Under the assumption (SM), and applying 6.10.(3), we obtain a monotonically nondecreasing sequence

$$0 \leq \int_X f_1 \,\mathrm{d}\mu \leq \int_X f_2 \,\mathrm{d}\mu \leq \cdots \leq \infty;$$

by standard results in analysis, such a sequence must have a limit in $[0, \infty]$. It is, moreover, not hard to show (s. e.g. [Elst, Kapitel IV, Korollar 2.2]) that this limit is actually independent of the choice of the sequence f_1, f_2, ...; therefore, the following definition is meaningful:

$$\int_X f \,\mathrm{d}\mu := \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu \in [0, \infty], \qquad (f_n)_{n\geq 1} \text{ as in (SM)} \qquad (\textstyle\int, \text{ PT. } 2)$$

(and agrees with ($\int$, PT. 1) if f happens to be simple). Going back to the introductory discussion in 6.1, this definition can be seen as crystallizing the intuition that the area of the region $A = A(f)$ under the graph of a generic nonnegative function f can be determined by gradually and successively "filling out" A by regions of a simpler form, whose area is computed as per ($\int$, PT. 1).

6.12. It is easily checked that properties (1)–(3) from 6.10 still hold true if 'nonnegative simple function(s)' is replaced by 'nonnegative function(s) satisfying (SM)' throughout.

6.13. Having defined the integral for a certain class of nonnegative functions, the next step is of course to extend the definition to suitable real- and complex-valued functions. We know from the discussion in 6.8 that this can be tricky even for simple functions, so our task will consist first and foremost in pinpointing a condition, on a real- or complex-valued function f, which ensures that the potential problems detected earlier do not arise. The functions satisfying this condition (yet to be formulated) will be the *Lebesgue-integrable functions.*

Thus, let $(X, \mathcal{A}, \mu)$ be a measure space. The definition of integrability, and indeed of the integral itself, may be given in successive stages, each building on and extending the previous one.

(1) A nonnegative function f on X is *(Lebesgue-)integrable* with respect to μ if it satisfies (SM) from 6.11 *and* its integral as per ($\int$, PT. 2) is finite (i.e., "strictly less than ∞").

(2) A real-valued function f on X is *(Lebesgue-)integrable* if both its positive part f^+ and negative part f^- are integrable in the sense of (1).[23] If this condition is satisfied, then the *integral of* f with respect to μ is, by definition, the real number

$$\int_X f \,\mathrm{d}\mu \coloneqq \int_X f^+ \,\mathrm{d}\mu - \int_X f^- \,\mathrm{d}\mu\,. \qquad (\textstyle\int, \text{PT. } 3)$$

(3) Finally, a complex-valued function f on X is *(Lebesgue-)integrable* if both its real part $\mathrm{Re}(f)$ and imaginary part $\mathrm{Im}(f)$ are integrable in the sense of (2).[24] The *integral of* f with respect to μ is then given by the formula

$$\int_X f \,\mathrm{d}\mu = \int_X \mathrm{Re}(f) \,\mathrm{d}\mu + \mathrm{i} \int_X \mathrm{Im}(f) \,\mathrm{d}\mu\,. \qquad (\textstyle\int, \text{PT. } 4)$$

(We leave it to the reader to check that each step is indeed consistent with the previous one.)

6.14. *Remarks on the definition.*

(1) Keeping the notations from the preceding paragraph, one can also consider the following extension of the notion of integrability (for *real*-valued functions): a function $f\colon X \to \mathbb{R}$ may be called *quasi-integrable* if both f^+ and f^- satisfy (SM) from 6.11 and *at least* one of them is integrable. For a quasi-integrable function f, we may again define $\int_X f \,\mathrm{d}\mu$ by the above formula ($\int$, PT. 3), with the caveat that this time the integral might equal $-\infty$ or $+\infty$.
The advantage of integrability over quasi-integrability is that, while it is easily checked that sums of integrable functions are again integrable (as follows easily from the definitions, cf. also 6.11 and 6.12), the same need not be true for quasi-integrable functions, see also E.25.

[23] Recall that the functions $f^\pm$ are defined by $f^\pm(x) = (|f(x)| \pm f(x))/2$ and thus satisfy $f = f^+ - f^-$. Observe that, despite the nomenclature, they are *both nonnegative*.

[24] Here, of course, $\mathrm{Re}(f)(x) \coloneqq \mathrm{Re}(f(x))$, and analogously for $\mathrm{Im}(f)$; in particular, both functions are real-valued and $f = \mathrm{Re}(f) + \mathrm{i}\,\mathrm{Im}(f)$.

(2) It is perhaps worth noting that, even though the Lebesgue integral involves functions with values in the real line (or the complex plane), the only measure which is relevant to the construction (and, potentially, computation) of the integral is that on the *domain*; one never uses any measure on the target space $\mathbb{R}$ (or $\mathbb{C}$). Thus, in spite of the terminology, the Lebesgue *integral* does not involve Lebesgue *measure* at all (except, of course, if the domain of the functions one wants to integrate is again the real line equipped with Lebesgue measure...). ◇

6.15. *Examples.*

(1) Let f be a complex-valued *simple* function on a measurable space $(X, \mathcal{A})$, and μ be a measure on this same space. It is the content of E.26 that f is integrable if and only if, for every *nonzero* y in the range of f, the inverse image $f^{-1}(y)$ has finite μ-measure, in which case the integral is given by the formula from 6.8: $\int_X f \,\mathrm{d}\mu := \sum_{y \in f(X)} y \cdot \mu(f^{-1}(y))$ (with the by-now-familiar convention $0 \cdot \infty := 0$).

(2) Let X be an arbitrary set and let μ denote *counting measure* on $(X, \mathcal{P}(X))$. Then (see E.27) a complex-valued function f on X is integrable if and only if

  (i) the set $N = N(f) = \{x \in X : f(x) \neq 0\} \subseteq X$ is at most countable, *and*

  (ii) the series $\sum_{x \in N} f(x)$ converges absolutely in $\mathbb{C}$,

in which case $\int_X f \,\mathrm{d}\mu = \sum_{x \in N} f(x)$. ◇

6.16. As was already mentioned in 6.14, sums of integrable functions are again integrable; it is moreover easily checked that the integral of the sum equals precisely the sum of the integrals of the individual summands (use 6.12). On the other hand, it is about as easy to see that, if f is a real- or complex-valued integrable function on a measure space $(X, \mathcal{A}, \mu)$ and c is a real or complex scalar, then cf is again integrable with integral equal to $c \int_X f \,\mathrm{d}\mu$. In more concise and more technical vocabulary, we have the following statement.

6.17. Proposition. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathcal{L}_1(X, \mathcal{A}, \mu; \mathbb{K})$ denote the set of $\mathbb{K}$-valued functions on X which are integrable with respect to*

μ. (Here, $\mathbb{K}$ may be either $\mathbb{R}$ or $\mathbb{C}$.) Then $\mathcal{L}_1(X, \mathcal{A}, \mu; \mathbb{K})$ is a vector space over $\mathbb{K}$ with the obvious (pointwise) operations, and the assignment

$$f \mapsto \int_X f \, \mathrm{d}\mu$$

is a $\mathbb{K}$-linear functional on $\mathcal{L}_1(X, \mathcal{A}, \mu; \mathbb{K})$.

6.18. With the preceding result, our discussion of the Lebesgue integral might be regarded as essentially complete, with one major exception: the condition (SM) on which our definition of integrability currently rests is actually quite cumbersome to check in practice. As a consequence, it is not at all clear—at least not without additional general results—which functions really occur as elements of $\mathcal{L}_1(X, \mathcal{A}, \mu; \mathbb{K})$ (cf. 6.17 for the notation) for a given measure space $(X, \mathcal{A}, \mu)$. This is not a fault of the Lebesgue integral, but rather of our presentation: in fact, (SM) may be equivalently replaced by the much more natural and easily-formulated condition that the nonnegative function f be *measurable*. In the next subsection, we shall introduce the notion of a measurable function and thus belatedly provide the appropriate theoretical framework for a systematic study of the Lebesgue integral.

6.3 Measurable functions

Abstractly, measurable functions are precisely the "structure-preserving maps" between measurable spaces; informally, we could say that they are to measurable spaces what continuous maps are to topological spaces. In fact, the definition of measurability, given immediately below, is even formally analogous to that of continuity.

6.19. A function between two measurable spaces is said to be a *measurable* function if preimages of measurable subsets of the target space are measurable in the domain. More precisely, if $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces and f is a function $X \to Y$, then f is *$\mathcal{A}$-$\mathcal{B}$-measurable* (or, if the σ-algebras are understood, simply *measurable*) if and only if $f^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}$.

6.20. *Remarks on the definition.*

(1) To provide some motivation for the above definition, recall that inverse images are much better behaved than ("regular") images with respect to set-theoretic operations. For instance, it is easily checked that, if $(Y, \mathcal{B})$ is a measurable space and $f\colon X \to Y$ is a set-theoretic function from a set X to Y, then

$$f^{-1}\mathcal{B} := \{f^{-1}(B) : B \in \mathcal{B}\} \subseteq \mathcal{P}(X)$$

is a σ-algebra on X. Thus, if X comes equipped with a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(X)$, then the condition that f be measurable is synonymous with the inclusion $f^{-1}\mathcal{B} \subseteq \mathcal{A}$.

(2) Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, and suppose that $\mathcal{E} \subset \mathcal{B}$ generates $\mathcal{B}$ in the sense that $\mathcal{B} = \sigma(\mathcal{E})$. (Cf. 3.7 for the notations.) An ingenious argument (s. e.g. [Elst, Kapitel I, Satz 4.4; Kapitel III, Satz 1.3]) can be used to show that, in order to check that a function $f\colon X \to Y$ is $\mathcal{A}$-$\mathcal{B}$-measurable, it suffices to check that $f^{-1}(E) \in \mathcal{A}$ for every $E \in \mathcal{E}$. $\diamond$

6.21. *Examples, pt. 1.*

(1) Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be arbitrary measurable spaces. Then it is easily checked that every constant function $X \to Y$ is measurable.

(2) Let X and Y be topological spaces, and $f\colon X \to Y$ be a continuous map from X to Y. By definition, if $U \subseteq Y$ is open, then $f^{-1}(U)$ is open in X, hence Borel; by 6.20.(2), this proves that f is $\mathcal{B}_X$-$\mathcal{B}_Y$-measurable, or *Borel measurable* for short.

(3) Compositions of measurable functions are again measurable. In more details, if $(X, \mathcal{A})$, $(Y, \mathcal{B})$ and $(Z, \mathcal{C})$ are measurable spaces and the maps $f\colon X \to Y$ and $g\colon Y \to Z$ are measurable, then so is $g \circ f$. $\diamond$

For the applications to integration theory, the measurable functions we will be interested in will be those whose target set Y is either the real line or the complex plane (while the domain $(X, \mathcal{A})$ shall remain arbitrary). In either case, the target set will *invariably* be considered with its *Borel* σ-algebra, $\mathcal{B}_{\mathbb{R}}$ or $\mathcal{B}_{\mathbb{C}}$ as the case may be.[25] More concisely, we stipulate the following

[25] In particular, the notion of measurability of a real-valued function does *not* make any reference to the σ-algebra of *Lebesgue* measurable sets.

6.22. *Convention.* A function on a measurable space $(X, \mathcal{A})$ with values in either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ is '*measurable*' if it is $\mathcal{A}$-$\mathcal{B}_{\mathbb{K}}$-measurable.

6.23. *Remark.* Of course, since $\mathbb{R} \in \mathcal{B}_{\mathbb{C}}$ and hence $\mathcal{B}_{\mathbb{R}} = \{B \in \mathcal{B}_{\mathbb{C}} : B \subseteq \mathbb{R}\}$ (as mentioned in footnote 21), a function $f \colon X \to \mathbb{R}$ is measurable if and only if it is measurable when viewed as a complex-valued function, cf. 6.6.(2). $\diamond$

In addition to the examples in 6.21, which were of a more general nature, we now provide some examples of real- and complex-valued measurable functions.

6.24. *Examples, pt. 2.* Throughout, we consider a fixed but arbitrary measurable space $(X, \mathcal{A})$. Whenever the symbol '$\mathbb{K}$' appears in a statement, this means that the statement is true whether '$\mathbb{K}$' is replaced by '$\mathbb{R}$' or '$\mathbb{C}$'.

(4) For a subset $A \subseteq X$, the indicator function $1_A \colon X \to \{0, 1\} \subset \mathbb{K}$ of A is measurable if and only if $A \in \mathcal{A}$.

(5) More generally, it is easily seen that simple functions on X, as defined in 6.5, are measurable. (S. also E.29.)

(6) For a function $f \colon X \to \mathbb{R}$, the following conditions are equivalent.
   (i) f is measurable.
   (ii) For all $b \in \mathbb{R}$, the set $\{x \in X : f(x) < b\} \subseteq X$ is ($\mathcal{A}$-)measurable.
   (iii) For all $b \in \mathbb{R}$, the set $\{x \in X : f(x) \leq b\} \subseteq X$ is ($\mathcal{A}$-)measurable.

   (The equivalence is seen to hold by 6.20.(2) and 3.9.)

(7) Because $(\mathbb{K}, +)$ is a topological group, the map $(x, y) \mapsto x + y$ from $\mathbb{K}^2$ to $\mathbb{K}$ is Borel-measurable by 6.21.(2). Using this in conjunction with 6.21.(3), one readily deduces that (pointwise) sums of measurable functions $X \to \mathbb{K}$ are measurable. Similar arguments (see [Elst, Kapitel III, §4]) show that the set of measurable functions $X \to \mathbb{K}$ is closed under scalar multiplication (by elements of $\mathbb{K}$) and taking pointwise products; in particular, it is a vector space over $\mathbb{K}$.

(8) A function $f \colon X \to \mathbb{C}$ is measurable if and only if both its real part $\mathrm{Re}(f)$ and its imaginary part $\mathrm{Im}(f)$ from footnote 24 are measurable. (This is E.30.)

(9) Similarly, a function $f \colon X \to \mathbb{R}$ is measurable if and only if both its positive part f^+ and its negative part f^-, as introduced in footnote 23, are measurable. (This is E.31.)

(10) It can be shown (s. e.g. [Elst, Kapitel III, Satz 4.3]) that, if a real- or complex-valued function f on X can be written as a pointwise limit of measurable functions—to wit, if there exist measurable functions f_1, f_2, ... on X such that $f(x) = \lim_{n\to\infty} f_n(x)$ for every $x \in X$—then f is itself measurable.

$\diamond$

6.25. Combining items (5) and (10) from 6.24, we see that a nonnegative function on a measurable space $(X, \mathcal{A})$ satisfying (SM) from 6.11 is certainly measurable. Most importantly, and perhaps surprisingly, the converse is also true.

6.26. PROPOSITION. *A nonnegative function f on a measurable space $(X, \mathcal{A})$ satisfies (SM) from 6.11 if and only if it is measurable.*

(We shall omit the proof, see e.g. [Elst, Kapitel III, Satz 4.13].)

6.27. The above result promptly implies that every integrable function (in the sense of 6.13) on a measure space $(X, \mathcal{A}, \mu)$ is measurable. (In fact, use 6.24.(8)–(9) above.) As for the "converse" problem of determining which measurable functions are integrable, the following result 6.28 yields a powerful characterization. In order to formulate it, we shall need one further item of notation.

For a real- or complex-valued function f on a set X, we let $|f|$ denote the nonnegative function on X given by $x \mapsto |f(x)|$; in other words: $|f| = |\cdot| \circ f$, where $|\cdot|$ denotes the Euclidean absolute value viewed as a map from $\mathbb{R}$, or $\mathbb{C}$, to $\mathbb{R}_{\geq 0}$. In particular, if X is equipped with a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ and f is a *measurable* function, then $|f|$ is also measurable. (Use 6.21.(2)–(3).) By 6.26, this means that $\int_X |f| \, \mathrm{d}\mu \in [0, \infty]$ is defined as per 6.11.

6.28. PROPOSITION. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathcal{L}_1(X, \mathcal{A}, \mu; \mathbb{K})$ denote the $\mathbb{K}$-linear space of $\mathbb{K}$-valued functions on X which are integrable with respect to μ. (Here, $\mathbb{K}$ may be either $\mathbb{R}$ or $\mathbb{C}$.) Then*

$$\mathcal{L}_1(X, \mathcal{A}, \mu; \mathbb{K}) = \{f \colon X \to \mathbb{K} \text{ measurable} : \int_X |f| \, \mathrm{d}\mu < \infty\},$$

where the integral on the right-hand side may be understood in the sense of 6.11. In other words, if f is a measurable real- or complex-valued function on X, then f is integrable if and only if $|f|$ is.

Proof. Let us first treat the case $\mathbb{K} = \mathbb{R}$. We shall use the fact that, for a real-valued function f on a set X, the equality $|f| = f^+ + f^-$ holds; in particular, the inequalities $|f| \geq f^+$ and $|f| \geq f^-$ follow.

Now let f be a real-valued measurable function. By 6.24.(9), both f^+ and f^- are measurable, hence their integrals exist (as elements of $[0, \infty]$) as per 6.11. But by the previous paragraph, the integral of $|f|$ is finite if and only if both $\int_X f^+ \, \mathrm{d}\mu$ and $\int_X f^- \, \mathrm{d}\mu$ are finite. (Use 6.12.) Thus, f is integrable if and only if $|f|$ is.

The complex case is settled by an analogous argument, by using the inequalities

$$\max\{|\mathrm{Re}(f)|, |\mathrm{Im}(f)|\} \leq |f| \leq |\mathrm{Re}(f)| + |\mathrm{Im}(f)|,$$

which, when combined with the real case handled above, show that, for a complex-valued measurable function f, the integral of $|f|$ is finite if and only if both $\int_X \mathrm{Re}(f) \, \mathrm{d}\mu$ and $\int_X \mathrm{Im}(f) \, \mathrm{d}\mu$ are. $\square$

6.29. Incidentally, it follows from the above proof that, for $f \colon X \to \mathbb{C}$ integrable, we have the estimate $\left|\int_X f \, \mathrm{d}\mu\right| \leq \int_X |f| \, \mathrm{d}\mu$.

We conclude with a few conventions on notation. Throughout, we consider a fixed measure space $(X, \mathcal{A}, \mu)$.

6.30. *Notation.*

(1) The notation $\int_X f \, \mathrm{d}\mu$ introduced earlier may also be replaced by the more verbose notation $\int_X f(x) \, \mathrm{d}\mu(x)$ if one wishes to emphasize the integration variable. In pratice, one often omits the measure altogether and simply writes $\int_X f(x) \, \mathrm{d}x$ (as for the Riemann integral); in the context of locally compact groups, it is even common to find such statements as "Let $\mathrm{d}g$ denote Haar measure on G". For the sake of clarity, we shall abstain from such abuse of notation in this note.

(2) If a measurable function f and a measurable subset $A \in \mathcal{A}$ are such that $f1_A$ is integrable on X, then we may set $\int_A f \, \mathrm{d}\mu := \int_X f 1_A \, \mathrm{d}\mu$. This is then called the *integral of f over A* (with respect to μ). Observe that the condition is automatically satisfied if f is itself integrable; more precisely, if f is integrable over X and $A \in \mathcal{A}$ is arbitrary, then $f1_A$ is again integrable over X. This is simply because $|f1_A| \leq |f|$ (and because of 6.28).

7 Uniqueness of Haar measure

Much like §5, this section deals exclusively with the proof of one result, namely:

7.1. THEOREM. *Any two left Haar measures on a locally compact topological group G are proportional. More precisely, if μ and ν are two left Haar measures on G, then there exists a positive real number $c = c(\mu, \nu)$ such that $\nu(B) = c\mu(B)$ for every Borel set $B \subseteq G$.*

While the claim is about measures, the proof we follow—which is given e.g. in [Cohn, Thm. 9.2.3] or [Gleason]—involves almost exclusively the (Lebesgue) integrals associated to the measures in question. We can provide some "preemptive motivation" for this change of perspective as follows.

7.2. Let $(X, \mathcal{A})$ be a measurable space consisting of a locally compact Hausdorff topological space X and its Borel-σ-algebra $\mathcal{A} = \mathcal{B}_X$. We saw in the previous section how to associate, to each measure μ on $(X, \mathcal{A})$, a real vector space $\mathcal{L}_1(X, \mathcal{A}, \mu; \mathbb{R})$, and how to view the Lebesgue integral with respect to μ as a linear functional on this space, which we shall now denote by I_μ.

On the other hand, as we shall see in §7.1, there exists a certain space of real-valued functions on X, denoted $C_c(X; \mathbb{R})$, with the following properties:

(1) the definition of $C_c(X; \mathbb{R})$ involves only the topology on X;

(2) $C_c(X; \mathbb{R})$ is contained in $\mathcal{L}_1(X, \mathcal{A}, \mu; \mathbb{R})$ for *every* Radon measure μ; and in this case,

(3) μ is uniquely determined by the restriction of the linear functional I_μ to $C_c(X; \mathbb{R})$.

This result essentially yields a "test" to check whether two given Radon measures μ and ν on the LCH space X are equal: it suffices to check whether integrating a function $f \in C_c(X; \mathbb{R})$ with respect to μ and ν gives the same results for every f. Thus, we may[26] (and will) establish 7.1 by proving the following intermediate assertion (where the symbols G, μ and ν retain the same meanings as in 7.1).

7.3. *There exists a positive real number $c = c(\mu, \nu)$ such that, for all $f \in C_c(G; \mathbb{R})$, the equality $\int_G f \,\mathrm{d}\nu = c \int_G f \,\mathrm{d}\mu$ holds.*

[26] For a more detailed argument, see §7.3.

Because of this change of perspective, it will benefit us to take a closer look at how the invariance properties of Haar measures translate into properties of the corresponding Lebesgue integrals: this will be the topic of §7.2. The results obtained in this subsection will become crucial in the proof of 7.1 (or 7.3) presented in §7.3.

7.1 Compactly supported functions

This subsection is essentially devoted to injecting mathematical rigour into the claims made in 7.2.

7.4. First of all, given topological spaces X and Y, we fix the notation $C(X;Y)$ for the set of all continuous functions $X \to Y$. We shall most often be concerned with the case $Y = \mathbb{K}$, where $\mathbb{K}$ denotes either of the two fields $\mathbb{R}$ and $\mathbb{C}$; in this special case, elements of $C(X;\mathbb{K})$ can be added and multiplied (be it by scalars or among themselves) pointwise, making $C(X;\mathbb{K})$ into an algebra (and in particular, a vector space) over $\mathbb{K}$. (This is well-known, but see also E.4.)

For a function $f \in C(X;\mathbb{K})$,[27] the *(topological) support* of f is defined to be the smallest closed subset of X outside of which f vanishes. In symbols,

$$\operatorname{supp} f := \overline{\{x \in X : f(x) \neq 0\}}.$$

We say that f has *compact support*, or is *compactly supported*, if $\operatorname{supp} f$ is compact in X.

One easily checks that pointwise sums and scalar multiples of [continuous and] compactly supported functions are again [continuous and] compactly supported. In other words,

$$\{f \in C(X;\mathbb{K}) : f \text{ has compact support}\}$$

is a $\mathbb{K}$-linear subspace of $C(X;\mathbb{K})$, which we may call the (sub-)*space of continuous compactly-supported* $\mathbb{K}$*-valued functions on* X. Throughout this note, this subspace will be denoted by $C_c(X;\mathbb{K})$.

[27] The definition is clearly meaningful in much more general contexts. For instance, continuity of f is not essential to the definition; moreover, one may replace $\mathbb{K}$ by any additive (topological) group A. However, we shall have no use for this added generality.

7.5. *Remark.* If X is a compact topological space and $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$, then $C_c(X;\mathbb{K})$ is all of $C(X;\mathbb{K})$. $\diamond$

7.6. If, in 7.4, we specialize to $\mathbb{K} = \mathbb{R}$, we obtain the space $C_c(X;\mathbb{R})$ whose role in the proof of our main theorem was touched upon in 7.2. It is clear that the definition of $C_c(X;\mathbb{R})$ only involves the topology on the space X, i.e., the first of the three claims made in 7.2 is settled. The second claim follows immediately from the upcoming proposition (together with the defining properties of Radon measures).

7.7. Proposition. *Let X be a Hausdorff topological space and let μ be a measure on $(X, \mathcal{B}_X)$ which is finite on compact sets. Then every continuous compactly-supported complex-valued function on X is Lebesgue-integrable with respect to μ.*

Proof. Let f be a continuous compactly-supported function $X \to \mathbb{C}$. By 6.21.(2), f is a ($\mathcal{B}_X$-$\mathcal{B}_{\mathbb{C}}$-)measurable function on X. By 6.28, it remains to show that $\int_X |f| \,\mathrm{d}\mu < \infty$. (See *loc. cit.* for the notation $|f|$.)

To that end, let K denote the support of f. (Observe that K is also the support of $|f|$.) Since K is compact in X and f is continuous, there exists a positive real number M such that $|f(x)| \leq M$ for all $x \in K$. Because $|f|$ is zero outside of K, we obtain $|f| \leq M1_K$ (recall that this notation means that the inequality holds everywhere pointwise). But then, by 6.10 and 6.12, $\int_X |f| \,\mathrm{d}\mu \leq \int_X M1_K \,\mathrm{d}\mu = M\mu(K) < \infty$, where the last inequality holds by our assumption that μ is finite on compact sets. Hence, f is indeed integrable, as claimed. $\square$

To prove the third claim, we have to use deeper facts about the topology of LCH spaces. The central result is the following version of *Urysohn's Lemma*.

7.8. Proposition. *Let X be a locally compact Hausdorff space, U be an open subset of X and K be a compact subset of U. Then there exists a continuous compactly-supported real-valued function f on X such that:*

(i) $0 \leq f(x) \leq 1$ *for all* $x \in X$*;*

(ii) $f(x) = 1$ *for all* $x \in K$*; and*

(iii) the support of f is contained in U.

(For a proof, see [Folland2, Urysohn's Lemma 4.32].)

7.9. To draw a first consequence of the previous result, it will be convenient to introduce some *ad hoc* items of notation, again following [Folland2]. Let X be a topological space, U be an open subset of X, and $f \in C_c(X;\mathbb{R})$. Then we write

$$f \prec U$$

if and only if f has values in the closed interval $[0,1]$ and the support of f is contained in U.

7.10. PROPOSITION. *Let X be a locally compact Hausdorff space and μ be a Radon measure on X. Then, for every open subset U of X,*

$$\sup\left\{\int_X f \,\mathrm{d}\mu : f \in C_c(X;\mathbb{R}), f \prec U\right\} \in [0,\infty]$$

is equal to $\mu(U)$.

Proof. First, one needs to convince oneself that the supremum exists and is meaningful; we leave this verification to the reader.

Next, fix an open subset $U \subseteq X$. Clearly, if $f \in C_c(X,\mathbb{R})$ is such that $f \prec U$, then $0 \leq \int_X f \,\mathrm{d}\mu \leq \int_X 1_U \,\mathrm{d}\mu = \mu(U)$, so the supremum appearing in the statement is certainly less than or equal to $\mu(U)$.

To show actual equality, recall that μ is inner regular by assumption, i.e.: $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{compact}\}$. Now whenever K is a compact subset of U, there exists, by the previous result, a function $f \in C_c(X;\mathbb{R})$ such that $f \prec U$ and additionally f is identically 1 on K. For such a function f, we have the inequalities

$$\mu(K) = \int_X 1_K \,\mathrm{d}\mu \leq \int_X f \,\mathrm{d}\mu \leq \mu(U);$$

thus, the integral of f is "squeezed" between $\mu(K)$ and $\mu(U)$. Inner regularity of μ completes the proof. □

With the help of the preceding result, the last and most important claim made in 7.2 follows immediately.

7.11. COROLLARY. *Let X be a locally compact Hausdorff space. and let μ and ν be Radon measures on X. Suppose that $\int_X f \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\nu$ for every $f \in C_c(X;\mathbb{R})$. Then $\mu = \nu$.*

Proof. We shall show that $\mu(U) = \nu(U)$ for every open subset U of X; by outer regularity, this is enough to prove that μ and ν agree on all Borel sets of X.

Thus, let $U \subseteq X$ be open. Applying the above proposition combined with our current assumptions, we obtain

$$\begin{aligned}\mu(U) &= \sup\left\{\int f \,\mathrm{d}\mu : f \in C_c(X;\mathbb{R}), f \prec U\right\}\\ &= \sup\left\{\int f \,\mathrm{d}\nu : f \in C_c(X;\mathbb{R}), f \prec U\right\} = \nu(U),\end{aligned}$$

and the proof is completed. □

7.2 Invariant integrals

The main goal of this subsection is to establish the following result, which, informally, reads as the assertion that the Lebesgue integral associated to a left Haar measure inherits an invariance property of sorts.

7.12. Theorem. *Let G be a locally compact group and μ be a left Haar measure on G. Then the Lebesgue integral with respect to μ is left-invariant in the following sense: for every $f \in \mathcal{L}_1(G, \mathcal{B}_G, \mu; \mathbb{C})$ and every $g \in G$, the function $f \circ l_g$ is again integrable (cf. 1.6 for the notation l_g) and has the same integral as f. In other words,*

$$\int_G f(gx) \,\mathrm{d}\mu(x) = \int_G f(x) \,\mathrm{d}\mu(x) \qquad \text{for all } f \in \mathcal{L}_1(G, \mathcal{B}_G, \mu; \mathbb{C}), g \in G.$$

Proof. Because the definitions of integrability and of the Lebesgue integral itself were given in several stages, it is natural to argue following those same stages: in other words, the claim may be first shown for nonnegative simple functions, then for nonnegative measurable functions, then for real-valued and finally for complex-valued integrable functions.

As a preliminary remark, observe that, if g is any element of G and f is a measurable complex-valued function on $(G, \mathcal{B}_G)$, then the same is true of $f \circ l_g$: this is because of 6.21.(2)–(3). (Recall that l_g is continuous by 1.7.) Moreover, since l_g is a bijection from G to itself, the functions f and $f \circ l_g$ have the same range; therefore, if f is real-valued, nonnegative, or simple, then the same is true of $f \circ l_g$. (For the last claim, recall E.29.)

In this paragraph, let f be a nonnegative simple function on $(G, \mathcal{B}_G)$, and $g \in G$ be arbitrary; our goal is to show that $\int_G (f \circ l_g)\,\mathrm{d}\mu = \int_G f\,\mathrm{d}\mu$. If f happens to be of the form 1_B for some $B \in \mathcal{B}_G$, then $f \circ l_g$ is none other than the indicator function $1_{g^{-1}B}$; using 6.10.(0), the left-invariance of μ immediately yields

$$\int_G f(gx)\,\mathrm{d}\mu(x) = \int_G 1_{g^{-1}B}\,\mathrm{d}\mu = \mu(g^{-1}B) = \mu(B) = \int_G f\,\mathrm{d}\mu\,,$$

as desired. In general, f is a finite linear combination of indicator functions of Borel sets (as seen in 6.7.(3)), so the claim readily follows by linearity from the above special case.

Now, let f denote a nonnegative measurable function on $(G, \mathcal{B}_G)$, and let g again denote an arbitrary element of G. Then, by 6.26, f satisfies (SM) from 6.11, so there exist nonnegative simple functions f_1, f_2, ... such that $f_1 \leq f_2 \leq \cdots$ and $f = \lim f_n$ (pointwise). By our preliminary remarks, we obtain that, for every $n \geq 1$, the function $\tilde{f}_n \coloneqq f_n \circ l_g$ is again nonnegative and simple; that $\tilde{f}_1 \leq \tilde{f}_2 \leq \cdots$ (pointwise); and that $f \circ l_g = \lim \tilde{f}_n$ (again pointwise). By the arguments in 6.11, the limit $\lim_{n\to\infty} \int_G \tilde{f}_n\,\mathrm{d}\mu$ yields the value of the integral of $f \circ l_g$ with respect to μ; but then, using what was already proved for simple functions,

$$\begin{aligned}\int_G (f \circ l_g)\,\mathrm{d}\mu &= \lim_{n\to\infty} \int_G (f_n \circ l_g)\,\mathrm{d}\mu \\ &= \lim_{n\to\infty} \int_G f_n\,\mathrm{d}\mu \\ &= \int_G f\,\mathrm{d}\mu\,,\end{aligned}$$

where the equality holds in $[0, \infty]$. In particular, $f \circ l_g$ is integrable if and only if f is.

The rest of the proof is straightforward: to extend the validity of the claim to all real-valued and then all-complex valued integrable functions, one takes advantage of the linearity of the integral to fall back on the nonnegative case. The details are omitted. $\square$

7.13. *Remark on the proof.* The equality $\int_G (f \circ l_g)\, \mathrm{d}\mu = \lim \int_G (f_n \circ l_g)\, \mathrm{d}\mu$ seen in the above proof involves interchanging an integral sign and a (pointwise) limit of functions; since changing the order of these operations might alter the end result in general, one has to argue with particular care here.

At this stage in the proof, some authors (cf. [Cohn, p. 305] and, expanding on this, [Gleason, Lemma 4.5]) invoke the *monotone convergence theorem*, a fundamental result of Lebesgue integration theory which gives a sufficient condition for when the interchange of limit and integral is permitted. Since $(f_n \circ l_g)_{n\geq 1}$ is a monotonically increasing sequence of nonnegative measurable function converging pointwise to $f \circ l_g$, the assumptions of the theorem are fulfilled and so any potential concerns are swiftly put to rest.

By contrast, in our approach we avoid applying the monotone convergence theorem by observing that each $f_n \circ l_g$ is additionally *simple*; thus, the fact that the integrals of $f_n \circ l_g$ converge to that of $f \circ l_g$ as n goes to ∞ follows directly from the "well-definedness" of the integral of a nonnegative measurable function discussed in 6.11. $\diamond$

7.14. Let G be a locally compact group and μ be a *right* Haar measure on G. Then the argument given in the above proof—with left translation maps l_g replaced by *right* translation maps r^g (cf. 1.6) throughout and a few other obvious changes— shows the "mirror version" of 7.12, i.e.: for every $f \in \mathcal{L}_1(G, \mathcal{B}_G, \mu; \mathbb{C})$ and every $g \in G$, the function $f \circ r^g$ is again integrable and has the same integral as f, or in symbols,

$$\int_G f(xg)\, \mathrm{d}\mu(x) = \int_G f(x)\, \mathrm{d}\mu(x) \qquad \text{for all } f \in \mathcal{L}_1(G, \mathcal{B}_G, \mu; \mathbb{C}), g \in G.$$

7.3 The core of the uniqueness proof

Having argued how to reduce the main theorem 7.1 to the assertion 7.3, we now set out to prove the latter. To this end, we first provide an elementary reformulation of the claim which will bring us one step closer to the actual (technical) core of the proof.

First, we observe that 7.3 entails—and is in fact equivalent to—the following pair of assertions (with notations being as in the original statement of 7.1):

(a) for every $f \in C_c(G; \mathbb{R})$, the integrals $\int_G f\, \mathrm{d}\mu$ and $\int_G f\, \mathrm{d}\nu$ are either both zero or both nonzero; and

(b) if for a certain concrete function $f_0 \in C_c(G;\mathbb{R})$ they are both nonzero, then

$$\int_G f \, \mathrm{d}\nu = \frac{\int_G f_0 \, \mathrm{d}\nu}{\int_G f_0 \, \mathrm{d}\mu} \int_G f \, \mathrm{d}\mu \,,$$

or equivalently

$$\frac{\int_G f \, \mathrm{d}\mu}{\int_G f_0 \, \mathrm{d}\mu} = \frac{\int_G f \, \mathrm{d}\nu}{\int_G f_0 \, \mathrm{d}\nu},$$

for every $f \in C_c(G;\mathbb{R})$.

In this connection, the usefulness of the following result should be apparent.

7.15. Proposition. *Let G be a locally compact group, and let $f \in C_c(X;\mathbb{R})$ be nonnegative and not identically zero. Then, for every left [right] Haar measure μ on G, the integral $\int_G f \, \mathrm{d}\mu$ is strictly greater than zero.*

Proof. By the assumptions on f, there exists an element x_0 of X such that $f(x_0)$ is strictly positive; if we let ε denote this function value and

$$U := \left\{ x \in X : f(x) > \frac{\varepsilon}{2} \right\} \subseteq X,$$

then U is open (by continuity of f) and non-empty, since it contains x_0. By construction, we have the inequality $f \geq \frac{\varepsilon}{2} 1_U$ pointwise.

Now let μ be a left [right] Haar measure on G. Then we know from 4.21.(i) that $\mu(U)$ is strictly positive. By the basic properties of the integral, we then find that

$$\int_X f \, \mathrm{d}\mu \geq \frac{\varepsilon}{2} \int_X 1_U \, \mathrm{d}\mu = \frac{\varepsilon}{2} \mu(U) > 0,$$

as claimed. □

Thanks to this result and the discussion preceding it, we conclude that, in order to prove 7.1, it suffices to establish the following.

7.16. Proposition. *Let G be a locally compact group and let f, f_0 be elements of $C_c(G;\mathbb{R})$, with f_0 nonnegative and not identically zero. Then there exists a real number $C = C(f, f_0)$ such that, for every left Haar measure μ on G, the equality*

$$\frac{\int_G f \, \mathrm{d}\mu}{\int_G f_0 \, \mathrm{d}\mu} = C$$

holds.

(The salient point is that C is *independent of the choice of Haar measure on G*.)

Proof. By 5.1, we may pick a left Haar measure μ_0 on G. For $x \in G$, set

$$\Psi(x) := \int_G f_0(tx)\,\mathrm{d}\mu_0(t)\,.$$

Because $t \mapsto f_0(tx)$ is (like f_0 itself) a nonnegative continuous compactly-supported function on G which is not identically zero, we have that $\Psi(x) > 0$ for every x by the previous result 7.15. This yields a function Ψ from G into the positive real numbers.

7.17. Lemma. *The function Ψ introduced above is continuous.*

(See [Cohn, Corollary 9.1.5] for the proof.)

Now consider the function $G \to \mathbb{R}$ given by $y \mapsto \frac{f(y^{-1})}{\Psi(y^{-1})}$. (This is well-defined because Ψ never vanishes.) It is readily verified that this function is again continuous and compactly supported, so its integral

$$\int_G \frac{f(y^{-1})}{\Psi(y^{-1})}\,\mathrm{d}\mu_0(y) =: C$$

exists. We claim that this is precisely the constant $C = C(f, f_0)$ whose existence is claimed in the statement, i.e., that for every left Haar measure μ on G, the equation $C \int_G f_0\,\mathrm{d}\mu = \int_G f\,\mathrm{d}\mu$ holds. To show this, we will first rewrite the left-hand side, which is a product of two integrals over G, as a *double integral*. This is achieved by applying (a suitable version of) *Fubini's Theorem*, namely:

7.18. Fubini's Theorem. *Let X and Y be locally compact Hausdorff spaces, equipped with Radon measures μ and ν respectively, and let f be an element of $C_c(X \times Y)$. Then:*

(i) for every $x \in X$, the map $y \mapsto f(x, y)$ belongs to $C_c(Y)$, and for every $y \in Y$, the map $x \mapsto f(x, y)$ belongs to $C_c(X)$;

(ii) the two maps

$$x \mapsto \int_Y f(x, y)\,\mathrm{d}\nu(y) \qquad \textit{and} \qquad y \mapsto \int_X f(x, y)\,\mathrm{d}\mu(x)$$

belong to $C_c(X)$ and $C_c(Y)$, respectively; and *(iii) the equality*

$$\int_X \int_Y f(x,y)\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x) = \int_Y \int_X f(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)$$

holds.

(See [Cohn, Prop. 7.6.4] for a proof.)

From now on, we choose a fixed but arbitrary left Haar measure μ on G (in addition to the left Haar measure μ_0 which was used to define Ψ and C above). Recall that our aim is to show that $C\int_G f_0\,\mathrm{d}\mu = \int_G f\,\mathrm{d}\mu$.

First, consider the function $h\colon G\times G\to\mathbb{R}$ given by

$$h(x,y) \coloneqq \frac{f(y^{-1})f_0(x)}{\Psi(y^{-1})}.$$

It is easy to see that h is continuous and compactly supported, so we may consider double integrals of h (with respect to μ and μ_0) as per Fubini's Theorem. More importantly, the variables in h can be "separated", so the integral can be computed as follows:

$$\begin{aligned}\int_G\int_G h(x,y)\,\mathrm{d}\mu_0(y)\,\mathrm{d}\mu(x) &= \int_G\int_G \frac{f(y^{-1})f_0(x)}{\Psi(y^{-1})}\,\mathrm{d}\mu_0(y)\,\mathrm{d}\mu(x)\\ &= \int_G f_0(x)\Big(\int_G \frac{f(y^{-1})}{\Psi(y^{-1})}\,\mathrm{d}\mu_0(y)\Big)\,\mathrm{d}\mu(x)\\ &= C\int_G f_0\,\mathrm{d}\mu\,.\end{aligned}$$

(Recall the definition of C.) In conclusion, we have indeed accomplished writing $C\int_G f_0\,\mathrm{d}\mu$ as a double integral.

Next, for $x,y\in G$, set

$$\tilde{h}(x,y) \coloneqq h(yx,x^{-1}) = \frac{f(x)f_0(yx)}{\Psi(x)};$$

then $\tilde{h}$ is again in $C_c(G\times G)$. The double integral of $\tilde{h}$ equals

$$\begin{aligned}\int_G\int_G \tilde{h}(x,y)\,\mathrm{d}\mu_0(y)\,\mathrm{d}\mu(x) &= \int_G\int_G \frac{f(x)f_0(yx)}{\Psi(x)}\,\mathrm{d}\mu_0(y)\,\mathrm{d}\mu(x)\\ &= \int_G f(x)\frac{\int_G f_0(yx)\,\mathrm{d}\mu_0(y)}{\Psi(x)}\,\mathrm{d}\mu(x)\\ &= \int_G f(x)\,\mathrm{d}\mu(x)\,.\end{aligned}$$

(Recall the definition of Ψ!) Therefore, the proof will be complete if we can show that $\iint \tilde{h}\,\mathrm{d}\mu_0\,\mathrm{d}\mu = \iint h\,\mathrm{d}\mu_0\,\mathrm{d}\mu$. This turns out to be a simple application of the left-invariance of the integrals corresponding to the left Haar measures μ and μ_0, as illustrated in 7.12; in particular, this does not use the concrete form of h but only the relation $\tilde{h}(x,y) = h(yx, x^{-1})$ linking h and $\tilde{h}$.

In more detail, consider $\varphi\colon G \to \mathbb{R}$ given by

$$\varphi(x) := \int_G \tilde{h}(x,y)\,\mathrm{d}\mu_0(y)\,;$$

by part (ii) of Fubini's Theorem 7.18, $\varphi \in C_c(G)$. Because μ is a left Haar measure, the Lebesgue integral with respect to μ is left-invariant as per 7.12, so

$$\int_G \varphi(y^{-1}x)\,\mathrm{d}\mu(x) = \int_G \varphi(x)\,\mathrm{d}\mu(x)\,.$$

But the integral on the right-hand side is none other than the double integral of $\tilde{h}$ as per part (iii) of Fubini's Theorem, whereas the left-hand side equals

$$\int_G \int_G h(x, x^{-1}y)\,\mathrm{d}\mu_0(y)\,\mathrm{d}\mu(x)\,.$$

Similar arguments yield

$$\int_G \int_G h(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu_0(y) = \int_G \int_G h(x, x^{-1}y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu_0(y)\,.$$

(It is left as an exercise to the reader to check the details.) Combining all these different equalities and recalling again part (iii) of Fubini's Theorem, we obtain the sought-after equality $\iint \tilde{h}\,\mathrm{d}\mu_0\,\mathrm{d}\mu = \iint h\,\mathrm{d}\mu_0\,\mathrm{d}\mu$ and, in so doing, complete the proof. $\square$

To wrap up the subsection, here is a summary of the complete argument which proves 7.1 (and, incidentally, 7.3).

Proof of 7.1. (Throughout the proof, the notations G, μ and ν retain the same meaning as in the statement of 7.1.)

Pick a nonnegative function $f_0 \in C_c(G;\mathbb{R})$ which is not identically zero (f_0 will be fixed for the entirety of the proof). Then, by 7.15, both $\int_G f_0\,\mathrm{d}\mu$ and $\int_G f_0\,\mathrm{d}\nu$ are strictly positive. Let c denote the ratio of the latter by the former.

Now let f be an arbitrary element of $C_c(G;\mathbb{R})$. By 7.16, we have the equality

$$\frac{\int_G f\,\mathrm{d}\mu}{\int_G f_0\,\mathrm{d}\mu} = \frac{\int_G f\,\mathrm{d}\nu}{\int_G f_0\,\mathrm{d}\nu},$$

or equivalently,

$$\int_G f\,\mathrm{d}\nu = \underbrace{\frac{\int_G f_0\,\mathrm{d}\nu}{\int_G f_0\,\mathrm{d}\mu}}_{=c}\int_G f\,\mathrm{d}\mu\,.$$

This proves 7.3.

Finally, let μ' denote the measure $\mu' \colon \mathcal{B}_G \to [0,\infty]$ defined by $\mu'(B) := \frac{1}{c}\nu(B)$ for every $B \in \mathcal{B}_G$. Then $\int_G f\,\mathrm{d}\mu = \int_G f\,\mathrm{d}\mu'$ for every $f \in C_c(G;\mathbb{R})$. By 7.11, it follows that μ and μ' are the same measure, i.e., $\nu(B) = c\mu(B)$ for every $B \in \mathcal{B}_G$, as desired. $\square$

(Essential) uniqueness of left Haar measure is clearly a pivotal statement which immediately makes accessible a plethora of useful consequences. This will become more and more apparent in the course of the next section. For the time being, we only record three immediate corollaries.

7.19. Corollary. *Any two right Haar measures on a locally compact topological group G are proportional.*

Proof. Let μ and ν be right Haar measures on G; we want to show that there exists a positive real number c such that $\nu = c\mu$, i.e.: $\nu(B) = c\mu(B)$ for all Borel sets $B \subseteq G$.

By 4.20.(1), we obtain left Haar measures $\mu^\vee$ and $\nu^\vee$ on G. Then, by 7.1, there exists a positive real number c such that $\nu^\vee = c\mu^\vee$. But then it follows that $\nu = c\mu$, as claimed. $\square$

7.20. Corollary. *The following statements are equivalent for a locally compact group G.*

(i) G possesses a bi-invariant Haar measure (as per 4.18).

(ii) Every left [right] Haar measure on G is bi-invariant.

The proof is an immediate application of the theorem and will therefore be omitted.

7.21. Corollary. *Let G be a locally compact group, μ and μ' two left [right] Haar measures on G. Then a [Borel-measurable] complex-valued function f on G is integrable with respect to μ if and only if it is integrable with respect to μ'.*

This is again immediate from 7.1.

7.4 Digression: the Riesz Representation Theorem

In the above discussion, we have brushed upon the intimate connection between Radon measures on a locally compact Hausdorff space X and their respective Lebesgue integrals, viewed as linear functionals on $C_c(X;\mathbb{R})$. In this connection, there is a central result which we deem absolutely worth mentioning, even though our proofs of existence and uniqueness of Haar measure do not require its full strength: the *Riesz Representation Theorem*.

This result can be put into context as follows. Recall that we have already noted that every Radon measure μ on X (with X being as above) yields a linear functional I_μ on $C_c(X;\mathbb{R})$, namely

$$f \mapsto I_\mu(f) := \int_X f \, \mathrm{d}\mu \, .$$

Every functional arising in this way is additionally *positive* in the sense that it maps nonnegative functions to nonnegative real numbers. (I.e.: if $f \in C_c(X;\mathbb{R})$ only attains nonnegative values, then $I_\mu(f) \geq 0$.) The Riesz Representation Theorem tells us that every positive linear functional I on $C_c(X;\mathbb{R})$ can be *represented* as I_μ for a unique Radon measure μ on X:

7.22. Riesz Representation Theorem. *Let X be a LCH topological space, and let I be a linear functional on $C_c(X;\mathbb{R})$ which is* positive *in the sense that $I(f) \geq 0$ whenever f is nonnegative. Then there exists a unique Radon measure μ on X such that $I(f) = \int_X f \, \mathrm{d}\mu$ for every $f \in C_c(X;\mathbb{R})$.*

7.23. *Remarks on terminology.*

(1) This result, more accurately named the *Riesz-Markov-Kakutani representation theorem*, is unrelated to the *Riesz* (or *Riesz-Fréchet*) *representation theorem* from functional analysis, which is a result on the dual spaces of Hilbert spaces.

(2) There exist other versions of the above result—also dubbed "the" Riesz Representation Theorem—in which the positive linear functionals on $C_c(X;\mathbb{R})$ (with X being a LCH space) are shown to be in one-to-one correspondence with a different set of measures than what we chose to call 'Radon measures' (recall that we dwelled on these terminological points in 4.14.(1)–(2)). For instance, the version proved in [Rudin, Thm. 2.14] shows that, for a LCH space X, the positive linear functionals on $C_c(X;\mathbb{R})$ are in bijection with pairs $(\mathfrak{M}, \overline{\mu})$ such that:

   (a) $\mathfrak{M} \subseteq \mathcal{P}(X)$ is a σ-algebra on X containing the Borel-σ-algebra $\mathcal{B}_X$;

   (b) $\overline{\mu}$ is a measure on $(X, \mathfrak{M})$ which is finite on compact sets, outer regular on all $E \in \mathfrak{M}$, and inner regular on all open sets *as well as* all sets $E \in \mathfrak{M}$ of finite $\overline{\mu}$-measure; and

   (c) $(X, \mathfrak{M}, \overline{\mu})$ is a *complete measure space.*

   (The bijection being again, at least in one direction, given by integrating: the positive linear functional corresponding to $(\mathfrak{M}, \overline{\mu})$ is $I: f \mapsto \int_X f \, \mathrm{d}\overline{\mu}$.) Intuitively, Rudin's version implies 7.22 simply by setting $\mu = \overline{\mu}|_{\mathcal{B}_X}$; conversely, if I and μ are as in 7.22, then one may take $\overline{\mu}$ to be the *saturation* of the *completion* of μ (cf. 3.16 and E.19). (This choice also determines $\mathfrak{M}$ as being, in the notation of 3.16, simply $\mathcal{M}(\mu^*)$.) Of course, one has to check the details, namely that $\int_X f \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\overline{\mu}$ for all $f \in C_c(X;\mathbb{R})$; this is essentially the content of [Cohn, Prop. 7.2.11].

(3) In Bourbaki's terminology, a *measure* on a locally compact Hausdorff topological space X is, by definition, a positive linear functional on $C_c(X;\mathbb{R})$. Thus, while our notion of a measure (cf. 3.10) was much more general, on a locally compact Hausdorff space Bourbaki's 'measures' are in one-to-one correspondence with "our" Radon measures.

   Notably, the discussion in the preceding item shows that the notion of a 'Radon measure' can be defined differently than in 4.13 (for instance, by allowing more flexibility in the domain) without losing the one-to-one correspondence with positive linear functionals on $C_c(X;\mathbb{R})$ and hence, by 7.22, with Radon measures in "our" sense. As a consequence, an approach such as Bourbaki's, in which measures are equated with positive linear functionals on $C_c(X;\mathbb{R})$ (for X a LCH space) from the very start, has the indisputable advantage that it elegantly bypasses the need to use the phrase 'Radon measure' and sidelines all the potential ambiguity which might otherwise arise. (The downside being, of course, that Bourbaki's definition is hard to fully motivate—or reconcile with intuition, or indeed most modern bibliographical sources—without assuming knowledge of such an advanced result as the Riesz Representation Theorem.) ◇

Sketch of proof of 7.22. One immediately recognizes that the uniqueness part of the statement is merely a reformulation of 7.11. To prove existence, one has to construct a Radon measure μ on X and show that I is indeed given by the Lebesgue integral with respect to μ.

For the first half, we can rely on our previous results 4.8, 4.17. Indeed, set

$$h(K) \coloneqq \inf\{I(f) : f \in C_c(X;\mathbb{R}), f \geq 1_K\}$$

for $K \subseteq X$ compact. Then h satisfies the assumptions (h1)–(h3) of 4.8. (To obtain (h3), one needs to use Urysohn's Lemma 7.8, cf. [Elst, Kapitel VIII, Lemma 2.2].) Accordingly, 4.17 yields a Radon measure μ on X. As for the proof that $I(f) = \int_X f \, \mathrm{d}\mu$ for all $f \in C_c(X;\mathbb{R})$, we refer the reader to [Cohn, Prop. 7.2.11] or claim (iv) in the proof of [Folland2, Thm. 7.2]. □

7.24. *Remark on the proof.* The argument given above makes use of the technical proposition 4.8, which in turn relies on Carathéodory's extension theorem 3.14. Alternatively, as explained in 7.23.(2), one can derive 7.22 from [Rudin, Thm. 2.14], whose proof (as given *ibidem*) remarkably does *not* invoke Carathéodory's result. ◇

7.25. The Riesz Representation Theorem is key in proving several results about Radon measures on LCH spaces, see e.g. [Folland2, §7.2]. Most of these are beyond the scope of this note, but we have opted to record one particular result which characterizes Radon measures on "sufficiently well-behaved" LCH topological spaces.

7.26. Proposition. *Let X be a LCH space, and suppose additionally that X* is second-countable*. Then any measure μ on $(X, \mathcal{B}_X)$ which is finite on compact sets is automatically inner regular on open sets and outer regular on Borel sets, i.e. a Radon measure.*

(See e.g. [Folland2, Thm. 7.8].)

7.27. Finally, to conclude our digression, we shall spend a few words on how an implicit or explicit application of the Riesz Representation Theorem can affect the way in which the main results of this chapter—namely: existence and essential uniqueness of Haar measure—are sometimes stated, proved or presented in the literature. To this end, we posit a definition.

Let G be a locally compact group, I be a positive linear functional on $C_c(G;\mathbb{R})$. We call I a *left Haar integral* on G if I is not identically zero and is *left-invariant* in the sense that $I(f \circ l_g) = I(f)$ for all $f \in C_c(G;\mathbb{R})$ and all $g \in G$ (with l_g being, as usual, as in 1.6). Analogously, a *right Haar integral* is a nonzero positive linear functional I on $C_c(G;\mathbb{R})$ which is *right-invariant* in the sense that $I(f \circ r^g) = I(f)$ for all f and all g.

By 7.12 [or 7.14 as the case may be] together with 7.7, every left [right] Haar measure μ yields a left [right] Haar integral via $f \mapsto \int_G f \, \mathrm{d}\mu$. Conversely, the Riesz Representation Theorem 7.22 implies the following:

7.28. Proposition. *Let G be a locally compact group, and let I be a left [right] Haar integral on G. Then there exists a unique left [right] Haar measure μ on G such that $I(f) = \int_X f \, \mathrm{d}\mu$ for all $f \in C_c(G;\mathbb{R})$.*

Sketch of proof. Clearly one takes μ to be the measure associated to I by the Riesz Representation Theorem; it only remains to check that μ is left- [right-]invariant and nonzero.

The latter claim is trivial: if μ is identically zero, then so is I (indeed, recall the step-by-step definition of the integral covered in §6), contradicting the assumption that I is a left [right] Haar integral. We thus set out to show that, if I is left-invariant, then μ is. (The arguments can easily be applied to the right-invariant case *mutatis mutandis*.) Thus, let B be a Borel set in G and $g \in G$; we need to show that $\mu(gB) = \mu(B)$.

We shall first restrict to the case where $B = U$ is an open set; by applying 7.10 (with the notations used *ibidem*), we obtain

$$\mu(U) = \sup\{I(f) : f \in C_c(G;\mathbb{R}), f \prec U\},$$
$$\mu(gU) = \sup\{I(f) : f \in C_c(G;\mathbb{R}), f \prec gU\}.$$

We now observe that, if $f \in C_c(G;\mathbb{R})$ and $g \in G$, then $f \circ l_g$ again belongs to $C_c(G;\mathbb{R})$, and that f satisfies $f \prec gU$ if and only if $f \circ l_g \prec U$. (The latter is an easy computation using the results of §1). Thus, $\mu(gU)$ can also be expressed as the supremum of the set $\{I(f \circ l_g) : f \in C_c(G;\mathbb{R}), f \prec U\}$, which by left-invariance of I is none other than the set appearing on the right-hand side the first of the two equations above. In conclusion, $\mu(gU) = \mu(U)$. Outer regularity then allows one to conclude that μ is indeed left-invariant on all Borel sets, as claimed. (The argument was given in more detail in the proof of 5.1, in §5.3.) $\square$

It is, moreover, self-evident that two Haar integrals are proportional if and only if the corresponding Haar measures are. Thus, in each of the four fundamental statements 5.1, 5.16, 7.1 and 7.19 (concerning existence and essential uniqueness of left [right] Haar measures), swapping out the words 'Haar measure(s)' for 'Haar integral(s)' yields an equivalent statement.

Authors may choose to take advantage of this equivalence in different ways. For instance, the proof of existence of Haar measure we gave in §5 has no need for integrals; on the other hand, our proof of [essential] uniqueness of left Haar measure in the present section was, arguably, a proof of [essential] uniqueness of the left Haar *integral* in disguise! (We then leveraged the "uniqueness half" of the Riesz Representation Theorem, which we proved in full in §7.1, to obtain the corresponding results for measures.) In this regard, our approach is similar to [Cohn]'s. By contrast, other authors prove both the existence and the uniqueness of Haar measure (almost) entirely at the level of linear functionals: as we already mentioned in passing at the beginning of §5, this is the choice made e.g. in [Folland2] and [Joys], and obviously (in view of 7.23.(3)) it is also the approach followed by [Bourbaki]. All in all, it is our hope that this discussion will be useful to the reader navigating the literature and will defuse potential sources of confusion early on.

8 Determining Haar measure

As a way to wrap up this chapter, we have chosen to devote a large part of the final section to the following question: given a locally compact group, how does one concretely and efficiently determine Haar measure on that group?

Of course, fully answering this question means not only exhibiting left [or right] Haar measure for a number of familiar groups (such as the ones that already appeared as examples in §§1–2), but most importantly providing general results which we can potentially apply again as new examples of locally compact groups appear and become relevant. Accordingly, in §8.1 we will present several results of this type, deferring applications to §8.2 for the most part.

Throughout these discussions, left and right Haar measures will be treated parallelly (but separately), as was already copiously done in previous sections. In the third subsection, we shall break from this tradition and take up the question of when the left and right Haar measures on a given group actually coincide—i.e., we will investigate which groups have bi-invariant Haar measure (cf. 7.20). As we shall see, all compact (Hausdorff) groups fall under this category, and so we will devote the final subsection §8.4 to presenting a few additional results which specifically hold true for compact groups, as well as an application which will become quite prominent in Chapter II.

All in all, the contents of this section prove to be both varied and quite involved. In fact, in many cases proofs will have to be sketched, and we will rely on external sources more than in previous sections. Accordingly, it seems impractical to include a list of references in this introductory discussion, as was done for other sections; instead, references will be provided for individual results wherever appropriate.

8.1 Subgroups, products, quotients

As the title suggests, the current subsection will mostly deal with how Haar measures "propagate" across standard constructions such as taking subgroups, products or quotients of locally compact groups.

We start with a general result relating Haar measure of a subgroup to Haar measure of the ambient group, for whose statement and proof we are indebted to SE1335588.

8.1. Proposition. *Let G be a locally compact group, H be a locally compact subgroup of G, and μ be a fixed left [right] Haar measure on G. Suppose that H has nonzero measure with respect to μ, i.e. that $\mu(H)$ is strictly positive. Then for every left [right] Haar measure ν on H there exists a positive real number c such that $\nu = c\mu|_{\mathcal{B}_H}$, i.e.: $\nu(B) = c\mu(B)$ for every Borel subset B of H.*

8.2. *Remarks on the statement.*

(1) To see that the notation $\mu|_{\mathcal{B}_H}$ is meaningful, recall from 2.10 that, in a Hausdorff topological group, a locally compact subgroup is necessarily closed, and hence in particular Borel; thus, if G and H are as in the above statement, then by footnote 21 it holds that $\mathcal{B}_H = \{B \in \mathcal{B}_G : B \subseteq H\} \subseteq \mathcal{B}_G$.

(2) Clearly the result fails in general if one lifts the assumption that $\mu(H) > 0$: it suffices to consider the inclusion $H = \mathbb{Z} \subset \mathbb{R} = G$, with the larger group equipped with Lebesgue measure $\mu = \lambda$ (and the discrete subgroup $\mathbb{Z}$ equipped with $\nu =$ counting measure). $\diamond$

Proof of 8.1. By 7.1 [or 7.19], it suffices to show that $\mu' := \mu|_{\mathcal{B}_H} : \mathcal{B}_H \to [0, \infty]$ is a left [right] Haar measure on H. Clearly μ' is again left- [right-]invariant, and μ' is not identically zero by the assumption $\mu'(H) = \mu(H) > 0$. To check that μ' is a Radon measure, it is convenient to use the following result.

8.3. Lemma. *In a locally compact group, a Borel-measurable subgroup of positive left [right] Haar measure is automatically open.*

(This is [a version of] the *Steinhaus theorem*, see SE1258647, cf. also [Cohn, Prop. 1.4.8].)

The proof is then completed by E.32.(b). $\square$

The next construction we consider is taking (finite)[28] products of locally compact groups. We start by considering binary products:

8.4. Theorem. *Let G_1 and G_2 be locally compact groups, and let G denote the product $G_1 \times G_2$. Then, given left [right] Haar measures μ_1 and μ_2 on G_1 and G_2, respectively, there exists precisely one left [right] Haar measure $\mu =: \mu_1 \times \mu_2$ on G with the property that, for all $f \in C_c(G; \mathbb{R})$,*

$$\int_G f \, \mathrm{d}\mu = \int_{G_1} \int_{G_2} f \, \mathrm{d}\mu_2 \, \mathrm{d}\mu_1 = \int_{G_2} \int_{G_1} f \, \mathrm{d}\mu_1 \, \mathrm{d}\mu_2 \qquad (\textstyle\int = \iint = \iint)$$

[28] Recall from 2.9–2.10 that *finite* products of locally compact groups are always locally compact, whereas products with an *arbitrary* (infinite) number of factors need not be. The situation is different when restricting to compact groups, see §8.4 and in particular 8.49.

(where the expressions on the right are indeed well-defined and equal to each other by Fubini's Theorem 7.18). Additionally, every left [right] Haar measure ν on G arises as $\nu_1 \times \nu_2$ for suitable left [right] Haar measures ν_1 and ν_2 on G_1 and G_2, respectively.

Proof. We start by proving the first claim, i.e. existence and uniqueness of μ for given μ_1 and μ_2.

Uniqueness is quite clear: suppose that both μ and μ' are left [right] Haar measures on G for which the claim holds true. Then by 7.1 [or 7.19], there exists a positive real number c such that $\mu' = c\mu$. Now pick a nonnegative $f \in C_c(G; \mathbb{R})$ which is not identically zero. Then f is integrable with respect to both μ and μ', by 7.7, and clearly $\int_G f \, \mathrm{d}\mu' = c \int_G f \, \mathrm{d}\mu$; both integrals are strictly positive by 7.15. But by our choice of μ and μ', both integrals must also be equal to, say, the double integral $\iint f \, \mathrm{d}\mu_2 \, \mathrm{d}\mu_1$. It follows that $c = 1$, i.e. $\mu' = \mu$, as claimed.

As for existence, consider the map $C_c(G; \mathbb{R}) \to \mathbb{R}$ given by

$$f \mapsto I(f) := \iint f \, \mathrm{d}\mu_2 \, \mathrm{d}\mu_1 = \iint f \, \mathrm{d}\mu_1 \, \mathrm{d}\mu_2 \, .$$

It is easily checked that I defines a positive linear functional on $C_c(G; \mathbb{R})$; by the Riesz Representation Theorem 7.22, I is associated to a Radon measure μ on G. But I is nonzero (ultimately as a consequence of 7.15), and left- [right-]invariant (by iterated application of 7.12 [7.14]). Thus, by 7.28, the Radon measure μ corresponding to I is a left [right] Haar measure on G. The equality of the three integrals now holds true by construction.

The final claim is an easy consequence of 7.1 [or 7.19]: if ν is a left [right] Haar measure on G, then $\nu = C\mu$ for some positive real number C, so $\nu = \nu_1 \times \nu_2$ for, say, $\nu_1 := C\mu_1$ and $\nu_2 := \mu_2$. $\square$

8.5. *Remark.* A part of the above argument—the one which invokes Fubini's Theorem 7.18 and the Riesz Representation Theorem 7.22—can actually be applied more generally to obtain, from Radon measures μ_1, μ_2 on LCH topological spaces X, Y respectively, a Radon measure $\mu = \mu_1 \times \mu_2$ on the product space $X \times Y$ satisfying the equality ($\int = \iint = \iint$) (with G_1, G_2 and G replaced by X, Y and $X \times Y$, respectively) for all $f \in C_c(X \times Y; \mathbb{R})$. In [Folland2, p. 227], $\mu_1 \times \mu_2$ is called the *Radon product* of μ_1 and μ_2; in [Cohn, p. 245], it is called the *regular Borel product* of μ_1 and μ_2. (This is consistent with the fact that "our" Radon measures are called *regular Borel measures* in Cohn's book, cf. *op. cit.*, p. 206.) $\diamond$

8.6. *Caveat.* In measure theory, there is a general theory of *product measures*—more precisely,

(a) given measurable spaces $(X, \mathcal{A})$, $(Y, \mathcal{B})$, the Cartesian product $X \times Y$ can be equipped with a *product σ-algebra*[29] $\mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{P}(X \times Y)$; and

(b) given measures μ_1, μ_2 on $(X, \mathcal{A})$, $(Y, \mathcal{B})$ respectively, there is a corresponding *product measure* on the measurable space $(X \times Y, \mathcal{A} \otimes \mathcal{B})$, also customarily denoted by $\mu_1 \times \mu_2$.

(For details, we refer to [Folland2, §2.5] or [Cohn, Chapter 5], but note that, in the latter source, the product measure is only defined in case both measure spaces $(X, \mathcal{A}, \mu_1)$ and $(Y, \mathcal{B}, \mu_2)$ are σ-finite.)

Our warning to the reader is that, in general, *the Radon product of two Radon measures* (as per the above remark 8.5) *is* not *the same as the product measure* discussed in (b) above. More precisely, if μ_1, μ_2 are Radon measures on LCH spaces X, Y, respectively, then the Radon product of μ_1 and μ_2, which is a measure on $(X \times Y, \mathcal{B}_{X \times Y})$, is *not* necessarily the same as the product measure, which "lives" on $(X \times Y, \mathcal{B}_X \otimes \mathcal{B}_Y)$—in fact, the two σ-algebras $\mathcal{B}_{X \times Y}$ and $\mathcal{B}_X \otimes \mathcal{B}_Y$ need not even be the same in general! (For more on this, we refer to [Cohn, Section 7.6] and [Folland2, §7.4].) This technical difficulty serendipitously disappears under the additional assumption that the spaces X and Y be *second-countable* (in addition to being locally compact Hausdorff). $\diamond$

Before moving onto the next topic, it is worthwhile to spend a few more words on the properties of the "product Haar measure" from the previous result. We shall use the notion of *σ-finite sets*, first introduced in the course of 3.17. We also adopt the following notational conventions: if X and Y are LCH

[29] Note that the symbol $\otimes$ is standard in this context and that some authors even use the terminology "*tensor product σ-algebra*", which might be disorienting at first.

spaces and B is a Borel subset of $X \times Y$, we set

$$B_x \coloneqq \{y \in Y : (x, y) \in B\} \subseteq Y \qquad (x \in X),$$
$$B^y \coloneqq \{x \in X : (x, y) \in B\} \subseteq X \qquad (y \in Y).$$

8.7. PROPOSITION. *Let G_1, G_2 be locally compact groups, equipped with left [right] Haar measures μ_1, μ_2, respectively, and let G denote the product $G_1 \times G_2$, equipped with the measure $\mu = \mu_1 \times \mu_2$ from 8.4. Furthermore let B be a Borel subset of G.*

(i) If $B = U$ is open, then the real-valued maps $x \mapsto \mu_2(B_x)$ and $y \mapsto \mu_1(B^y)$ (on G_1 and G_2 respectively) are Borel-measurable and

$$\mu(B) = \int_{G_1} \mu_2(B_x)\, \mathrm{d}\mu_1(x) = \int_{G_2} \mu_1(B^y)\, \mathrm{d}\mu_2(y)\,.$$

(ii) The conclusion of (i) also holds if B is contained in a set of the form $B_1 \times B_2$ where $B_1 \in \mathcal{B}_{G_1}$ [resp., $B_2 \in \mathcal{B}_{G_2}$] is σ-finite for μ_1 [resp., μ_2].

(iii) If G_1 and G_2 are σ-finite (e.g., if they are second-countable), then the conclusion of (i) holds for all Borel subsets $B \subseteq G$.

References. All three claims are special cases of results that hold more generally for Radon products (defined as in 8.5). Claim (i) is [Cohn, Prop. 7.6.5] or [Folland2, Prop. 7.25], claim (ii) is [Cohn, Cor. 7.6.6], and [the bulk of] claim (iii) can either be derived from (ii) or proved independently as in [Folland2, Thm. 7.26]. As for the other claim implicitly made in (iii)—namely that a second-countable LCH space is σ-finite—this follows from [Cohn, Prop. 7.1.5] together with 4.16. $\square$

8.8. *Remark.* Resume the notations of the above statement and pick a Borel subset $B \subseteq G$ which is of the form $B_1 \times B_2$ for σ-finite Borel subsets $B_i \subseteq G_i$, where $i = 1, 2$. Then the sections B_x and B^y (for arbitrary $x \in G_1$, resp. $y \in G_2$) are easily computed, and applying 8.7.(ii) yields

$$(\mu_1 \times \mu_2)(B_1 \times B_2) = \mu_1(B_1)\mu_2(B_2).$$

(Here, we again need the convention $0 \cdot \infty = \infty \cdot 0 \coloneqq 0$ for this to be well-defined.) If both G_1 and G_2 are second-countable, then

(i) the above equality holds for *all* $B_1 \in \mathcal{B}_{G_1}$, $B_2 \in \mathcal{B}_{G_2}$, and

(ii) $\mu_1 \times \mu_2$ is the *unique* measure on $(G, \mathcal{B}_G)$ with this property.

(Here is a brief explanation for the uniqueness statement. We mentioned in 8.6 that, under the assumption that G_1 and G_2 are second-countable, we have the equality $\mathcal{B}_G = \mathcal{B}_{G_1} \otimes \mathcal{B}_{G_2}$. But by definition, the right-hand side is the σ-algebra $\sigma(\mathcal{H})$ generated by the collection $\mathcal{H}$ of all "rectangles" $B_1 \times B_2$ with $B_1 \in \mathcal{B}_{G_1}$, $B_2 \in \mathcal{B}_{G_2}$. Moreover, we have just proved that $\mu_1 \times \mu_2$ is an extension of the map $\ell \colon \mathcal{H} \to [0, \infty]$ sending a "rectangle" $B_1 \times B_2$ to $\mu_1(B_1)\mu(B_2)$. Since G_1 and G_2 are σ-finite for μ_1 and μ_2, respectively, it follows that G is σ-finite for ℓ. Uniqueness now follows from Carathéodory's uniqueness theorem 3.18 since $\mathcal{H} \subset \mathcal{P}(G)$ is a semiring.) ◇

Iteratively applying the above results allows one to retrieve—and carry out computations with—Haar measures on finite products $G_1 \times \cdots \times G_n$ of locally compact groups for any $n \geq 2$. The precise statements read *verbatim* as those above, with the obvious modifications, and can easily be derived by the interested reader.

As for us, we shall continue our discussion by stating the third major result of this subsection, 8.10, which concerns Haar measures on quotients.

8.9. *Bibliographical remark.* Our phrasing of the statement is essentially identical to that of [Bourbaki, Chapter VII, §2, no. 7, Prop. 10]. In *op. cit.*, the claim is derived from a result (§2, no. 6, Thm. 3) regarding *relatively invariant measures* (defined in *op. cit.*, Chapter VII, §1, no. 1, Def. 1) on *homogeneous spaces*, i.e. quotient spaces of the form G/H where H is a closed but not necessarily normal subgroup of the locally compact group G. By contrast, the proof we present will specialize arguments coming from the theory of *quasi-invariant* measures on homogeneous spaces, as covered e.g. in [Folland, §2.6] or [Kan-Tay, Section 1.3] (but also in [Bourbaki, Chapter VII, §2, no. 5]). This latter theory is indispensable when dealing with *induced representations* of locally compact groups in full generality, cf. [Kan-Tay, Chapter 2] and [Folland, Chapter 6]. ◇

8.10. Theorem. *Let G be a locally compact group, H be a closed normal subgroup, and let μ and ν denote fixed left Haar measures on G and H, re-*

spectively. Additionally, choose a left Haar measure $\dot{\mu}$ on G/H. Then, upon scaling $\dot{\mu}$ by a positive scalar if necessary, we have Weil's integration formula*:*

$$\int_G f(g)\,\mathrm{d}\mu(g) = \int_{G/H}\int_H f(gh)\,\mathrm{d}\nu(h)\,\mathrm{d}\dot{\mu}(\dot{g}) \qquad \textit{for all } f \in C_c(G;\mathbb{R}),$$

where $\dot{g}$ denotes the integration variable ranging over the elements of G/H.

8.11. *Remarks on the statement.*

(1) Recall that the quotient G/H is locally compact by 2.9, so the choice of a left Haar measure $\dot{\mu}$ on G/H is possible by 5.1.

(2) The statement requires some clarification as to the meaning of the nested integrals on the right-hand side: the integration variable for the outer integral is $\dot{g}$, but the integrand (i.e., in this case, the inner integral) does not ostensibly depend on $\dot{g}$.
First, for $f \in C_c(G;\mathbb{R})$, consider the map $G \to \mathbb{R}$ defined by

$$g \mapsto \int_H f(gh)\,\mathrm{d}\nu(h)\,.$$

This map is constant on (left) H-cosets by construction: indeed, if g' lies in the coset gH, i.e. if $g' = gh'$ for some $h' \in H$, then

$$\begin{aligned}\int_H f(g'h)\,\mathrm{d}\nu(h) &= \int_H f(gh'h)\,\mathrm{d}\nu(h)\\ &= \int_H f(gh)\,\mathrm{d}\nu(h)\,.\end{aligned}$$

(The crucial step uses the invariance of the integral, see 7.12, applied to the function $h \mapsto f(gh)$, with g fixed.) Accordingly, it descends to a well-defined map

$$gH \mapsto \int_H f(gh)\,\mathrm{d}\nu(h)$$

from G/H to $\mathbb{R}$, which we shall denote by $\Phi(f)$. One can easily convince oneself ([Folland, p. 56], [Kan-Tay, p. 12]) that $\Phi(f)$, like f, is again continuous and compactly supported, so it is integrable with respect to any Haar measure on G/H by 7.7. The equality in the above statement can then more accurately be phrased as:

$$\int_G f(g)\,\mathrm{d}\mu(g) = \int_{G/H} \Phi(f)(\dot{g})\,\mathrm{d}\dot{\mu}(\dot{g}) \qquad \text{for all } f \in C_c(G;\mathbb{R}).$$

(3) As always, there is a "right-invariant counterpart" of the above statement: if μ, ν and $\dot{\mu}$ are *right* Haar measures on G, H and G/H respectively, then, up to a scalar, we have

$$\int_G f(g)\,\mathrm{d}\mu(g) = \int_{G/H}\int_H f(hg)\,\mathrm{d}\nu(h)\,\mathrm{d}\dot{\mu}(\dot{g}) \qquad \text{for all } f \in C_c(G;\mathbb{R}).$$

(To prove, e.g., that the inner integral on the right-hand side indeed descends to a function of $\dot{g}$, use the same argument as in the preceding item, except with *right* cosets.) $\diamond$

Sketch of proof of 8.10. One can provide an elegant argument by exploiting the intimate relation between $C_c(G;\mathbb{R})$ and $C_c(G/H;\mathbb{R})$. Recall that, in the preceding paragraph, we defined a map $\Phi\colon C_c(G;\mathbb{R}) \to C_c(G/H;\mathbb{R})$; it is easily seen from the definition that Φ is $\mathbb{R}$-linear (because the integral $f \mapsto \int_G f\,\mathrm{d}\mu$ is, as a map $I\colon C_c(G;\mathbb{R}) \to \mathbb{R}$). More importantly, as shown e.g. in [Folland, Prop. 2.48] or [Kan-Tay, Prop. 1.9], Φ is *surjective*, so it has a section (i.e.: a right-inverse)[30] $\Sigma\colon C_c(G/H;\mathbb{R}) \to C_c(G;\mathbb{R})$. One can, moreover, show that the kernel of Φ consists precisely of those $f \in C_c(G;\mathbb{R})$ with $\int_G f\,\mathrm{d}\mu = 0$. (In [Folland], this is shown essentially in the course of the proof of Thm. 2.49.) From this, it follows that $J = I \circ \Sigma$, i.e.

$$\begin{aligned} J\colon C_c(G/H;\mathbb{R}) &\to \mathbb{R}, \\ f &\mapsto \int_G \Sigma(f)\,\mathrm{d}\mu\,, \end{aligned}$$

is actually independent of the choice of Σ. This, in turn, together with linearity of Φ and of I, implies that J is itself linear. Finally, because Σ can be chosen so that $\Sigma(f)$ is nonnegative whenever $f \in C_c(G/H;\mathbb{R})$ is nonnegative ([Kan-Tay, Prop. 1.9] or [Folland, Prop. 2.48]), we conclude that J is a positive linear functional on $C_c(G/H;\mathbb{R})$. One can check that J is nonzero (use 7.15) and left-invariant (using the fact that I is, by 7.12), i.e. a left Haar integral on G/H. We can then define $\dot{\mu}$ to be the left Haar measure on G/H corresponding to J via 7.28, and Weil's integration formula becomes an immediate reformulation of the definition of J. $\square$

[30] This is an application of the axiom of choice.

8.12. *Example.* Let us better illustrate some aspects of the above discussion with the help of a concrete example. To that end, take G to be the additive group of the reals and let H denote the closed normal subgroup $\mathbb{Z} \subset \mathbb{R}$; then, the natural choice of Haar measure ν on the discrete subgroup $\mathbb{Z}$ is *counting measure*. With this choice, for $f \in C_c(\mathbb{R};\mathbb{R})$, the map $\Phi(f) \in C_c(\mathbb{R}/\mathbb{Z};\mathbb{R})$ (as defined in 8.11) is given by

$$x + \mathbb{Z} \mapsto \sum_{m \in \mathbb{Z}} f(x+m).$$

(Recall 6.15.(2); to see why the right-hand side is well-defined, observe that $\{m \in \mathbb{Z} : f(x+m) \neq 0\}$ is finite, being contained in the intersection of the compact subset $\operatorname{supp} f \subset \mathbb{R}$ with the discrete subset $x + \mathbb{Z} \subset \mathbb{R}$.)

Now recall that, by 1.17.(3), the quotient $G/H = \mathbb{R}/\mathbb{Z}$ is isomorphic to the circle group $\mathbb{T}$ from 1.2.(2), hence in particular compact. It follows (from 7.5) that $C_c(\mathbb{R}/\mathbb{Z};\mathbb{R}) = C(\mathbb{R}/\mathbb{Z};\mathbb{R})$; since Φ is surjective (s. the proof *supra*), this means that *every* continuous real-valued function on $\mathbb{R}/\mathbb{Z}$ is of the form $\Phi(f)$ for some $f \in C_c(\mathbb{R};\mathbb{R})$.

Consider for example the constant function on $\mathbb{R}/\mathbb{Z}$ which sends every element to $1 \in \mathbb{R}$. This function is continuous, hence is equal to $\Phi(f)$ for some $f \in C_c(\mathbb{R};\mathbb{R})$; in fact, one may take e.g.

- $f(x) = x 1_{[0,1)}(x) + (2-x) 1_{[1,2)}(x)$; or
- $f(x) = \cos^2(\frac{\pi}{2}x) 1_{[-1,1]}$.

In accordance with the claims made in the proof of 8.10, these functions have the same integral (over $\mathbb{R}$, with respect to Lebesgue measure λ): in both cases, $\int_{\mathbb{R}} f \, d\lambda = 1$. Thus, if $\dot{\lambda}$ is the unique Haar measure on $\mathbb{R}/\mathbb{Z}$ for which Weil's integration formula holds without scaling factor, then $\int_{\mathbb{R}/\mathbb{Z}} 1 \, d\dot{\lambda} = 1$, i.e., $\dot{\lambda}(\mathbb{R}/\mathbb{Z}) = 1$. The reader is invited to compare these findings with 8.26. $\diamond$

There is one more construction with topological groups which appeared in earlier sections, namely taking *projective limits* (cf. §2.3). A general discussion of Haar measures for projective limits is given e.g. in [Bourbaki, Chapter VII, §1, no. 6]; however, we will not discuss this in detail in this note.

We conclude the subsection with an apparently harmless result which does not technically involve any *constructions* with locally compact groups, but nevertheless belongs in this discussion from a conceptual viewpoint and will prove quite useful in the following subsection. In order to state the result, we shall first add a few new terms to our vocabulary.

8.13. A map $\varphi\colon G \to H$ between topological groups G and H will be called a *topological group isomorphism* if it is both a group homomorphism and a homoeomorphism. (As is easily seen, a topological group isomorphism from G to H may equivalently be described as

(i) a continuous group isomorphism φ with continuous inverse φ^{-1}, or

(ii) a homoeomorphism φ which simultaneously satisfies $\varphi(gg') = \varphi(g)\varphi(g')$ for all $g,\ g' \in G$.)

A topological group isomorphism from a topological group G to itself is also called a *(topological) automorphism* of G.

8.14. *Remarks.*

(1) Clearly if $\varphi\colon G \to H$ is a topological group isomorphism then the same is true of $\varphi^{-1}\colon H \to G$.

(2) Restating an earlier definition (given in passing in 1.17), we may say that two topological groups are *isomorphic (as topological groups)* if (and only if) there exists a topological group isomorphism between them. ⋄

8.15. Proposition. *Let H be a topological group which affords a topological group isomorphism $\varphi\colon H \to G$ to a locally compact group G. Then H is itself locally compact, and every left [right] Haar measure ν on H is of the form*

$$\nu(B) = \mu(\varphi(B)), \qquad B \in \mathcal{B}_H$$

for a unique left [right] Haar measure μ on G.

8.16. *Remark.* The relationship between ν and μ in the above statement can also be expressed concisely using the notion of the *pushforward* of a measure. Indeed, set $\psi := \varphi^{-1}\colon G \to H$, the inverse map of φ. (By 8.14.(1), ψ is again a topological group isomorphism.) Then ν is simply the pushforward of μ along ψ; in symbols, $\nu = \psi_*\mu$. For more on pushforwards, see also E.13 and E.32. ⋄

Proof of 8.15. The first claim is clear since local compactness and "Hausdorffness" are topological properties (i.e.: preserved by homoeomorphisms). The bulk of the remaining claim is contained in the following fact, which we shall state and prove separately—and slightly more generally than needed right now—for later convenience.

8.17. Lemma. *Let $\varphi\colon H \to G$ be a continuous group homomorphism between locally compact groups H and G, and let μ be a left [right] Haar measure on G. Suppose that φ is open and injective, i.e. a homoeomorphism onto an open subgroup $\varphi(H) \subseteq G$. Then*

$$\begin{aligned}\nu\colon \mathcal{B}_H &\to [0,\infty],\\ B &\mapsto \mu(\varphi(B))\end{aligned}$$

is a left [right] Haar measure on H.

Proof of the lemma. The heavy lifting is done by E.32.(c), which ensures that ν is a Radon measure under our assumptions on μ and φ. Next, to check that ν is nonzero, one may proceed as follows: because $\varphi(H)$ is open in G by assumption, and μ is strictly positive on non-empty open subsets of G (this is 4.21.(i)), we find that $\nu(H) = \mu(\varphi(H)) > 0$; in particular, ν is nonzero.

Finally, because φ is a group homomorphism, left- [right-]invariance of μ immediately carries over to ν. In more detail, suppose μ is *left*-invariant (the other case being fully analogous). Then, for any Borel subset $B \subseteq H$ and every $h \in H$, we have $\varphi(hB) = \varphi(h)\varphi(B)$, so

$$\begin{aligned}\nu(hB) &= \mu(\varphi(hB))\\ &= \mu(\varphi(h)\varphi(B)) \qquad (\mu(\varphi(\cdot)))\\ &= \mu(\varphi(B)) = \nu(B),\end{aligned}$$

i.e.: ν is a left Haar measure on H. □

Applying the above lemma to "our" φ (which is even a homoeomorphism onto all of G), we obtain a way to assign, to each left [right] Haar measure on H, a left [right] Haar measure on G. We shall now show that this assignment is a one-to-one correspondence, concluding the proof.

Thus, let ν be a left [right] Haar measure on H. Now note the following

8.18. Lemma. *Let X, Y be LCH spaces, and let $f\colon X \to Y$ be a homoeomorphism. Then every [nonzero] Radon measure ν on Y is of the form $f_*\mu$ for a unique [nonzero] Radon measure μ on X, namely $\mu = (f^{-1})_*\nu$.*

(The proof has been left to the reader: it is E.33.)

Applying the lemma (with $X = G$, $Y = H$, and $f = \varphi^{-1}$) to the measure ν we fixed earlier, we conclude that there exists a unique nonzero Radon measure μ on G such that $\nu(B) = \mu(\varphi(B))$ for all $B \in \mathcal{B}_H$. It remains to prove that μ is left- [right-]invariant. But, again by the lemma, μ is given by the formula $\mu(B) = \nu(\psi(B))$ for all $B \in \mathcal{B}_G$, where $\psi := \varphi^{-1}$ is a topological group isomorphism $G \to H$, so the left- [right-]invariance of μ can be established again by the previous lemma 8.17—one need only replace φ by ψ and interchange the roles of G and H (and obviously, those of μ and ν). □

We conclude our discussion with a few observations about the above proof which will be useful in §8.3.

8.19. *Remark on the proof of 8.17.* Let us resume the notations of the statement and proof of 8.17, and let us take a closer look at the chain of equalities ($\mu(\varphi(\cdot))$). In order to apply the left-invariance of μ in the final step, we used the fact that φ "commutes with left translation", or more precisely that $\varphi \circ l_h = l_{\varphi(h)} \circ \varphi$ for all $h \in H$. But, on closer inspection, it is irrelevant *which* left translation map appears on the right-hand side of this equality: all that matters is that it is a map of the form l_g for *some* $g \in G$, so that we can conclude $\mu(g\varphi(B)) = \mu(\varphi(B)) = \nu(B)$ in the final steps. Thus, the arguments given in the proof above actually establish the following stronger claim. ◇

8.20. Proposition. *Let G, H be locally compact groups, $\varphi\colon H \to G$ be a homoeomorphism onto an open subspace $\varphi(H) \subseteq G$, and suppose that*

(φ-γ) *for all $h \in H$ there exists an element $\gamma(h) \in G$ such that $\varphi \circ l_h = l_{\gamma(h)} \circ \varphi$.*

Then, for every left Haar measure μ on G,

$$\begin{aligned} \nu\colon \mathcal{B}_H &\to [0,\infty], \\ B &\mapsto \mu(\varphi(B)) \end{aligned}$$

is a left Haar measure on H.

8.21. *Remarks.*

(1) Observe that (φ-γ) can equivalently be rephrased as the condition that

$$\varphi(hx) = \gamma(h)\varphi(x) \qquad \text{for all } x \in H \tag{φ-γ}$$

for some (set-theoretic) function $\gamma\colon H \to G$. One can check (E.34) that, *if* this condition is verified for *some* γ, then there is *precisely* one γ for which it is verified, and this γ is necessarily a group homomorphism $H \to G$.

(2) Clearly every map $\varphi\colon H \to G$ as in the statement of 8.17 satisfies the assumptions of 8.20 (with $\gamma = \varphi$). To see that the newer condition is more general, consider for a moment the special case $H = G$. Then, for any $g \in G$, the right translation map $\varphi = r^g\colon G \to G$ is a homoeomorphism which satisfies (φ-γ) with γ equal to the identity map (i.e., $\gamma(h) = h$ for all $h \in G$); however, r^g is clearly *not* a group homomorphism in general. Similarly, the left translation map $\varphi = l_g$ is a homoeomorphism $G \to G$ which satisfies (φ-γ) with $\gamma = l_g \circ (r^g)^{-1}$ and is not a group homomorphism.

(3) There is a rather obvious counterpart of the above result which holds true for *right* Haar measures: to obtain the statement, it suffices to replace left translation maps by right translation maps throughout. The proof is again obtained by a straightforward modification of the arguments seen in the proof of 8.17. ◇

8.2 Concrete examples

In this subsection, we will successively put several locally compact groups under our figurative microscope, approximately in the same order in which they appeared in the previous sections, and use the results from §8.1 to determine Haar measure on each of them. All the groups considered in this subsection (except in 8.28) are abelian; for such groups, we will always simply speak of 'Haar measure' (or 'Haar integral') without either of the qualifiers 'left' or 'right'. Our starting point—the Haar measures we already know—will be 4.19, that is:

- on the additive group of the reals, the restriction $\lambda|_{\mathcal{B}_{\mathbb{R}}}$ of Lebesgue measure λ to the Borel-σ-algebra $\mathcal{B}_{\mathbb{R}}$ is a (bi-invariant) Haar measure on $(\mathbb{R}, +)$; and
- on *any* discrete group G, counting measure on $\mathcal{B}_G = \mathcal{P}(G)$ is a (bi-invariant) Haar measure on G.

8.22. *Example*. Consider the group $(\mathbb{R}^d, +)$, with d a positive integer greater than or equal to 2, and let $\lambda^d := \lambda \times \cdots \times \lambda$ be the Haar measure on $\mathbb{R}^d$ obtained by iteratively applying 8.4. Then, by 8.8, λ^d satisfies

$$\lambda^d(B_1 \times \cdots \times B_d) = \lambda(B_1)\lambda(B_2)\cdots\lambda(B_d) \qquad \text{whenever } B_1, \ldots, B_d \in \mathcal{B}_{\mathbb{R}},$$

and is in fact the *unique* measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$ which possesses this property. One calls λ^d *(d-dimensional) Lebesgue measure*, or *Lebesgue measure on* $\mathbb{R}^d$. (However, see also the remark immediately below.)

As a special case, the above discussion yields a Haar measure on the complex numbers $\mathbb{C}$, and indeed on $\mathbb{C}^d$ for any $d \geq 1$; more precisely, because for every $d \geq 1$ we (trivially) have an isomorphism of topological groups between $(\mathbb{C}^d, +)$ and $(\mathbb{R}^{2d}, +)$, the measure is given by 8.15. ◇

8.23. *Remark*. As was the case for $(\mathbb{R}, +)$, Lebesgue measure on $\mathbb{R}^d$ can actually be defined on more than "just" Borel sets: there is a σ-algebra of *Lebesgue-measurable subsets* of $\mathbb{R}^d$, containing all Borel subsets of $\mathbb{R}^d$, to which λ^d can be uniquely extended, and the extension may again be called *Lebesgue measure* and again denoted by λ^d.

One way to carry out this extension is via E.19. Alternatively, we can retrace the steps taken in 3.3 and 3.17 and define, for every $d \geq 1$,

$$\mathcal{H}^d = \{[a_1, b_1) \times \cdots \times [a_d, b_d) : a_1, \ldots, a_d, b_1, \ldots, b_d \in \mathbb{R}\} \subset \mathcal{P}(\mathbb{R}^d);$$

this is again a semiring (cf. 3.4), and we may again speak of *Lebesgue outer measure* on $\mathbb{R}^d$ (defined again as in OUT.MEAS., with $\mathcal{H}^d$ in place of $\mathcal{H}$) and again obtain Lebesgue measure by applying 3.14 as in 3.15. As before, 3.18 guarantees that these two methods yield the same measure; an advantage of the second method is that it works even without previous knowledge of the properties of "one-dimensional" Lebesgue measure λ, and in fact it even allows one to construct Lebesgue measure in all dimensions simultaneously. ◇

8.24. *Example.* Consider now the multiplicative groups $\mathbb{R}_{>0}$ and $\mathbb{R}^*$. Recall that these are considered with the respective subspace topology coming from the inclusions $\mathbb{R}_{>0} \subset \mathbb{R}^* \subset \mathbb{R}$; in particular, $\mathcal{B}_{\mathbb{R}_{>0}} = \{B \in \mathcal{B}_{\mathbb{R}} : B \subseteq \mathbb{R}_{>0}\}$ and analogously for $\mathcal{B}_{\mathbb{R}^*}$. (Cf. E.13 and footnote 21.)

A concise, if somewhat abstract, way to exhibit a Haar measure on $\mathbb{R}_{>0}$ is to exploit the maps $\exp\colon \mathbb{R} \to \mathbb{R}_{>0}$ and $\log\colon \mathbb{R}_{>0} \to \mathbb{R}$ familiar from basic real analysis. Indeed, these are well-known to be continuous, inverse to each other and to yield group homomorphisms between $(\mathbb{R}, +)$ and $(\mathbb{R}_{>0}, \cdot)$—i.e., each map defines an isomorphism of topological groups. By our earlier result 8.15, we may then conclude that

$$\lambda^\times(B) := \lambda(\log(B)), \qquad B \in \mathcal{B}_{\mathbb{R}_{>0}}$$

defines a left Haar measure on $\mathbb{R}_{>0}$; the invariance property reads

$$\lambda^\times(cB) = \lambda^\times(B) \qquad \text{for all } B \in \mathcal{B}_{\mathbb{R}_{>0}}, c \in \mathbb{R}_{>0}$$

(where we have used the notation $cB = \{cx : x \in B\}$ from 1.6). Finally, because $\mathbb{R}^* \cong \{\pm 1\} \times \mathbb{R}_{>0}$ (as topological groups!), where $\{\pm 1\}$ denotes the essentially unique group with two elements, we obtain a Haar measure on $\mathbb{R}^*$ via

$$B \mapsto \lambda^\times(B \cap \mathbb{R}_{>0}) + \lambda^\times(-B \cap \mathbb{R}_{>0})$$

(where of course $-B = \{-b : b \in B\}$). It is immediate that this measure is an extension of the measure $\lambda^\times$ we defined earlier on $\mathcal{B}_{\mathbb{R}_{>0}}$; we shall not differentiate in the notation between the previously-defined measure and its extension to $\mathcal{B}_{\mathbb{R}^*}$.

There is a different approach to the construction of $\lambda^\times$ which we shall now sketch (expanding on [Bourbaki, Chapter VII, §1, no. 2, Example 2)]). As a starting point for this, we may go all the way back to 3.3 (and resume the notations therein) and observe that the original length function ℓ (defined *a priori* only on half-open intervals $I = [a, b) \in \mathcal{H}$) has the property that

$$\ell(cI) = |c|\ell(I) \qquad \text{for all } c \in \mathbb{R}, I \in \mathcal{H}.$$

It is then easy to check that Lebesgue outer measure λ^*—and hence also Lebesgue measure λ, being simply the restriction of λ^*—inherits this property.

(See also [Folland2, Thm. 1.21] for an outline of the argument.) Using this, one may carry out the following computation for any nonnegative measurable function f on $\mathbb{R}^*$ and any nonzero real number c:

$$\begin{aligned}\int_{\mathbb{R}\setminus\{0\}} \frac{f(cx)}{|x|}\,\mathrm{d}\lambda(x) &= \int_{\mathbb{R}\setminus\{0\}} \frac{f(cx)}{|c||x|}|c|\,\mathrm{d}\lambda(x)\\ &= \int_{\mathbb{R}\setminus\{0\}} \frac{f(cx)}{|cx|}\,\mathrm{d}\lambda(cx)\\ &= \int_{\mathbb{R}\setminus\{0\}} \frac{f(x)}{|x|}\,\mathrm{d}\lambda(x)\,.\end{aligned}$$

(More rigorously, the last two steps of this computation should instead be justified by a "change-of-variables formula", such as [Cohn, Prop. 2.6.5].)

The above computation shows in particular that the map sending $f \in C_c(\mathbb{R}^*;\mathbb{R})$ to $\int_{\mathbb{R}\setminus\{0\}} \frac{f(x)}{|x|}\,\mathrm{d}\lambda(x)$ is a left Haar integral, which thus yields a Haar measure $\widetilde{\lambda^\times}$ on $\mathbb{R}^*$ by 7.28. Applying [Folland2, Chapter 7, Exercise 9, p. 220] (jointly with E.32.(b)), we can describe $\widetilde{\lambda^\times}$ more concretely as

$$\begin{aligned}\widetilde{\lambda^\times}(B) &= \int_{\mathbb{R}\setminus\{0\}} \frac{1_B(x)}{|x|}\,\mathrm{d}\lambda(x)\\ &= \int_B \frac{\mathrm{d}\lambda(x)}{|x|},\end{aligned}$$

where $B \in \mathcal{B}_{\mathbb{R}^*}$.

It remains, of course, to check that this alternative method indeed yields the same measure as above. For Borel sets of a very special kind, this is verified easily by using elementary calculus to explicitly compute the Lebesgue integral (as a Riemann integral): more precisely, if B is an interval $[a,b)$ with $0 < a < b$, the second method yields $\widetilde{\lambda^\times}(B) = \int_a^b \frac{\mathrm{d}x}{x} = \log(b) - \log(a)$, which is the same result provided by the earlier definition: $\lambda^\times(B) = \lambda(\log(B)) = \lambda([\log(a), \log(b)) = \log(b) - \log(a)$. It is then easy to see that $\lambda^\times$ and $\widetilde{\lambda^\times}$ agree on the collection of all intervals $[a,b)$ with $ab > 0$, at which point one may use 3.18 since this collection is certainly a semiring which generates the Borel-σ-algebra of $\mathcal{B}_{\mathbb{R}^*}$. $\diamond$

8.25. *Remark*. It is not uncommon to find the notation '$\mathrm{d}^\times x$' in reference to the measure which we have opted to denote by $\lambda^\times$. (Cf. also 6.30.(1).) For the sake of clarity authors may then also use the analogous notation '$\mathrm{d}^+ x$' for the "*additive*" Haar measure (i.e. simply Lebesgue measure λ), and explain the relationship between the two by means of an equality such as: $\mathrm{d}^\times x = \frac{\mathrm{d}^+ x}{|x|}$. $\diamond$

8.26. *Example.* We now consider the circle group $\mathbb{T}$ introduced in 1.2.(2) and set out to determine its Haar measure.

The strategy used in [Bourbaki, Chapter VII, §1, no. 2, Example 3)] takes advantage of the properties of the map $q\colon \mathbb{R} \to \mathbb{T}$ given by $x \mapsto \mathrm{e}^{2\pi \mathrm{i} x}$. (We can regard q as the composition of the canonical quotient map $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ (cf. 1.15.(iii)) with the isomorphism $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$ from 1.17.(3).) Indeed, let f belong to $C_c(\mathbb{T};\mathbb{R})$—then $f \circ q$ is a continuous function on $\mathbb{R}$, and as such, integrable (with respect to Lebesgue measure λ) over any interval $I \subset \mathbb{R}$. But $f \circ q$ is also periodic, with $(f \circ q)(x+1) = (f \circ q)(x)$ for all $x \in \mathbb{R}$; accordingly, for *any* interval $I \subset \mathbb{R}$ of length 1, the integral $\int_I (f \circ q)\,\mathrm{d}\lambda$ yields the same value (independently of the choice of I). It is readily seen that $f \mapsto \int_I (f \circ q)\,\mathrm{d}\lambda$ defines a Haar integral on $\mathbb{T}$ and hence, by 7.28, a Haar measure μ on $\mathbb{T}$.

As is often the case, μ has an alternative description. To see this, let us choose once and for all $I = [0,1] \subset \mathbb{R}$, and let us agree to henceforth denote the restriction of λ to I (more precisely: to $\mathcal{B}_I$, cf. footnote 21) again just by λ, and the restriction of q to I again just by q. Now consider the pushforward $q_*\lambda$ of λ along q (cf. E.13), which is a measure on $(\mathbb{T}, \mathcal{B}_\mathbb{T})$. By a standard result on integrals with respect to pushforward measures (see e.g. [Cohn, Prop. 2.6.5]), a measurable complex-valued function f on $\mathbb{T}$ is integrable with respect to $q_*\lambda$ if and only if $f \circ q$ is integrable with respect to λ, in which case

$$\int_\mathbb{T} f\,\mathrm{d}q_*\lambda = \int_I (f \circ q)\,\mathrm{d}\lambda\,.$$

But, by compactness of I, *any* continuous function on I is (Riemann-, hence Lebesgue-)integrable. It follows that, if $f \in C_c(\mathbb{T};\mathbb{R})$, then f is integrable with respect to $q_*\lambda$, and the integral is given by $\int_I (f \circ q)\,\mathrm{d}\lambda$, which in turn, by the definition of μ, is precisely $\int_\mathbb{T} f\,\mathrm{d}\mu$. We now want to conclude that $q_*\lambda$ and μ are one and the same.

To that end, observe first that $q_*\lambda$ is finite on compact sets of $\mathbb{T}$. Next, note that $\mathbb{T}$ is surely second-countable (being a subspace of the second-countable space $\mathbb{C} \cong \mathbb{R}^2$). Then, by 7.26, $q_*\lambda$ is a Radon measure on $\mathbb{T}$; the uniqueness part of the Riesz Representation Theorem 7.22 now tells us that μ is none other than $q_*\lambda$, as desired.

As an application of this alternative description, let $A \subseteq \mathbb{T}$ be a connected arc—then $\mu(A) = \lambda(q^{-1}(A))$, which, as is readily seen, is just $\frac{\alpha}{2\pi}$, where α is the length of A in the usual sense, i.e. the size of the angle subtended by A expressed in radians. In particular, $\mu(\mathbb{T}) = 1$. (In all fairness, though, this was also evident from the previous description.) It is then easy to see that, upon denoting by φ the usual isomorphism $\mathbb{R}/\mathbb{Z} \to \mathbb{T}$, the Haar measures μ (on $\mathbb{T}$) and $\dot{\lambda}$ (on $\mathbb{R}/\mathbb{Z}$) from 8.12 correspond to each other as per the statement of 8.15. $\diamond$

8.27. *Example.* Building on the previous examples, we may now provide two ways to determine Haar measure on the multiplicative group $\mathbb{C}^*$. One method is similar to the (second) one seen in 8.24: first, one establishes that the "additive" Haar measure on $\mathbb{C}$ (which, modulo the isomorphism $\mathbb{C} \cong \mathbb{R}^2$, is just 2-dimensional Lebesgue measure λ^2, as seen in 8.22) has the property that $\lambda^2(zB) = |z|^2\lambda^2(B)$ for every complex number z and every Borel set B, from which one concludes that

$$f \mapsto \int_{\mathbb{C}\setminus\{0\}} f(z)\frac{\mathrm{d}\lambda^2(z)}{|z|^2}$$

defines a Haar integral on $\mathbb{C}^*$. (Cf. also [Folland, Chapter 11, Exercise 4a, p. 347].)

The other method, which is essentially identical to that of [Bourbaki, Chapter VII, §1, no. 5, Example 2)], takes advantage of *polar coordinates*. Recall that every nonzero complex number z can be written as $re^{\mathrm{i}\vartheta}$ for a unique positive real number r and a real number $\vartheta \in \mathbb{R}$, unique up to integral multiples of 2π. If we identify the quotient group $\mathbb{R}/2\pi\mathbb{Z}$ with the circle group $\mathbb{T}$ (via $\vartheta + 2\pi\mathbb{Z} \mapsto e^{\mathrm{i}\vartheta}$), then we conclude that there is a bijection between $\mathbb{C}^*$ and $\mathbb{R}_{>0} \times \mathbb{T}$, given (in the forward direction) by $z \mapsto \varphi(z) := \left(|z|, \frac{z}{|z|}\right)$. But it is easily checked that this bijection is even an isomorphism of topological groups, hence, by combining 8.4, 8.24 and the considerations in 8.26, we obtain that a

Haar integral on $\mathbb{C}^*$ is given by

$$\begin{aligned} f \mapsto & \int_0^\infty \int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}\vartheta}) \,\mathrm{d}\lambda(\vartheta) \,\mathrm{d}\lambda^\times(r) \\ &= \int_0^\infty \int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}\vartheta}) \,\mathrm{d}\lambda(\vartheta) \,\frac{\mathrm{d}\lambda(r)}{r}. \end{aligned}$$

Standard transformation formulae for polar coordinates (as can be found e.g. in [Folland2, Thm. 2.49]) show that the two integrals we have considered actually agree for any $f \in C_c(\mathbb{C}^*; \mathbb{R})$. The corresponding Haar measure is given by

$$B \mapsto \int_B \frac{\mathrm{d}\lambda^2(z)}{|z|^2} = \iint_{\varphi(B)} \frac{\mathrm{d}\lambda(\vartheta)\,\mathrm{d}\lambda(r)}{r}, \qquad B \in \mathcal{B}_{\mathbb{C}^*}.$$

(This follows, as in 8.22, from [Folland2, Chapter 7, Exercise 9, p. 220].) ◇

8.28. The attentive reader might have noticed that we have now discussed at least one way to compute Haar measure for each of the groups listed in items (1) and (2) of our earlier list 2.6 of locally compact groups. (In fact, we have also covered item (4), that is, discrete groups.) The next natural step would thus be to provide Haar measure for item (3), i.e. for Lie groups. We shall not go into details here, since we don't assume familiarity with differential geometry; let us simply mention for the initiated that left [right] Haar measures μ on a real Lie group G can be obtained from left- [right-]invariant volume forms (i.e.: top-degree differential forms) ω on G: more precisely, G affords such an ω, and for every such ω which is additionally *positive* with respect to a chosen orientation on G, there exists a μ such that integrating against ω is the same as integrating with respect to μ. Details can be found in [Knapp], esp. Thm. 8.21.

Fortunately, for many Lie groups which happen to be *matrix groups*, one can determine left and right Haar measures without resorting to the aforementioned result, s. e.g. [Folland2, Chapter 11, Exercises 3–4, p. 347]. As an example, consider $G = \mathrm{GL}_n(\mathbb{R})$, the group of invertible $n \times n$-matrices with real entries. Since $n \times n$-matrices with entries in $\mathbb{R}$ may be identified with vectors in $\mathbb{R}^{n^2}$ (simply by rearranging the entries into a single column), one can also regard G as a subset of $\mathbb{R}^{n^2}$. Modulo this identification, G is in fact the complement in $\mathbb{R}^{n^2}$ of the vanishing set of the polynomial (hence: continuous) map which sends a matrix A to its determinant $\det A \in \mathbb{R}$. It follows that G is open in $\mathbb{R}^{n^2}$. If we now let λ^{n^2} denote Lebesgue measure on $\mathbb{R}^{n^2}$ (as in 8.22), then the aforementioned exercises in Folland's book tell us that

$$f \mapsto \int_G f(g) \frac{\mathrm{d}\lambda^{n^2}(g)}{|\det g|^n}$$

is a bi-invariant Haar integral on G; applying once again [Folland2, Chapter 7, Exercise 9, p. 220], we conclude that the corresponding Haar measure is the map sending $B \in \mathcal{B}_G$ to $\int_B \frac{\mathrm{d}\lambda^{n^2}(g)}{|\det g|^n}$.

8.3 Bi-invariance and the modular function

So far, our discussions of left and right Haar measures have taken place parallelly to one another: every result which can be proved for left Haar measures also has a "mirror version" (less figuratively: a counterpart for right Haar measures) which can be proved with entirely analogous methods, and we often only carry out the proof in the former case (with notable exceptions, such as 5.16 and 7.19, where one case was used to prove the other by exploiting the natural bijection between left and right Haar measures first seen in 4.20). In this subsection, we will explore the intersection of these two worlds; in other words, we will tackle the question of determining when left Haar measures are also right-invariant (or viceversa). The key to our discussion will be expressing left- and right-invariance in a slightly new way (with the help of 8.20).

8.29. To start things off, recall the notation $f_*\mu$ used in the latter part of §8.1 for the *pushforward* of a measure μ along a measurable function f. It costs little effort to recognize that a nonzero Radon measure μ on a locally compact group G is a left Haar measure if and only $(l_g)_*\mu$ is equal to μ for all $g \in G$; analogously, μ is a right Haar measure if and only if $(r^g)_*\mu = \mu$ for all $g \in G$. More importantly, if we *fix* a left Haar measure μ on G and we apply 8.20 with $H = G$ and φ equal to a right translation map r^g for some $g \in G$ (cf. 8.21.(2)), then we obtain a new left Haar measure $\nu = (r^{g^{-1}})_*\mu$ on G, and by the above, μ is bi-invariant if and only if $(r^{g^{-1}})_*\mu = \mu$ for all $g \in G$. As it turns out, this characterization is quite fruitful when it comes to tackling the general problem of finding groups which possess bi-invariant Haar measure.

Having provided some motivation, let us now attempt a more systematic treatment. First, we state the following result, which can be obtained essentially by combining 8.20 (with $H = G$) and 7.1.

8.30. Proposition. *Let G be a locally compact group and let $\varphi\colon G \to G$ be a homoeomorphism onto an open subspace $\varphi(G) \subseteq G$ with the property that*

$$\varphi(gx) = \gamma(g)\varphi(x) \qquad \text{for all } x \in G \tag{φ-γ}$$

for a suitable $\gamma\colon G \to G$. Then there exists a positive real number $c = c(\varphi)$ such that, for every *left Haar measure μ on G,*

$$\mu(\varphi(B)) = c\mu(B) \qquad \text{for all } B \in \mathcal{B}_G.$$

8.31. *Remark.* Observe that the scalar c can be seen as a way of quantifying how much φ "dilates" (or "shrinks") subsets of G in terms of (any) Haar measure on G. This is made even more apparent by the following example: let G denote the additive group of the reals, and $\varphi\colon G \to G$ be given by $x \mapsto ax$ for some nonzero real number a. Then φ is a topological group isomorphism, hence satisfies (φ-γ) with $\gamma = \varphi$ (cf. 8.21.(2)), and we have already seen (in 8.24) that the conclusion of the statement holds (for μ equal to Lebesgue measure λ) with c equal to the absolute value $|a|$ of a.

Keeping in mind that the absolute value of a real number is also sometimes called its *modulus*, it is perhaps not surprising that the following terminology has entered common use. ◇

8.32. Let G and φ be as in the above statement. The scalar $c = c(\varphi)$ provided by 8.30 is called the *modulus*[31] of the homoeomorphism φ and denoted by $\mathrm{mod}_G(\varphi)$ or simply $\mathrm{mod}(\varphi)$.

Proof of 8.30. Fix a left Haar measure μ_0 on G, and let μ_1 denote the measure $B \mapsto \mu_0(\varphi(B))$ on $(G, \mathcal{B}_G)$. Then μ_1 is a left Haar measure on G by 8.20. Because both μ_0 and μ_1 are left Haar measures on G, there exists a positive real number c such that $\mu_1 = c\mu_0$ by 7.1. We claim that this scalar c has the required property.

[31] Some authors use "*module*".

Indeed, let μ be *any* left Haar measure on G. Then 7.1 guarantees that $\mu = C\mu_0$ for a suitable positive real number C. But then, for every $B \in \mathcal{B}_G$,

$$\begin{aligned}\mu(\varphi(B)) &= C\mu_0(\varphi(B)) \\ &= Cc\mu_0(B) = c\mu(B),\end{aligned}$$

as desired. □

8.33. *Recap.* The example from 8.31 can now be rephrased as follows: for $G = (\mathbb{R}, +)$ and φ of the form $x \mapsto ax$, where $a \in \mathbb{R}^*$, one has $\operatorname{mod}(\varphi) = |a|$.

Similarly, it follows from the contents of 8.27 that, for $G = (\mathbb{C}, +)$ and φ of the form $w \mapsto zw$, where $z \in \mathbb{C}^*$, one has $\operatorname{mod}(\varphi) = |z|^2$.

For a general locally compact group G, we have already seen that $\operatorname{mod}(l_g) = 1$ for every $g \in G$—recall that the modulus is defined relative to *left* Haar measure!—and that G has a bi-invariant Haar measure (i.e., satisfies the equivalent conditions of 7.20) if and only if $\operatorname{mod}(r^g) = 1$ for all $g \in G$.

8.34. Our latest observation suggests that, to determine whether a locally compact group G possesses bi-invariant Haar measures, one could study the map $G \to \mathbb{R}_{>0}$ sending $g \mapsto \operatorname{mod}(r^g)$. In this note, this map will be denoted by Δ_G and called the *modular function* of the group G. Unravelling the definition, this means that the modular function is the unique map $G \to \mathbb{R}_{>0}$ satisfying

$$\mu(Bg) = \Delta_G(g)\mu(B) \tag{Δ}$$

whenever μ is a left Haar measure on G, B is a Borel subset of G and $g \in G$; in particular, if one fixes a Borel subset B which satisfies $0 < \mu(B) < \infty$, then $\Delta_G(g)$ can be expressed (and even defined) as the ratio $\mu(Bg)/\mu(B)$ for every $g \in G$. (Observe that these assumptions are automatically satisfied if B is, say, a compact subset of G with non-empty interior, by 4.21.(i).)

8.35. *Remarks on terminology.*

(1) The phrase "modular function" is also commonly used to refer to certain functions of a complex variable which transform in a certain peculiar way under certain discrete groups of automorphisms of $\mathbb{C}$. The two notions are entirely unrelated.

(2) Some authors (such as [Wallach] and [Knapp]) use the term "modular function" to refer to the function which, in our notation, would be denoted by

$$g \mapsto \Delta_G(g^{-1}) = \Delta_G(g)^{-1}.$$

(The equality will follow from 8.36.(i) below.) Unfortunately, the term is often thrown around without an accompanying remark or formula which might disambiguate the meaning right away; it would seem that many authors are not even aware that two different conventions exist. An egregious example is provided by the entry for "Haar measure" in the online Encyclopedia of Mathematics:[32] in the former half of the article, the modular function is defined in the same way as in this note (to see this, cf. also 8.36 below), whereas in the latter half, where the discussion is specialized to Lie groups, the opposite convention is used. ◇

Here are a couple of fundamental properties of the modular function.

8.36. Proposition. *Let G be a locally compact group, and let Δ_G denote the modular function of G as per 8.34. Then the following hold.*

(i) $\Delta_G \colon G \to \mathbb{R}_{>0}$ *is a group homomorphism.*

(ii) Δ_G *is continuous.*

(iii) For every left Haar measure μ on G and every $f \colon G \to \mathbb{C}$ which is integrable with respect to μ,

$$\int_G f(xg)\,\mathrm{d}\mu(x) = \Delta_G(g)^{-1} \int_G f(x)\,\mathrm{d}\mu(x) \qquad \text{for all } g \in G.$$

(iv) For every left Haar measure μ on G and every $f \colon G \to \mathbb{C}$ which is integrable with respect to the right *Haar measure $\mu^\vee$ (cf. 4.20.(1)),*

$$\begin{aligned} \int_G f(x^{-1})\,\mathrm{d}\mu(x) &= \int_G f(x)\,\mathrm{d}\mu^\vee(x) \\ &= \int_G f(x)\Delta_G(x)^{-1}\,\mathrm{d}\mu(x) \end{aligned}$$

(in particular, all integrals in question are indeed defined).

[32] https://encyclopediaofmath.org/wiki/Haar_measure.

Proof. To prove the first claim, observe first that, if φ, ψ are homoeomorphisms $G \to G$ which satisfy the equation (φ-γ) from 8.30 for suitable functions γ, γ' respectively, then $\varphi \circ \psi$ has the same property (for $\gamma \circ \gamma'$) and $\operatorname{mod}(\varphi \circ \psi) = \operatorname{mod}(\varphi) \operatorname{mod}(\psi)$. In particular, $\Delta_G(gh) = \operatorname{mod}(r^{gh}) = \operatorname{mod}(r^h) \operatorname{mod}(r^g) = \Delta_G(g)\Delta_G(h)$ (recall that $\mathbb{R}_{>0}$ is commutative!), and the claim is proved.

Now we establish continuity of Δ_G, following the argument given in SE668827. By (i) and E.5, it suffices to prove that Δ_G is continuous at the neutral element e, i.e. that for every $\varepsilon > 0$ there exists a neighbourhood W of e such that $\Delta_G(g) \in (1-\varepsilon, 1+\varepsilon)$ for all $g \in W$.

Thus, fix a left Haar measure μ on G and an $\varepsilon > 0$; additionally, fix a compact set $K \subseteq G$ with non-empty interior. Then, by outer regularity of μ, there exists an open set $U \supseteq K$ such that $\mu(U) < (1+\varepsilon)\mu(K)$. By 5.15, there also exists a neighbourhood V of e such that $KV \subseteq U$; then, for $g \in V$, we have $\Delta_G(g) = \mu(Kg)/\mu(K) \leq \mu(U)/\mu(K) < 1 + \varepsilon$ using 8.34 and the monotonicity of μ. On the other hand, if $h \in V^{-1}$, then $K \subseteq Uh$, so

$$\Delta_G(h) = \frac{\mu(Uh)}{\mu(U)} \geq \frac{\mu(K)}{\mu(U)} \geq \frac{1}{1+\varepsilon} > 1 - \varepsilon$$

(again using 8.34 and monotonicity.) Thus, $W = V \cap V^{-1}$ has the sought-after properties, and this concludes the proof of (ii).

The third claim may be proved "in stages" like 7.12, so we omit the details. To understand the appearance of the inverse $^{-1}$, observe that, if f is an indicator function 1_B (and $g \in G$), then $f \circ r^g = 1_{Bg^{-1}}$, so

$$\int_G f(xg)\,\mathrm{d}\mu(x) = \mu(Bg^{-1}) = \Delta_G(g)^{-1}\mu(B) = \Delta_G(g)^{-1} \int_G f\,\mathrm{d}\mu\,.$$

We now turn to the final claim (iv). Part of it is settled by the following

8.37. Lemma. *Let G be a locally compact group, μ be a left [right] Haar measure, and f be a [Borel-measurable] complex-valued function on G. Then f is integrable with respect to the right [left] Haar measure $\mu^\vee$ if and only if the function $x \mapsto f(x^{-1})$ is integrable with respect to μ, in which case*

$$\int_G f(x)\,\mathrm{d}\mu^\vee(x) = \int_G f(x^{-1})\,\mathrm{d}\mu(x)\,.$$

Proof of the lemma. As was the case for (iii), one may again proceed as in the proof of 7.12: start with simple functions, ... □

To conclude the proof of the claim, fix first of all a *left* Haar measure μ. If $f \in C_c(G;\mathbb{R})$, then $x \mapsto f(x)\Delta_G(x^{-1})$ is again a continuous compactly-supported function on G (use (ii)) and hence (by 7.7) integrable with respect to any fixed left Haar measure on G. Thus, we may define a map $I\colon C_c(G;\mathbb{R}) \to \mathbb{R}$ by assigning, to $f \in C_c(G;\mathbb{R})$, the real number $I(f) = \int_G f(x)\Delta_G(x^{-1})\,\mathrm{d}\mu(x)$. It is immediate that I is a nonzero positive linear functional on $C_c(G;\mathbb{R})$. We shall now show that I is a right Haar integral: to that end, pick $f \in C_c(G;\mathbb{R})$ and $g \in G$, and let h denote the function $x \mapsto f(x)\Delta_G(gx^{-1})$—then, by (iii),

$$\begin{aligned}\int_G f(xg)\Delta_G(x^{-1})\,\mathrm{d}\mu(x) &= \int_G h(xg)\,\mathrm{d}\mu(x)\\ &= \int_G h(x)\Delta_G(g^{-1})\,\mathrm{d}\mu(x)\\ &= \int_G f(x)\Delta_G(x^{-1})\,\mathrm{d}\mu(x)\end{aligned}$$

(where we used (i) in the last step to simplify), proving $I(f \circ r^g) = I(f)$, as required. By 7.28, I defines a right Haar measure ν on G; for $B \in \mathcal{B}_G$, we have

$$\nu(B) = \int_B \Delta_G(x^{-1})\,\mathrm{d}\mu(x) \qquad (\nu(B))$$

by [Folland2, Chapter 7, Exercise 9, p. 220].

Now, by 7.19, ν is a multiple $c\mu^\vee$ of $\mu^\vee$. Thus, all it remains to show is that $\nu = \mu^\vee$, i.e.: $c = 1$. We argue by contradiction, following [Folland2, Prop. 11.14].

Thus, suppose that $c \neq 1$. Then $\varepsilon := \frac{1}{2}|c-1| > 0$. Because Δ_G is continuous, the same is true of $x \mapsto \Delta_G(x)^{-1}$, so in particular there exists a neighbourhood U of e such that $|\Delta_G(x)^{-1} - 1| < \varepsilon$ for all $x \in U$. For such a neighbourhood

U, we find that

$$\begin{aligned}
|\nu(U) - \mu(U)| &= \left| \int_U \Delta_G(x)^{-1} \,\mathrm{d}\mu(x) - \int_U 1 \,\mathrm{d}\mu \right| \\
&= \left| \int_U \big(\Delta_G(x)^{-1} - 1\big) \,\mathrm{d}\mu(x) \right| \\
&\le \int_U \big|\Delta_G(x)^{-1} - 1\big| \,\mathrm{d}\mu(x) \\
&< \int_U \varepsilon \,\mathrm{d}\mu = \frac{1}{2}|c-1|\mu(U).
\end{aligned}$$

(This computation uses fundamental properties of the integral, ($\nu(B)$), and 6.29.)

Now recall that every neighbourhood of e contains a symmetric neighbourhood of e (1.14); thus, we may even choose U to be symmetric in the above discussion. But if U is symmetric then $\mu(U) = \mu^\vee(U)$, so the left-hand side of the above computation is also equal to

$$\begin{aligned}
|\nu(U) - \mu(U)| &= |c\mu^\vee(U) - \mu(U)| \\
&= |c\mu(U) - \mu(U)| \\
&= |c-1|\mu(U).
\end{aligned}$$

Together with the previous computation, this yields the sought-after contradiction to our assumption that $|c-1|$ is strictly positive. □

For the sake of completeness, we record how the above findings translate into assertions about the "opposite modular function", i.e. the map $g \mapsto \Delta_G(g^{-1}) = \Delta_G(g)^{-1}$ which some authors call "the" modular function (see 8.35.(2)).

8.38. Corollary. *Let G be a locally compact group, and let δ_G denote the map $g \mapsto \Delta_G(g^{-1}) = \Delta_G(g)^{-1}$, with Δ_G as in 8.34. Then the following hold.*

(i) δ_G is the unique map $G \to \mathbb{R}_{>0}$ satisfying

$$\nu(gB) = \delta_G(g)\nu(B) \tag{δ}$$

whenever ν is a right *Haar measure on G, B is a Borel subset of G and $g \in G$.*

(ii) δ_G is a continuous group homomorphism from G to $\mathbb{R}_{>0}$.

(iii) For every right Haar measure ν on G and every $f\colon G \to \mathbb{C}$ which is integrable with respect to ν,

$$\int_G f(gx)\,\mathrm{d}\nu(x) = \delta_G(g)^{-1}\int_G f(x)\,\mathrm{d}\nu(x) \qquad \textit{for all } g \in G.$$

(iv) For every right Haar measure ν on G and every $f\colon G \to \mathbb{C}$ which is integrable with respect to the left *Haar measure $\nu^\vee$ (cf. 4.20.(1)),*

$$\int_G f(x^{-1})\,\mathrm{d}\nu(x) = \int_G f(x)\delta_G(x)^{-1}\,\mathrm{d}\nu(x)$$

(in particular, both integrals in question are indeed defined).

Sketch of proof. The equality (δ) follows immediately from (Δ) (from 8.34) upon passing to the *left* Haar measure $\mu := \nu^\vee$ (recall 4.20.(1)).

The assertion about δ_G being a group homomorphism follows immediately from 8.36.(i) since $\mathbb{R}_{>0}$ is abelian. Continuity of δ_G is clear from continuity of Δ_G (8.36.(ii)) together with continuity of the inversion map $g \mapsto g^{-1}$.

The third claim (iii) can easily be derived from 8.36.(iii) with the help of 8.37:

$$\begin{aligned}\int_G f(gx)\,\mathrm{d}\nu(x) &= \int_G f(x^{-1}g^{-1})\,\mathrm{d}\nu^\vee(x)\\ &= \Delta_G(g^{-1})^{-1}\int_G f(x^{-1})\,\mathrm{d}\nu^\vee(x)\\ &= \delta_G(g)^{-1}\int_G f(x)\,\mathrm{d}\nu(x)\,.\end{aligned}$$

Finally, (iv) follows essentially from 8.36.(iv), again with the help of 8.37. Indeed, let f be as in the statement, and consider the auxiliary function $h\colon G \to \mathbb{C}$ given by $h(x) := f(x)\Delta_G(x)$ for $x \in G$. Then:

$$\begin{aligned}\int_G f(x)\delta_G(x)^{-1}\,\mathrm{d}\nu(x) &= \int_G h(x)\,\mathrm{d}\nu(x)\\ &= \int_G h(x^{-1})\,\mathrm{d}\nu^\vee(x)\\ &= \int_G h(x)\Delta_G(x)^{-1}\,\mathrm{d}\nu^\vee(x)\\ &= \int_G h(x^{-1})\Delta_G(x)\,\mathrm{d}\nu(x)\\ &= \int_G f(x^{-1})\,\mathrm{d}\nu(x)\,,\end{aligned}$$

as claimed. □

We may now conclude the section with the application we have been striving towards all along: finding bi-invariant Haar measures (or rather, groups which possess them).

8.39. A locally compact group G is *unimodular* if it has a bi-invariant Haar measure, or equivalently, if its modular function Δ_G is identically 1.

8.40. *Remark*. The term "unimodular" is used in several different contexts with the same basic meaning of "having modulus equal to 1"—for instance, a *unimodular number* is a (complex) number of absolute value, and a *unimodular matrix* is a matrix of determinant 1. In a slight extension of this latter usage, some authors use the term '*unimodular group*' to refer to the group of $n \times n$–matrices of determinant 1 (for some n, and with entries in some ring R). To avoid confusion, this group will be called the *special linear group* instead in this note and denoted accordingly by $\mathrm{SL}_n(R)$. ◇

8.41. *Examples*.

(1) It has already been noted several times that every (locally compact) abelian group is necessarily unimodular.

(2) All discrete groups are unimodular in light of 4.19.

(3) Our result 8.4 concerning Haar measures on products readily implies that any finite product of unimodular locally compact groups is again unimodular. Similarly, it follows from 8.1 and its proof that an *open*[33] subgroup of a unimodular group is again unimodular. (A different proof of this can be found e.g. in [Kallman, Lemma 5].)

To obtain more interesting (classes of) examples, we will have to put in more work. As the first entirely nontrivial consequence of the introduction of the modular function, we have the following:

(4) *Every compact [Hausdorff] group is unimodular.*

We shall provide two proofs for this fact.

[33]By contrast, general subgroups of unimodular subgroups *need not be* unimodular, cf. also 8.42 below.

Proof n. 1. (As in [Folland2, Prop. 11.13].) We start with a compact Hausdorff group G and with the observation that, for all $g \in G$, the right translate $Gg = r^g(G)$ is simply G. Now let μ be a left Haar measure on G—then $0 < \mu(G) < \infty$, e.g. by 4.21.(iii), so 8.34 implies that $\Delta_G(g) = \frac{\mu(Gg)}{\mu(G)} = \frac{\mu(G)}{\mu(G)} = 1$ for all $g \in G$, i.e. that Δ_G is identically 1, proving the claim. □

Proof n. 2. We shall now show more generally that, if K is a *compact subgroup* of a locally compact group G, then $\Delta_G(K) = \{1\}$.

First, recall from 8.36 that Δ_G is a continuous group homomorphism $G \to \mathbb{R}_{>0}$. Since K is a compact subgroup of G, it follows that $\Delta_G(K)$ is a compact subgroup of $\mathbb{R}_{>0}$. But the only compact subgroup of $\mathbb{R}_{>0}$ is, in fact, the trivial subgroup $\{1\}$.[34] □

Before we can state the next item, we shall need a bit of preparation. To fix notation, let G be any locally compact group. Since Δ_G has values in the abelian group $\mathbb{R}_{>0}$, clearly $\Delta_G(x) = 1$ whenever $x \in G$ is a *commutator*, i.e. an element of the form $[g, h] := ghg^{-1}h^{-1}$ for some g, $h \in G$. It follows that Δ_G is certainly identically 1 on the subgroup generated by all the commutators.

In abstract group theory, the subgroup we just mentioned is known as the *commutator subgroup* (or *derived subgroup*) of G and often denoted by $[G, G]$. For *topological* groups, however, it makes more sense to define the commutator subgroup $[G, G]$ to be the smallest *closed* subgroup which contains all commutators, as in [Folland2, p. 346]. (Of course, the two definitions agree for discrete groups.) For the sake of clarity, we may denote the commutator subgroup according to the first (purely algebraic) definition as $[G, G]_{\mathrm{alg}}$, and reserve the unadorned notation $[G, G]$ for the subgroup yielded by the second definition.

With this notational convention, it follows from 1.18.(i) that $[G, G]$ is the closure in G of $[G, G]_{\mathrm{alg}}$. Because Δ_G is continuous and identically 1 on $[G, G]_{\mathrm{alg}}$, we deduce that it is even identically 1 on all of $[G, G]$. Hence:

(5) Any locally compact group G satisfying $G = [G, G] = \overline{[G, G]_{\mathrm{alg}}}$ is unimodular. For instance, the group $G = \mathrm{SL}_n(\mathbb{R})$ (cf. 8.40) is unimodular (for every $n \geq 1$) since $G = [G, G]_{\mathrm{alg}}$, see e.g. SE2372936.

[34] If H is a nontrivial subgroup of $\mathbb{R}_{>0}$, then H contains some $x \neq 1$, and with it, all of $x^{\mathbb{Z}} = \{x^n : n \in \mathbb{Z}\}$, which is unbounded, so H cannot be compact.

(6) More generally, [Folland2, Prop. 11.12] shows that G is unimodular if $[G, G]$ is of finite index in G.

(7) Also generalizing the example of $\mathrm{SL}_n(\mathbb{R})$ from (5), it holds ([Knapp, Cor. 8.31]) that *every semisimple Lie group* and in fact *every reductive Lie group*[35] *is unimodular.*

To prove the claim for (connected) semisimple groups, one exploits commutators as in the previous items—but this time, at the level of Lie algebras, see e.g. ([Knapp, Lemma 4.28])—to show that every continuous group homomorphism $G \to \mathbb{C}^*$ has to be identically 1. This result is then used to prove the claim for reductive Lie groups.

(8) The general linear group $\mathrm{GL}_n(\mathbb{R})$ is unimodular (for every $n \geq 1$).

This was mentioned at the very end of the previous subsection; it also follows from (7) since $\mathrm{GL}_n(\mathbb{R})$ is a reductive Lie group. (On the other hand, $G = \mathrm{GL}_n(\mathbb{R})$ does not fall in any of the other categories from the previous items: it is neither abelian (for $n > 1$), nor discrete, nor compact, nor semisimple, and its commutator subgroup $[G, G] = \mathrm{SL}_n(\mathbb{R})$ is not of finite index.) Unimodularity of $\mathrm{GL}_n(\mathbb{R})$ can also be derived from that of $\mathrm{SL}_n(\mathbb{R})$ as follows ([Nachbin, p. 92f.]): first, if $\mathrm{GL}_n^+(\mathbb{R})$ denotes the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of matrices of *positive* determinant, then $\mathrm{GL}_n^+(\mathbb{R}) \cong \mathbb{R}_{>0} \times \mathrm{SL}_n(\mathbb{R})$ and is hence unimodular by item (3) above (since $\mathbb{R}_{>0}$ is abelian). One then uses the fact that $\mathrm{GL}_n^+(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ is a closed normal subgroup of finite index[36] to conclude that $\mathrm{GL}_n(\mathbb{R})$ must itself be unimodular (e.g., by applying [Nachbin, Chapter II, Prop. 23] to $G = \mathrm{GL}_n(\mathbb{R})$, $H = \mathrm{GL}_n^+(\mathbb{R})$ and $K = G/H$, cf. also 1.15).

As a final class of examples, of a slightly different flavour, we have:

(9) (SE323017) Suppose that a locally compact group G is *nilpotent*, i.e. has a finite central series. Then G is unimodular. $\diamond$

[35] The notion of a reductive Lie group varies from author to author. In the reference we have given, the claim is proved for Lie groups which are *reductive* in the sense of [Knapp, §VII.2].

[36] More precisely, the index is two: there are two cosets, corresponding to matrices of positive [resp., negative] sign.

8.42. *Non-example.* The standard example of a locally compact group which is *not* unimodular is given by the group

$$\left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}_{>0}, y \in \mathbb{R} \right\} \subset \mathrm{GL}_2(\mathbb{R}),$$

see e.g. [Folland2, Chapter 11, Exercise 4d]. Since $\mathrm{GL}_2(\mathbb{R})$ is unimodular by 8.41.(8) above, this example also shows that *subgroups of unimodular groups need not be unimodular themselves.* ◇

Let us record a final characterization of unimodularity for the sake of completeness:

8.43. PROPOSITION. *A locally compact group G is unimodular if and only if, for every left [right] Haar measure μ on G, the corresponding right [left] Haar measure $\mu^\vee$ (cf. 4.20.(1)) is equal to μ.*

Proof. Sufficiency is obvious: if $\mu^\vee = \mu$ for some left [right] Haar measure μ on G, then μ is bi-invariant, hence G is unimodular. As for necessity, we shall first prove the statement for an arbitrary *left* Haar measure μ. Accordingly, pick such a μ, and consider $\mu^\vee$. Since we now assume G to be unimodular, $\mu^\vee$ is proportional to μ by 7.1; let c denote the scaling factor.

Now let K be a compact neighbourhood of e in G. By 1.14.(i), K contains a symmetric open neighbourhood U of e; by 4.21.(i), monotonicity of μ and finiteness of μ on compact sets, we infer that $0 < \mu(U) < \infty$. On the other hand, $\mu^\vee(U) = \mu(U^{-1}) = \mu(U)$ since U is symmetric. But then the proportionality relation $\mu^\vee(U) = c\mu(U)$ can only hold if $c = 1$, and the claim follows.

The proof for a *right* Haar measure ν is, of course, entirely analogous; alternatively, one may write $\nu = \mu^\vee$ for a left Haar measure μ (by 4.20) and use the above argument to conclude that $\mu = \mu^\vee = \nu$; but by 4.4, $\mu = \nu^\vee$, so in conclusion $\nu = \nu^\vee$, as desired. □

8.4 Compact groups

In this final subsection, we specialize to those (topological) groups within the realm of locally compact groups which are additionally *compact.* Strictly speaking, these are (precisely) the *compact Hausdorff topological groups*, but following a convention similar to the one adopted in 2.5, we shall just refer to them as *compact groups*. (Thus, our usage will be the same as in [SWarner] and [Bourbaki].)

First of all, for compact groups the main theorems of existence and uniqueness of Haar measure take on the following form.

8.44. Theorem. *Let K be a compact group. Then there exists precisely one bi-invariant Radon measure μ on K such that $\mu(K) = 1$. Every nonzero left- or right-invariant Radon measure ν on K is proportional to μ in the sense that there exists a positive real number c such that $\nu(B) = c\mu(B)$ for all $B \in \mathcal{B}_K$.*

Proof. By the convention laid out at the beginning of the subsection, K is a locally compact group (cf. 2.6.(5)); by what was proved in 8.41.(4), K is unimodular. It follows that there exists a bi-invariant Haar measure $\tilde{\mu}$ on K. We know that $0 < \tilde{\mu}(K) < \infty$ (because $\tilde{\mu}$ is nonzero and finite on compact sets), so we may scale $\tilde{\mu}$ by the positive real number $\frac{1}{\tilde{\mu}(K)}$, thus obtaining a new bi-invariant Haar measure μ on K such that $\mu(K) = 1$. The last claim follows from 7.1 (recalling also unimodularity of K). □

8.45. *Remark*. While we derived 8.44 above from the earlier results on general locally compact groups, it turns out that this is not necessary: the result can actually be proved independently in the special case of compact groups, as is done e.g. in [Joys, Chapter 5]. ◇

8.46. For a compact group K, the unique measure μ mentioned in 8.44 is often called *normalized Haar measure* on K. An integral over a compact group K where the differential is denoted by $\mathrm{d}k$ or similar expressions (cf. 6.30.(1)) is almost always to be understood as an integral with respect to normalized Haar measure μ on K; in our notation, "$\mathrm{d}k$" would be denoted less ambiguously by '$\mathrm{d}\mu(k)$'.

8.47. *Examples.*

(1) The measure $\dot{\lambda}$ from 8.12 is normalized Haar measure on the compact group $\mathbb{R}/\mathbb{Z}$.

(2) Consider the compact group $\mathbb{T}$ (the circle group from 1.2.(2)). Then normalized Haar measure on $\mathbb{T}$ is precisely the μ constructed in 8.26.

(3) Let G be a finite group (equipped with the discrete topology). Then G is compact, and normalized Haar measure on G is given by

$$\mu(A) = \frac{\#A}{\#G}, \qquad A \subseteq G,$$

where $\#X$ denotes cardinality of the finite set X. In other words, normalized Haar measure on a finite group is counting measure divided by the cardinality of the group. $\diamond$

8.48. Recall that, in contrast with locally compact groups, products of compact [Hausdorff] groups are always again compact [Hausdorff] regardless of the "number" of factors.[37] It is then natural to ask whether, for compact groups, our earlier result 8.4 concerning Haar measures on finite products can be extended to infinite products.

As usual, the major technical problems come from the measure-theoretic side. To prove our earlier result, we (implicitly) leaned on the notion of the *Radon product* of two Radon measures (see 8.5); the challenge now lies in extending the scope of this notion to arbitrary products (under the additional assumption that all factors are *compact*). This is accomplished e.g. in [Folland2, Thm. 7.28], to which we refer for details; in this note, we shall content ourselves with restating the result in our special context of Haar measures.

8.49. Proposition. *Let K be a compact group which can be written as a product $\prod_{i\in I} K_i$ for compact groups K_i, $i \in I$, with I an arbitrary index set. For each $i \in I$, let μ_i denote normalized Haar measure on K_i. Then normalized Haar measure on K is the unique Radon measure μ on K such that $\mu(K) = 1$ and such that,*

- *for every finite subset $J \subseteq I$,*
- *every ordering $J = \{i_1, \dots, i_r\}$ of the elements of J,*
- *and every Borel subset $E \subseteq \prod_{j=1}^{r} K_{i_j}$,*

the set

$$E^* = \{k = (k_i)_{i\in I} : (k_{i_1}, \dots, k_{i_r}) \in E\} \in \mathcal{B}_K$$

has measure $\mu(E^*) = (\mu_{i_1} \times \cdots \times \mu_{i_r})(E)$.

[37] This is, of course, Tychonoff's theorem, reliant on the axiom of choice.

(As was mentioned above, the bulk of the proof is [Folland2, Thm. 7.28].)

8.50. *Remark.* Resume the notations of the above statement, and pick a Borel subset $E \subseteq K$ which is of the form $\prod_{i\in I} E_i$ for Borel subsets $E_i \subseteq K_i$, with equality $E_i = K_i$ for all but at most finitely many $i \in I$. Then

$$\mu(E) = \prod_{i\in I} \mu_i(E_i)$$

(where the product on the right-hand side converges since only at most finitely many factors are allowed to be $\neq 1$). In analogy with the case of finite products (cf. 8.8), we have that, if all K_i are second-countable and the index set I is countable (so that K is itself second-countable), then μ is the *unique* measure with the property described immediately above. ◇

As an application of the proposition, we compute normalized Haar measure on our paradigmatic profinite group $A^{\mathbb{N}}$, first introduced in 2.11.

8.51. *Example.* Let A be a finite abelian group equipped with the discrete topology, and set $K := A^{\mathbb{N}}$ (equipped with the product topology, cf. 2.11). Then the assumptions of the proposition are met (with $I = \mathbb{N}$ and $K_i = A$ for all $i \in I$). It follows from 8.50 that, if μ denotes normalized Haar measure on G and μ_0 denotes normalized Haar measure on A, then $\mu(\prod_i E_i) = \prod_i \mu_0(E_i)$ whenever E_i, $i \in I$, are subsets of A such that all but finitely many of them are equal to A. Combining this with the explicit description of μ_0 from 8.47.(3), we can then compute $\mu(E)$ for several subsets $E \subseteq K$ of interest.

As a baby example, consider e.g. the set $B = \{(a_n)_{n\in\mathbb{N}} \in K : a_0 = 0\} \subset K$. By the above, its measure is given by

$$\mu(B) = \frac{1}{\#A} \cdot 1 \cdot 1 \cdots = \frac{1}{\#A}.$$

More generally, if $(b_0, \dots, b_{m-1})$ is a finite sequence of elements of A, then

$$B' = \{(a_n)_{n\in\mathbb{N}} \in K : a_i = b_i \text{ for } i = 0, \dots, m-1\} \subset K \tag{*}$$

is a Borel set with measure given by $(\#A)^{-m}$. This computation is especially useful in view of the fact that *every open ball in K of radius $r \leq 1$ is of the form* (*) *for a unique finite sequence $(b_0, \dots, b_{m-1})$ of elements of A*, as seen in E.8. One then concludes that every open ball of radius $r = 2^{-m}$ $(m \geq 0)$ in K has normalized Haar measure equal to $(\#A)^{-(m+1)}$.

Knowing the measure of all open balls is sufficient, at least in theory, to compute the Haar measure of *any* Borel subset of K. Indeed, let $E \in \mathcal{B}_K$. Then, for every $\varepsilon > 0$, there exists an open set $U \supseteq E$ with $\mu(U) \leq \mu(E) + \varepsilon$ (by outer regularity of μ). From the explicit description of the open balls in K (again, see E.8), we know that the collection of open balls in K is countable, so U can be written as a *countable* union of open balls in K. But the same exercise also tells us that any two open balls in K are either disjoint or one is contained in the other; it follows that U can actually be written as a *disjoint* union $\bigcup_{j \in J} B_j$ of open balls in K, where the index set J is at most countable. But then $\mu(U) = \sum_{j \in J} \mu(B_j)$ by countable—or even finite—additivity.

Let us conclude by mentioning in passing that the measures of open balls can also be determined without resorting to 8.49. Indeed, let $B \subseteq K$ be an open ball of some radius $r \leq 1$—by the explicit form of the metric on K, we may and shall assume that r is the reciprocal of an integral power of 2, say $r = 2^{-m}$. For $k \in K$, the translate $k + B$ of B by k is again an open ball of radius r in K. One readily checks that there are precisely $(\#A)^{m+1}$ distinct translates of B; that they are all disjoint; and that they cover K. Since $\mu(K) = 1$, finite additivity forces $\mu(B) = (\#A)^{-(m+1)}$, as we saw above. ◇

In the next chapter, we shall use the above example to determine normalized Haar measure on the ring $\mathbb{F}[\![x]\!]$ of *formal power series* in one variable with coefficients in a finite field $\mathbb{F}$ (cf. the preliminaries), which will be one of our main examples of *topological rings* and, most importantly, will accompany us throughout our discussion of *locally compact fields.*

E Exercises

E.1. *Exercise.* Consider the additive group of the reals, and consider the topology on $\mathbb{R}$ generated by the half-open intervals $[a, b)$. Prove:

(a) the map

$$\begin{aligned} \mu\colon \mathbb{R} \times \mathbb{R} &\to \mathbb{R}, \\ (x, y) &\mapsto x + y, \end{aligned}$$

is continuous (the domain being equipped with the product topology), but

(b) the inversion map

$$\begin{aligned} \nu\colon \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto -x, \end{aligned}$$

is not continuous.

E.2. *Exercise.* Prove 1.16, whose statement is reproduced below for convenience.

Let G be a group and let a topology be given on G. In order for both μ_G and ν_G to be continuous, it is necessary and sufficient that $q_G = \mu_G \circ (\mathrm{id}_G \times \nu_G)$ be continuous.

E.3. *Exercise.* Prove[38] that $(\mathbb{R}, +)$ and $(\mathbb{R}^2, +)$ are isomorphic as (abstract) groups but not as topological groups; in other words,

(a) show that there exists a bijective group homomorphism $\mathbb{R}^2 \to \mathbb{R}$; and

(b) show that there exists no map $\mathbb{R}^2 \to \mathbb{R}$ which is both a homoeomorphism and a group homomorphism.

(In fact, $\mathbb{R}$ and $\mathbb{R}^2$ are not homoeomorphic.)

E.4. *Exercise.* This problem is about pointwise operations with functions; the reader is invited to check the preliminaries for more on this topic.

[38] In showing the first assertion, the axiom of choice is key.

(a) Let X, Y, Z be topological spaces, and let $\beta\colon Z \times Y \to Y$ be a continuous map. Check that, if $f\colon X \to Y$ is continuous and $z \in Z$, then the map $X \to Y$ sending x to $\beta(z, f(x))$ is again continuous.

(b) Resume the notations X, Y, Z and β from the previous item, and suppose that $Z = Y$. Then, if f and g are continuous maps from X to Y, then so is the map $x \mapsto \beta(f(x), g(x))$.

(c) Use (b) to show that, for any topological group $(G, \cdot)$ and any topological space X, the set $C(X; G)$ of continuous maps $X \to G$ becomes a group—in fact, a subgroup of $G^X = \{f\colon X \to G\}$—with pointwise multiplication.

(d) Use the previous items to show that, if $\mathbb{K}$ denotes either the real field $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$, and X denotes any topological space, then the set $C(X; \mathbb{K})$ of continuous maps $X \to \mathbb{K}$ becomes a $\mathbb{K}$-linear space—in fact, a subspace of $\mathbb{K}^X$—with pointwise addition and pointwise scalar multiplication.

E.5. *Exercise.* A group homomorphism between topological groups is continuous if and only if it is continuous at the identity.

E.6. *Exercise.* Let G be a group, Σ be a family of normal subgroups which is closed under finite intersections (e.g., a nested family). Then there is a unique topology on G which makes G into a topological group and such that the subgroups $H \in \Sigma$ form a neighbourhood base around the neutral element. Show:

(a) In this topology, each $H \in \Sigma$ is both open and closed in G.

(b) The topology is Hausdorff if and only if $\bigcap_{H \in \Sigma} H$ is the trivial subgroup.

(c) The topology is discrete if and only if the trivial subgroup belongs to Σ.

(*Hint.* Cf. 1.19.(i).)

E.7. *Exercise.* Prove 1.21. In fact, prove the following strengthening of the third claim: a quotient of *any* topological group by a closed normal subgroup is Hausdorff.

(*Hint.* Use (i) of 1.19.)

E.8. *Exercise.* Let A be an abelian group, $G = A^{\mathbb{N}}$ be as described in 2.11. For $x \in G$ and $r \in \mathbb{R}_{>0}$, we let $B_r(x)$ [resp., $D_r(x)$] denote the *open* [resp., *closed*] ball of radius r with centre x, where of course we view G as equipped with the metric d from 2.11.(i).

(a) Verify that the closed unit ball $D_1(x)$ around *any* point $x \in G$ is equal to all of G.

(b) Let $B \subset G$ be an open ball $B = B_r(x)$ for some $x \in G$ and some $0 < r \leq 1$. Then B is of the form

$$\{(a_n)_{n\in\mathbb{N}} \in G : a_i = b_i \text{ for } i = 0, \ldots, m-1\} \qquad \text{(EQ. E.8)}$$

for a unique positive integer m and a unique finite sequence $(b_0, \ldots, b_{m-1})$ of elements of A.

(c) Let $D \subset G$ be a closed ball $D_r(x)$ for some $x \in G$ and some $0 < r < 1$. Then D is of the form (EQ. E.8) for a unique positive integer m and a unique finite sequence $(b_0, \ldots, b_{m-1})$ of elements of A.

(d) In fact, check that, for every $x \in G$ and every $r > 0$, there exists a $\rho > 0$ such that $B_r(x) = D_\rho(x)$.
(*Hint.* It is always possible to choose $\rho \in \{r, \frac{r}{2}\}$.)

Conclude that G satisfies the following condition (cf. 2.11).

(ZD) Every point in G has a neighbourhood basis consisting of neighbourhoods which are *both* open *and* closed in G.

E.9. *Exercise.* Let X be a topological space satisfying

(ZD) every point in X has a neighbourhood basis consisting of neighbourhoods which are *both* open *and* closed in X.

Show that, if X is T_1, then:

(TD) the only non-empty connected subsets of X are the singletons $\{x\}$, $x \in X$.

E.10. *Exercise.* Let A be an abelian group, $G = A^{\mathbb{N}}$ be as described in 2.11. Show that, in order for G to be locally compact, it is necessary and sufficient that A be a finite group, and that if this is the case, then G is even compact.

E.11. *Exercise.* Prove 2.22.

E.12. *Exercise.* We adopt the following notations: for a topological space X, let $\mathcal{B} = \mathcal{B}_X \subseteq \mathcal{P}(X)$ denote the σ-algebra generated by the open subsets of X, and let $\mathcal{C} = \mathcal{C}_X \subseteq \mathcal{P}(X)$ denote the σ-algebra generated by the compact subsets of X. Prove the following statements.

(a) If X is a finite set with at least three distinct elements, then there exists a topology on X such that $\mathcal{C} \nsubseteq \mathcal{B}$.

(b) If X is the real line with the discrete topology, then $\mathcal{B} \nsubseteq \mathcal{C}$.

E.13. *Exercise.* Given a measure space $(X, \mathcal{A}, \mu)$, prove the following claims.

(a) Let $(Y, \mathcal{B})$ be a measurable space and f be a function $X \to Y$. Suppose that f is $\mathcal{A}$-$\mathcal{B}$-*measurable*, i.e. that the preimage $f^{-1}(B)$ lies in $\mathcal{A}$ for all $B \in \mathcal{B}$. (Cf. 6.19.) Then

$$\begin{aligned} \nu\colon \mathcal{B} &\to [0, \infty], \\ B &\mapsto \mu\big(f^{-1}(B)\big) \end{aligned}$$

is a measure on $(Y, \mathcal{B})$, often denoted by $\nu = f_*\mu$ in the literature and called the *pushforward* of μ along f.

(b) Let Y be an arbitrary subset of X, and set

$$Y \cap \mathcal{A} \coloneqq \{Y \cap A : A \in \mathcal{A}\} \subseteq \mathcal{P}(Y).$$

Then $Y \cap \mathcal{A}$ is a σ-algebra (the *trace σ-algebra*). If, moreover, $Y \in \mathcal{A}$, then $Y \cap \mathcal{A} = \{A \in \mathcal{A} : A \subseteq Y\} \subseteq \mathcal{A}$. In this case the restriction $\nu \coloneqq \mu|_{Y \cap \mathcal{A}}$ is a measure on $(Y, Y \cap \mathcal{A})$.

(*Hint.* In both cases, we can write $\nu = \mu \circ \Phi$ for a suitable Φ; indeed,

- in (a), $\Phi\colon \mathcal{B} \to \mathcal{A}$ is the map $B \mapsto f^{-1}(B)$; and
- in (b), Φ is the inclusion $Y \cap \mathcal{A} \hookrightarrow \mathcal{A}$.

But Φ commutes with arbitrary unions and intersections and sends $\emptyset$ to $\emptyset$.)

E.14. *Exercise.* (Addendum to part (a) of the previous exercise.) Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be measurable spaces, and let $f\colon X \to Y$ be a bijective, $\mathcal{A}$-$\mathcal{B}$-measurable function with $\mathcal{B}$-$\mathcal{A}$-measurable inverse f^{-1}. (Cf. E.13, 6.19.) Then *every* [nonzero] measure ν on $(Y, \mathcal{B})$ is of the form $f_*\mu$ for a unique [nonzero] measure μ on $(X, \mathcal{A})$, namely $\mu = (f^{-1})_*\nu$.

E.15. *Exercise.* Prove that the Borel-σ-algebra of a topological group is both left-invariant and right-invariant.

(*Hint.* Recall 4.2.(2) and 6.21.(2).)

E.16. *Exercise.* Prove claims (i)–(iv) in 4.4.

(*Remark.* The first part of claim (iii) can be seen as a special case of E.13.)

E.17. *Exercise.* Prove that, if X is a discrete space, then counting measure on $(X, \mathcal{P}(X))$ has properties (a)–(b) from 4.7.

E.18. *Exercise.* Prove that, if X is a discrete space and μ denotes counting measure on $(X, \mathcal{P}(X))$, then μ is both inner regular and outer regular on all measurable sets, i.e.,

$$\begin{aligned}\mu(E) &= \sup\{\mu(K) : K \subseteq E \text{ compact}\}\\ &= \inf\{\mu(U) : U \supseteq E \text{ open}\}\end{aligned}$$

for every $E \subseteq X$.

E.19. *Exercise.* Let μ be a Radon measure (as per 4.13) on a locally compact Hausdorff space X. Let h denote the restriction of μ to $\mathcal{K}(X)$, the collection of compact subsets of X, and let $\mu^*\colon \mathcal{P}(X) \to [0, \infty]$ be the outer measure obtained from h by an application of 4.8. Show that μ^* is none other than the outer measure obtained from μ as explained in 3.4 (i.e. essentially by the formula (OUT.MEAS.)); in particular, μ^* agrees with μ on all Borel sets (hence with h on all compact sets) and is outer regular on *all* μ^*-measurable subsets (not just Borel sets).

E.20. *Exercise.* Let X be a Hausdorff space and $\mu\colon \mathcal{B}_X \to [0,\infty]$ be a measure defined on the Borel-σ-algebra of X. Suppose that μ is *locally finite* in the sense that every point $x \in X$ has a neighbourhood $U \in \mathcal{B}_X$ such that $\mu(U) < \infty$. Then for every compact set $K \subseteq X$ it holds that $\mu(K) < \infty$.

E.21. *Exercise.* Let G be a topological group and let $\mathcal{B}_G$ denote its Borel-σ-algebra. Then, using the notations introduced in 4.4, the following hold.

(a) ${\mathcal{B}_G}^\vee$ is precisely $\mathcal{B}_G$.

Suppose now that G is locally compact. Then:

(b) for every Radon measure μ on G, $\mu^\vee$ is again a Radon measure on G, and

(c) if, in (b), μ is not the zero measure, then $\mu^\vee$ is also not identically zero.

(*Hint.* To prove claim (a), cf. also the hint given for E.15. Claim (b) can be proved directly or regarded as a special case of E.32.(a).)

E.22. *Exercise.* Consider the additive group of the reals $(\mathbb{R},+)$. The following are given:

(a) K is a compact interval in $\mathbb{R}$;

(b) U is a (bounded) open interval in $\mathbb{R}$, with $U \neq \emptyset$.

Denoting by $\ell(K)$ and $\ell(U)$ the length of K and U, respectively, find a closed formula for the Haar covering number $(K:U)$ in terms of $\ell(K)$ and $\ell(U)$.

(See 5.3 for the definition of $(K:U)$.)

E.23. *Exercise.* Prove (i)–(iii) from 5.5.

E.24. *Exercise.* Prove that, if K, V and K_0 are as in 5.8, then $(K:V) \leq (K:K_0)(K_0:V)$.

(See 5.3 for the definition of $(K:V)$.)

E.25. *Exercise.* Consider the measure space $(X,\mathcal{A},\mu) = (\mathbb{R},\mathcal{B}_\mathbb{R},\lambda)$, where $\lambda\colon \mathcal{B}_\mathbb{R} \to [0,\infty]$ denotes (the suitable restriction of) Lebesgue measure. Fix the notations $B_+ = (0,\infty)$ and $B_- = (-\infty,0)$; then B_+ and B_- are Borel sets in $\mathbb{R}$. Show that 1_{B_+} and -1_{B_-} are both quasi-integrable, but their sum (which is the sign function from 6.8.(2)) is not.

E.26. *Exercise.* Prove the claims made in 6.15.(1).

E.27. *Exercise.* Prove the claims made in 6.15.(2).

(*Hint.* It is convenient to use the equivalent characterization of integrability given in 6.28.)

E.28. *Exercise.* Let X, Y be topological spaces. Use 6.21.(2) to infer the following.

(a) Let $f\colon X \to Y$ be a homoeomorphism. Then, for a subset $E \subseteq X$, the image $f(E)$ is Borel in Y if and only if E is Borel in X.

(b) Let $\varphi\colon Y \to X$ be a continuous, open injection—i.e., a homoeomorphism onto an open subset $\varphi(Y) \subseteq X$. Then, for $B \in \mathcal{B}_Y$, the image $f(B)$ is Borel in X. (Recall also footnote 21.)

E.29. *Exercise.* Let $(X, \mathcal{A})$ be a measurable space, f be a complex-valued function on X. Then the following are equivalent for f:

(i) f is simple in the sense of 6.5.

(ii) f is measurable (in the sense of 6.19) and the range $f(X)$ of f is a finite set.

E.30. *Exercise.* Prove the claim made in 6.24.(8).

(*Hint.* For one implication, use 6.24.(7). For the other, recall that $\mathrm{Re}\colon \mathbb{C} \to \mathbb{R}$ and $\mathrm{Im}\colon \mathbb{C} \to \mathbb{R}$ are continuous maps and use 6.21.)

E.31. *Exercise.* Prove the claim made in 6.24.(9). (Cf. the previous exercise.)

E.32. *Exercise.* Given a locally compact Hausdorff space X and a Radon measure μ on X, prove the following strengthenings of the claims from E.13.

(a) Let $f\colon X \to Y$ be a homoeomorphism to another LCH space Y. Then the pushforward $\nu = f_*\mu$ (cf. E.13.(a)) is a Radon measure on Y.

(b) Let Y be an *open* subset of X, so that Y is LCH with the subspace topology (by 2.4) and $\mathcal{B}_Y = Y \cap \mathcal{B}_X \subseteq \mathcal{B}_X$ (cf. E.13.(b) as well as footnote 21). Then the restriction $\nu = \mu|_{\mathcal{B}_Y}$ is a Radon measure on Y.

Moreover,

(c) if $\varphi\colon Y \to X$ is a continuous, open injection—i.e., a homoeomorphism onto an open subset $\varphi(Y) \subseteq X$—then

$$\begin{aligned} \nu\colon \mathcal{B}_Y &\to [0,\infty], \\ B &\mapsto \mu(\varphi(B)) \end{aligned}$$

is a Radon measure on Y.

(*Hint.* Observe that (c) follows from combining (a) and (b); conversely, (c) entails both (a) and (b) as special cases. Any of the three claims can be proved by writing $\nu = \mu \circ \Phi$ for the obvious choice of $\Phi\colon \mathcal{B}_Y \to \mathcal{B}_X$—cf. the hint given for E.13, and recall E.28—and observing that Φ commutes with arbitrary unions and intersections, sends $\emptyset$ to $\emptyset$, and, under the assumptions of either (a), (b) or (c), has the following additional properties.

(Φ1) For $B \in \mathcal{B}_Y$, we have that $\Phi(B)$ is compact [resp., open] in X if and only if B is compact [resp., open] in Y.

(Φ2) If $A \in \mathcal{B}_X$ and $B \in \mathcal{B}_Y$ are such that $A \subseteq \Phi(B)$, then $A = \Phi(B')$ for some $B' \in \mathcal{B}_Y$ with $B' \subseteq B$.

(Φ3) If $\Phi(B)$ (with $B \in \mathcal{B}_Y$) is contained in some open subset $U \subseteq X$, then there exists an open subset $V \subseteq Y$ such that $B \subseteq V$ and $\Phi(V) \subseteq U$.)

E.33. *Exercise.* (Addendum to part (a) of the previous exercise.) Let X, Y be LCH spaces, and let $f\colon X \to Y$ be a homoeomorphism. Then every [nonzero] Radon measure ν on Y is of the form $f_*\mu$ for a unique [nonzero] Radon measure μ on X, namely $\mu = (f^{-1})_*\nu$.

(*Hint.* Recall E.14.)

E.34. *Exercise.* Let G, H be topological groups, and let $\varphi\colon H \to G$ be a homoeomorphism of H onto an open subspace $\varphi(H) \subseteq G$. Suppose that there exists a function $\gamma\colon H \to G$ such that

$$\varphi(hx) = \gamma(h)\varphi(x) \qquad \text{for all } x \in H.$$

Then:

(a) there is a *unique* γ for which the above equality holds;

(b) γ is a group homomorphism $H \to G$; and finally,

(c) γ is again a homoeomorphism onto an open subspace $\gamma(H) \subseteq G$, and is surjective (onto G) if and only if φ is.

(*Hint.* $\gamma = (r^{\varphi(e)})^{-1} \circ \varphi$.)

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