Endogenous credit constraints: the role of informational non-uniqueness

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Abstract

We point out that the equilibrium definition applied by Miao and Wang [7] in their model of stock price bubbles involves an implicit assumption about the formulation of an endogenous credit constraint. By dropping this assumption, one can construct infinitely many additional equilibria for the Miao Wang economy, all of which exhibit stock price bubbles. The underlying reason for this result is informational non-uniqueness, a phenomenon known from the literature on dynamic games. Neither the original equilibria discussed by Miao and Wang [7] nor the additional ones shown to exist due to informational non-uniqueness are Markov-perfect. For this reason we propose a recursive equilibrium definition for the Miao Wang economy and show how it can be used to construct Markov-perfect equilibria with stock price bubbles.

Keywords: endogenous credit constraint; informational non-uniqueness, stock price bubble, Markov-perfect equilibrium

JEL Classification: D25, E22, E44, G10

1 Introduction

During the last two decades, macroeconomists have increasingly analyzed how financial market imperfections affect the business cycle and the functioning of monetary policy. In many of these studies, the market imperfections take the form of an endogenous credit constraint; see, e.g., Albuquerque and Hopenhayn [1], Alvarez and Jermann [2], Gertler and Karadi [4], Jermann and Quadrini [5], Kiyotaki and Moore [6], and Miao and Wang [7]. Such a constraint limits the size of a loan by an endogenous function of the borrower's own choice variables. The aforementioned papers differ from each other, however, in the way how this constraint is formulated and how equilibria are defined. Jermann and Quadrini [5], for example, use a recursive equilibrium definition, according to which the decisions of all agents are described by functions of the individual and aggregate states of the economy. Similarly, Albuquerque and Hopenhayn [1] consider fully state contingent contracts. Equilibria defined in this way are Markov-perfect by construction, that is, the agents' decision rules describe their optimal behavior not only in equilibrium but also off the equilibrium path. Kiyotaki and Moore [6] and Miao and Wang [7], on the other hand, define equilibria as sets of time-dependent functions, which satisfy the optimality and market clearing conditions in all periods. These equilibria normally fail to be Markov-perfect and, in addition, the phenomenon of informational non-uniqueness may arise.

Informational non-uniqueness was first detected by Başar [3] in dynamic games but has not received too much attention. It originates from the fact that, in a deterministic model, an agent's action can be described as a function of both time and individual states in infinitely many different ways. The information set is so big that it contains redundancies. But this is not the end of the story. Because different representations of the actions of one agent lead to different incentives for other agents, the resulting equilibrium paths differ from each other as well. In the case of endogenous credit constraints, for example, the lenders can – through the choice of the representation of the constraint – affect the behavior of the borrowers. We explain and analyze the effects of informational non-uniqueness in a slightly modified version of the model from Miao and Wang [7].

In contrast to most of the other papers in the macroeconomic literature on bubbles the paper by Miao and Wang [7] deals with stock price bubbles, i.e., with bubbles on productive assets. Moreover, it is a model with infinitely lived agents. The endogenous credit constraint restricts a borrowing firm's investment rate, $i(t)$, by the stock price value of a hypothetical firm that owns the fraction ξ of the borrowing firm's capital stock, $k(t)$:

$$i(t) \leq V(\xi k(t), t). \tag{1}$$

The function V is the endogenously determined optimal value function of the firm's optimization problem. Miao and Wang [7] show that there exist two different stationary equilibria in which the credit constraint (1) is binding. In the first one, the value of each firm coincides with the value of its capital stock. In this case, Tobin's average Q equals marginal Q. In the second stationary equilibrium the firm's value exceeds the market value of its capital stock and marginal Q falls short of average Q. Now suppose that the constraint (1) is replaced by

$$i(t) \leq I(k(t), t)$$

for some function I. If this function is chosen in such a way that

$$I(k(t), t) = V(\xi k(t), t) \tag{2}$$

holds for all t, then the endogenous credit constraint (1) is obviously satisfied. Miao and Wang [7] implicitly assume that the function I is given by $I(k, t) = V(\xi k, t)$, in which case (2) holds trivially. However, condition (2) can be ensured for infinitely many other functions I as well. Moreover, since different choices of I generate different incentives for the borrowers, one obtains infinitely many different equilibria. This is the essence of informational non-uniqueness.

The paper is organized as follows. In section 2 we review the model of Miao and Wang [7] and describe how we modify it. The modification is solely done for analytical convenience and does not affect the underlying economic assumptions. After describing the decision problems of households and firms and stating all market clearing conditions, we provide a detailed discussion of the endogenous credit constraint. To highlight the origin of informational non-uniqueness, we first present the original equilibrium definition used by Miao and Wang [7] and point out that it makes an implicit assumption about the representation of the credit constraint. Then we state an equilibrium definition, which we call relaxed Miao-Wang equilibrium and which does not impose the aforementioned implicit assumption. In section 3 we show that there exists a continuum of stationary relaxed Miao-Wang equilibria that are mutually different from each other and also different from the equilibria in [7]. Interestingly, all of them have the property that there exists a stock price bubble. Finally, in section 4 we argue that none of the original or relaxed Miao Wang equilibria are Markov-perfect. To obtain Markov-perfect equilibria we propose a recursive equilibrium definition for the Miao Wang economy, in which decision rules are functions on the state space of the model rather than functions on the time domain. Section 5 concludes the paper. All proofs are relegated to the appendix.

2 Model formulation

In this section we describe the model which will be analyzed in the rest of the paper. It is very similar to the model used by Miao and Wang [7]. The only essential difference is that we do not assume that investment opportunities arrive randomly at discrete instants of time (lumpy investment) but that we use a simpler and fully deterministic approach.

Time is modelled as a continuous variable on the domain $\mathbf{T} = \mathbb{R}_+$. The economy is populated by a unit interval of households and a unit interval of firms. Firms use the input factors capital and labor to produce a single output good. The latter can be used for consumption and for investment and it serves as numeraire. Households are endowed with labor and they own the firms. Firms own their capital and rent labor services from the households. There a two assets in the economy: bonds, which are available in zero net supply, and firm equity. Only the households have access to the two asset markets.

2.1 Households

There exists a unit interval of identical and infinitely-lived households. The representative household is endowed with a constant flow of one unit of labor per period. Initially at time $t = 0$ it owns equally many shares of all firms in the economy and holds no bonds. The household has the instantaneous utility function $U : \mathbb{R}_+ \mapsto \mathbb{R}$ and the time-preference rate $\rho > 0$.[1] Let us denote by $c(t)$, $s(t)$, and $b(t)$ the rate of consumption, the share holdings, and the bond holdings of the representative household at time $t \in \mathbf{T}$. Furthermore, we denote by $r(t)$, $w(t)$, $\pi(t)$, and $v(t)$ the real interest rate, the wage rate, the dividend flow, and the share price at time $t \in \mathbf{T}$. The household takes the functions $r : \mathbf{T} \mapsto \mathbb{R}$, $w : \mathbf{T} \mapsto \mathbb{R}$, $\pi : \mathbf{T} \mapsto \mathbb{R}$, and $v : \mathbf{T} \mapsto \mathbb{R}$ as given and chooses the functions $c : \mathbf{T} \mapsto \mathbb{R}$, $s : \mathbf{T} \mapsto \mathbb{R}$, and $b : \mathbf{T} \mapsto \mathbb{R}$ so as to maximize its lifetime utility

$$\int_0^{+\infty} e^{-\rho t} U(c(t))\, \mathrm{d}t \tag{3}$$

subject to the budget constraint

$$\dot{b}(t) + v(t)\dot{s}(t) + c(t) = r(t)b(t) + \pi(t)s(t) + w(t) \tag{4}$$

and the initial conditions

$$s(0) = 1 \text{ and } b(0) = 0. \tag{5}$$

2.2 Firms

There exists a unit interval of identical and infinitely-lived firms which produce output from capital and labor. The firms own their capital stock and they rent the labor services from the households. We denote the capital stock and the labor demand of the representative firm at time $t \in \mathbf{T}$ by $k(t)$ and $\ell(t)$, respectively. Output of the representative firm at time $t \in \mathbf{T}$ is given by $F(k(t), \ell(t))$ and its rate of investment is denoted by $i(t)$. The production function

[1] Later on we will assume the utility function to be linear.

$F : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ satisfies all standard assumptions and will later be chosen to be of Cobb-Douglas form. The flow of profits of the firm at time $t \in \mathbf{T}$ is therefore given by

$$\pi(t) = F(k(t), \ell(t)) - w(t)\ell(t) - i(t) \tag{6}$$

and the capital stock evolves according to

$$\dot{k}(t) = i(t) - \delta k(t), \tag{7}$$

where $\delta > 0$ is the rate of capital depreciation. The initial capital stock is given by

$$k(0) = \bar{k}, \tag{8}$$

where $\bar{k} > 0$ is an exogenous parameter. It is assumed that investment is non-negative and that it is bounded from above by a constraint of the form

$$i(t) \leq I(k(t), t). \tag{9}$$

This constraint is the result of (unmodelled) credit market imperfections and will be discussed in much more detail in subsection 2.4.

For notational convenience we introduce the state space $\mathbf{K} = [0, \bar{K}]$ and assume that the initial capital endowment satisfies $\bar{k} \in \mathbf{K}$.[2] The representative firm takes the functions $r : \mathbf{T} \mapsto \mathbb{R}$, $w : \mathbf{T} \mapsto \mathbb{R}$, and $I : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ as given and chooses functions $k : \mathbf{T} \mapsto \mathbb{R}_+$, $\ell : \mathbf{T} \mapsto \mathbb{R}_+$, and $i : \mathbf{T} \mapsto \mathbb{R}_+$ so as to maximize its shareholder value

$$\int_0^{+\infty} e^{-\int_0^t r(\tau)\,\mathrm{d}\tau} \pi(t)\,\mathrm{d}t \tag{10}$$

subject to (6)-(9). The optimal value function of this optimization problem will be denoted by $V : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$, that is,

$$V(\kappa, \tau) = \max \left\{ \int_\tau^{+\infty} e^{-\int_\tau^t r(t')\,\mathrm{d}t'} \pi(t)\,\mathrm{d}t \,\middle|\, \text{subject to (6)-(7), (9), and } k(\tau) = \kappa \right\}. \tag{11}$$

2.3 Market clearing

The labor market clears at time $t \in \mathbf{T}$ if

$$\ell(t) = 1, \tag{12}$$

[2] For the time being, the reader may assume that $\bar{K}$ is a sufficiently large real number or even that $\bar{K} = +\infty$. In the latter case, one can identify $\mathbf{K}$ with $\mathbb{R}_+$. The actual value of $\bar{K}$ will only be relevant in theorem 3.

the asset markets clear if

$$b(t) = 0, \tag{13}$$

$$s(t) = 1, \tag{14}$$

$$v(t) = V(k(t), t), \tag{15}$$

and the output market clears if

$$F(k(t), \ell(t)) = c(t) + i(t).$$

Due to Walras' law, one of the market clearing conditions is redundant and we will therefore disregard the output market clearing condition in the analysis.

Finally, we have to make sure that households keep their firms running. This will be the case if the market value of the firms, which are held by the representative household, is at least as large as the market value of the capital installed in those firms. Since the value of capital outside the firms is equal to 1, this means that

$$V(k(t), t) \geq k(t) \tag{16}$$

must hold for all $t \in \mathbf{T}$.

2.4 The endogenous credit constraint and equilibrium definitions

The most crucial element of the model is the investment constraint (9). Miao and Wang [7] devote considerable space to the foundation of this constraint by explicitly modelling a credit market on which firms can get loans to finance their investment projects. We do not include the credit market in our model at all, but simply impose condition (9) on the firms. We shall refer to (9) alternatively as a credit constraint or an investment constraint. The present subsection points out that there are different ways of implementing this constraint and that they lead to different equilibrium definitions.

It will be helpful to start with the case (neither considered by Miao and Wang [7] nor treated later in the present paper) where the function I appearing in (9) is exogenous to the model.

Definition 1 Let $I : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ be a given function and let $\bar{k} \in \mathbf{K} \backslash \{0\}$ be a given initial capital stock. An I-equilibrium from $\bar{k}$ is a 10-tuple of real-valued functions $(c, s, b, k, \ell, i, r, w, \pi, v)$ with domain $\mathbf{T}$ and a function $V : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ such that the following conditions hold.
(a) Given (r, w, π, v) it holds that (c, s, b) maximizes (3) subject to (4)-(5).
(b) Given (r, w) it holds that (k, ℓ, i) maximizes (10) subject to (6)-(9) and the function V satisfies (11).
(c) The market clearing conditions (12)-(16) hold.

In the model of Miao and Wang [7] condition (9) takes the form

$$i(t) \leq V(\xi k(t), t), \tag{17}$$

where V is the optimal value function from (11) and $\xi \in (0, 1]$ is a fixed parameter. More specifically, Miao and Wang [7] assume that the function I appearing in (9) satisfies

$$I(\kappa, t) = V(\xi\kappa, t) \tag{18}$$

for all $(\kappa, t) \in \mathbf{K} \times \mathbf{T}$. This turns the function I into an endogenous element of the model and, correspondingly, condition (9) is called an endogenous credit constraint. Note, however, that even if the function I is endogenously determined in the model, it is still taken as exogenous by the firms when they maximize their shareholder value subject to (6)-(9).

The justification of the constraint (17) provided by Miao and Wang [7] is that the firms can finance their investments only via an imperfect credit market. Investment opportunities arrive in the form of idiosyncratic shocks at the firms, and firms with an investment opportunity can borrow from those without such an opportunity. But they can do so only by pledging part of their capital as collateral. Miao and Wang [7] assume that the maximal fraction of capital that can be used as collateral is ξ. If a borrowing firm with capital stock $k(t)$ defaults on the loan, the creditor can seize the amount $\xi k(t)$ of the borrowing firm's capital and can set up its own firm, which has the market value $V(\xi k(t), t)$. For this reason, the creditors are not willing to lend more than $V(\xi k(t), t)$.

We apply a modelling shortcut by using a completely deterministic framework in which firms can invest at all times but are restricted to do so by condition (17). An equilibrium in this setting which is based on assumption (18) from Miao and Wang [7] can therefore be defined as follows.

Definition 2 Let $\bar{k} \in \mathbf{K} \setminus \{0\}$ be a given initial capital stock. A Miao-Wang equilibrium from $\bar{k}$ is a 10-tuple of real-valued functions $(c, s, b, k, \ell, i, r, w, \pi, v)$ with domain $\mathbf{T}$ and a function $V : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ such that there exists a function $I : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ satisfying the following conditions:
(a) $(c, s, b, k, \ell, i, r, w, \pi, v, V)$ is an I-equilibrium from $\bar{k}$.
(b) Equation (18) holds for all $(\kappa, t) \in \mathbf{K} \times \mathbf{T}$.

The key point made in the present paper is that specifying the function I according to (18) is not the only way to ensure that (17) holds.[3] Indeed, there exist infinitely many alternative

[3] In this regard, it is essential to point out that Miao and Wang [7] state condition (17) explicitly (see their equations (16)-(17)), but that they assume our condition (18) only implicitly (for example in their formula (24)).

specifications of I which are equally suitable. This is a consequence of two properties, namely, (i) that the function V on the right-hand side of (17) is an endogenous object and (ii) that the right-hand side of (17) depends on the representative firm's individual capital stock at time t, $k(t)$, and on the time variable t itself. Moreover, because (9) does not only constrain current investment at time t but determines also the incentives for capital accumulation after time t, it turns out that different specifications of I typically generate different equilibrium trajectories. This phenomenon is known from the dynamic games literature, where it is referred to as informational non-uniqueness; see, Başar [3] for the original reference and Sorger [8, chapter 7] for a more recent textbook presentation. Before we demonstrate the effects of informational non-uniqueness in the present model, we need to provide the formal equilibrium definition that we are going to use.

Definition 3 Let $\bar{k} \in \mathbf{K}\backslash\{0\}$ be a given initial capital stock. A relaxed Miao-Wang equilibrium from $\bar{k}$ is a 10-tuple of real-valued functions $(c, s, b, k, \ell, i, r, w, \pi, v)$ with domain $\mathbf{T}$ and a function $V : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ such that there exists a function $I : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ satisfying the following conditions:
(a) $(c, s, b, k, \ell, i, r, w, \pi, v, V)$ is an I-equilibrium from $\bar{k}$.
(b) The equation $I(k(t), t) = V(\xi k(t), t)$ holds for all $t \in \mathbf{T}$.

The only difference between definitions 2 and 3 is that, in a Miao-Wang equilibrium, the condition $I(\kappa, t) = V(\xi\kappa, t)$ must hold for all feasible pairs $(\kappa, t) \in \mathbf{K} \times \mathbf{T}$ whereas, in a relaxed Miao-Wang equilibrium, it has to hold only along the equilibrium trajectory, i.e, for all $(\kappa, t) \in \{(k(\tau), \tau) \,|\, \tau \in \mathbf{T}\}$. It is obvious that every Miao-Wang equilibrium is a relaxed Miao-Wang equilibrium. The converse, however, is not true as will become evident in the following section.

3 Informational non-uniqueness

In this section we construct relaxed Miao-Wang equilibria under the assumptions of risk neutral households and a Cobb-Douglas technology.[4] More specifically, we assume that $U(\gamma) = \gamma$ and $F(\kappa, \lambda) = \kappa^\alpha \lambda^{1-\alpha}$ hold for all $(\gamma, \kappa, \lambda) \in \mathbb{R}^3_+$, where $\alpha \in (0, 1)$ is a constant. Following Miao and Wang [7] we look for equilibria in which the value function $V : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ takes the linear form

$$V(\kappa, t) = Q(t)\kappa + q(t) \tag{19}$$

[4] The main part of the original paper by Miao and Wang [7] also restricts the analysis to the case of risk neutrality. The case of risk averse households is treated in appendix D of Miao and Wang [7] and in Sorger [9]. Both of these papers assume also a Cobb-Douglas technology.

with $Q : \mathbf{T} \mapsto \mathbb{R}$ and $q : \mathbf{T} \mapsto \mathbb{R}$ satisfying

$$Q(t) \geq 1 \text{ and } q(t) \geq 0 \tag{20}$$

for all $t \in \mathbf{T}$. As for the constraint function I, we restrict ourselves to the linearly parameterized family $\{I_\mu : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}_+ \,|\, \mu \in \mathbb{R}\}$ defined by

$$I_\mu(\kappa, t) = (1-\mu)V(\xi k(t), t) + \mu V(\xi\kappa, t) = \xi Q(t)[(1-\mu)k(t) + \mu\kappa] + q(t). \tag{21}$$

In order to avoid misunderstanding, let us emphasize that the borrowing firm's individual capital stock enters the constraint function I_μ as its first argument; see the investment constraint (9). The variable in definition (21) that corresponds to the individual capital stock is therefore κ. The variable $k(t)$ on the right-hand side of (21), on the other hand, refers to the *equilibrium value* of the borrowing firm's capital stock at time t as it is perfectly foreseen by the creditors. Since there exists a unit interval of identical firms, $k(t)$ in (21) could also be interpreted as the aggregate capital stock in the economy. In the present section it does not make a difference whether we interpret $k(t)$ as the representative firm's capital stock or the aggregate capital stock. What is important, though, is that $k : \mathbf{T} \mapsto \mathbb{R}$ is the equilibrium trajectory, i.e., a known function of time.

Since the representative firm's own capital stock enters the right-hand side of the credit constraint (9), capital accumulation does not only influence the firm's production possibilities but also its investment possibilities. According to (21), the marginal relaxation of constraint (9) by one additional unit of capital in period t is given by $\mu\xi Q(t)$, which is the partial derivative of $I_\mu(\kappa, t)$ with respect to κ. Note that the marginal relaxation is an increasing function of μ and that it is negative whenever μ is negative. In other words, if the parameter μ is positive, the firm can enhance its future investment possibilities via capital accumulation. In the case where $\mu = 0$ holds, the firm cannot affect its investment possibilities at all, as the function $I_0(\kappa, t)$ is a pure time-function that is independent of κ. In accordance with the dynamic games literature one could refer to $\mu = 0$ as the open-loop case. Finally, if $\mu < 0$ holds, accumulating more capital makes the credit constraint for the borrowers even tighter, i.e., capital accumulation now reduces the maximal investment volume in the future.

It is obvious that the specification (21) ensures the validity of condition (b) in definition 3 and that the Miao-Wang equilibrium from definition 2 arises as the special case $\mu = 1$. The following theorem presents a characterization of relaxed Miao-Wang equilibria by means of a three-dimensional boundary value problem.

Theorem 1 *Let a triple of real-valued functions (k, Q, q) on the time domain $\mathbf{T}$ and a function $V : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ be given such that (19)-(20) hold for all $t \in \mathbf{T}$.*

(a) *If $(c, s, b, k, \ell, i, r, w, \pi, v, V)$ is a relaxed Miao-Wang equilibrium and the constraint function*

I is given by I_μ from (21), then it follows that the functions k, Q, and q satisfy the dynamical system

$$\dot{k}(t) \le \xi Q(t)k(t) + q(t) - \delta k(t), \tag{22}$$

$$[Q(t)-1][\xi Q(t)k(t) + q(t) - \delta k(t) - \dot{k}(t)] = 0, \tag{23}$$

$$\dot{Q}(t) = (\delta+\rho)Q(t) - \mu\xi Q(t)[Q(t)-1] - \alpha k(t)^{\alpha-1}, \tag{24}$$

$$\dot{q}(t) = \rho q(t) - [Q(t)-1][\xi(1-\mu)k(t)Q(t) + q(t)] \tag{25}$$

for all $t \in \mathbf{T}$ as well as the boundary conditions

$$k(0) = \bar{k}, \tag{26}$$

$$\lim_{t\to+\infty} e^{-\rho t}[Q(t)k(t) + q(t)] = 0. \tag{27}$$

(b) *Conversely, if the functions k, Q, and q satisfy (22)-(27), then there exist real-valued functions $(c, s, b, \ell, i, r, w, \pi, v)$ on the domain $\mathbf{T}$ such that $(c, s, b, k, \ell, i, r, w, \pi, v, V)$ is a relaxed Miao-Wang equilibrium with the constraint function $I = I_\mu$ from (21).*

With the help of the above theorem we can now study for which values of the parameter μ a relaxed Miao-Wang equilibrium with constraint function I_μ exists. In a first step, we disregard the initial condition (26) and look for constant solutions of (20) and (22)-(25).[5] We shall refer to such solutions as stationary equilibria. To formulate our next main result, we need the following auxiliary lemma.

Lemma 1 *Assume that $\xi \le \delta$ is satisfied and define*

$$\bar{\mu} = \frac{\delta(1+2\rho) - \rho\xi - 2\sqrt{\delta\rho(1+\rho)(\delta-\xi)}}{\xi},$$

$$Q_-(\mu) = \begin{cases} \dfrac{\delta + (\mu+\rho)\xi - \sqrt{[\delta+(\mu+\rho)\xi]^2 - 4\delta\mu\xi(1+\rho)}}{2\mu\xi} & \text{if } \mu \neq 0 \\ \dfrac{\delta(1+\rho)}{\delta+\rho\xi} & \text{if } \mu = 0, \end{cases}$$

$$Q_+(\mu) = \frac{\delta + (\mu+\rho)\xi + \sqrt{[\delta+(\mu+\rho)\xi]^2 - 4\delta\mu\xi(1+\rho)}}{2\mu\xi} \quad \text{for } \mu \neq 0.$$

(a) *The inequality $\bar{\mu} \ge 1$ holds, and it holds with equality if and only if $\xi = \delta/(1+\rho)$.*
(b) *For all $\mu \le \bar{\mu}$ it holds that $Q_-(\mu)$ is a real number and the function $Q_- : (-\infty, \bar{\mu}] \mapsto \mathbb{R}$ is continuous and strictly increasing. Moreover, it holds that*

$$\lim_{\mu\to-\infty} Q_-(\mu) = 1,$$

$$Q_-(1) = \begin{cases} \dfrac{\delta}{\xi} & \text{if } \xi \ge \dfrac{\delta}{1+\rho}, \\ 1+\rho & \text{if } \xi < \dfrac{\delta}{1+\rho}. \end{cases}$$

[5]Note that any constant solution trivially satisfies the transversality condition (27).

(c) *For all* $\mu \in (-\infty, 0) \cup (0, \bar{\mu}]$ *it holds that* $Q_+(\mu)$ *is a real number and the function* $Q_+ : (0, \bar{\mu}] \mapsto \mathbb{R}$ *is continuous and strictly decreasing. Moreover, it holds that*

$$\lim_{\mu \searrow 0} Q_+(\mu) = +\infty,$$

$$Q_+(1) = \begin{cases} \dfrac{\delta}{\xi} & \text{if } \xi < \dfrac{\delta}{1+\rho}, \\ 1+\rho & \text{if } \xi \geq \dfrac{\delta}{1+\rho}. \end{cases}$$

(d) *It holds that* $Q_-(\bar{\mu}) = Q_+(\bar{\mu})$.

The graphs of the functions Q_- and Q_+ are shown in figures 1 and 2, respectively. Figure 1 applies to the case $\xi > \delta/(1+\rho)$ whereas figure 2 illustrates the situation when $\xi < \delta/(1+\rho)$ holds. The bold curves in these two figures represent the continuum of stationary equilibria that will be discussed in the following theorem.

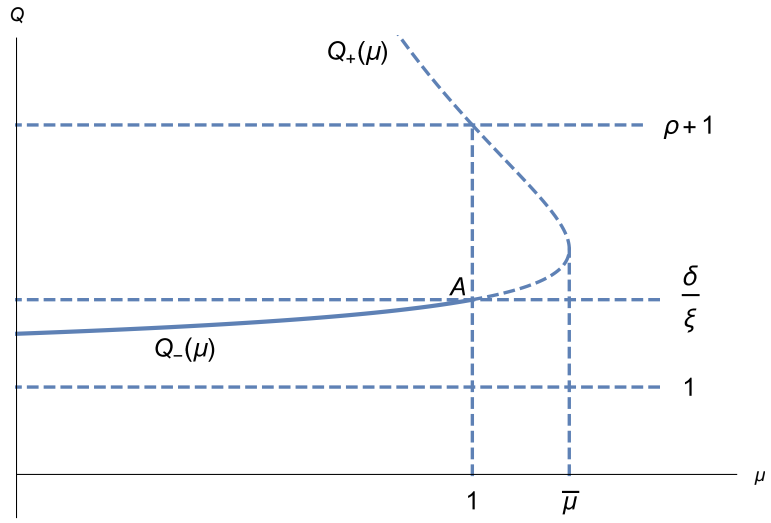

Figure 1: Illustration of lemma 1 in the case where $\delta/(1+\rho) < \xi < \delta/\rho$.

Theorem 2 **(a)** *If* $\delta \leq \xi$ *is satisfied then there exists a stationary equilibrium defined by*

$$k(t) = k_* = \left(\frac{\alpha}{\delta+\rho}\right)^{1/(1-\alpha)}, \; Q(t) = 1\,, \; q(t) = 0.$$

This equilibrium is independent of the parameter μ *and it is the only stationary equilibrium satisfying* $Q(t) = 1$.

(b) *If* $\xi < \delta$ *holds, then there exists a stationary equilibrium for every* $\mu \in (-\infty, 1]$. *This stationary equilibrium is given by*

$$k(t) = k_-(\mu)\,, \; Q(t) = Q_-(\mu)\,, \; q(t) = q_-(\mu),$$

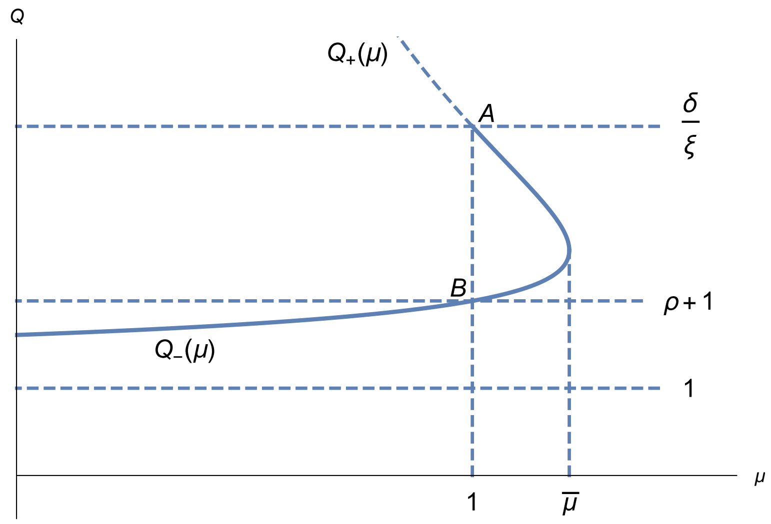

Figure 2: Illustration of lemma 1 in the case where $\xi < \delta/(1+\rho)$.

where

$$k_-(\mu) = \left[\frac{\alpha}{\delta(1+\rho)+\rho(1-\xi)Q_-(\mu)}\right]^{1/(1-\alpha)},$$
$$q_-(\mu) = k_-(\mu)[\delta - \xi Q_-(\mu)].$$

(c) *If $\xi < \delta/(1+\rho)$ holds, then there exist in addition to the stationary equilibria from part (b) two stationary equilibria for every $\mu \in (1, \bar{\mu}]$. These stationary equilibria are given by*

$$k(t) = k_-(\mu)\,,\ Q(t) = Q_-(\mu)\,,\ q(t) = q_-(\mu),$$

and

$$k(t) = k_+(\mu)\,,\ Q(t) = Q_+(\mu)\,,\ q(t) = q_+(\mu),$$

respectively, where $k_-(\mu)$ and $q_-(\mu)$ have been defined in part (b) above and where

$$k_+(\mu) = \left[\frac{\alpha}{\delta(1+\rho)+\rho(1-\xi)Q_+(\mu)}\right]^{1/(1-\alpha)},$$
$$q_+(\mu) = k_+(\mu)[\delta - \xi Q_+(\mu)].$$

For $\mu \in (1, \bar{\mu})$ it holds that the two stationary equilibria differ from each other whereas for $\mu = \bar{\mu}$ they coincide.

Let us pause for a moment to make a number of comments on the above theorem. In the stationary equilibrium from part (a) of the theorem, the credit constraint is not binding and Tobin's Q is equal to 1. Hence, capital inside a firm is as valuable as capital outside of firms. The more interesting cases are those in which the credit constrain is binding. These equilibria are described in parts (b) and (c) of the theorem. For $\mu = 1$ we obtain the same stationary equilibria that have already been identified by Miao and Wang [7], which is no surprise, because $\mu = 1$ corresponds to Miao-Wang equilibria. Whenever $\xi < \delta$ holds, there exists a stationary Miao-Wang equilibrium with $Q(t) = \delta/\xi$. This equilibrium has the label A in figures 1 and 2. If the credit constraint is so tight that $\xi < \delta/(1+\rho)$ is satisfied, then a second stationary Miao-Wang equilibrium occurs, in which $Q(t) = 1+\rho$ holds and which has the label B in figure 2. But theorem 2 demonstrates that there exist infinitely many other stationary, albeit relaxed Miao-Wang equilibria. As a matter of fact, there exist stationary relaxed Miao-Wang equilibria for all $\mu \in (-\infty, \bar{\mu}]$, and they all differ from each other. In figures 1 and 2 these stationary equilibria are indicated by the bold curves.

Next we add a few observations on the possible stationary equilibrium values of the key variables, whereby we restrict the discussion to those equilibria with $Q(t) > 1$. We see from lemma 1 that Tobin's marginal Q can take any value between 1 and δ/ξ. Correspondingly, the stationary equilibrium capital stocks can take any value between

$$\left[\frac{\alpha}{\delta(1+\rho)+\rho(1-\xi)}\right]^{1/(1-\alpha)} \quad \text{and} \quad \left[\frac{\alpha\xi}{\delta(\rho+\xi)}\right]^{1/(1-\alpha)}.$$

As in Miao and Wang [7] it holds that higher values of $Q(t)$ go along with lower values of the capital stock. It is moreover straightforward to see that consumption in the stationary equilibrium is also a decreasing function of $Q(t)$. Hence, we obtain the interesting result that both consumption and the capital stock in a stationary equilibrium can be increased by making the parameter μ in the constraint function I_μ smaller. More specifically, very negative values of μ generate consumption and capital values close to their respective suprema. This is counterintuitive, because a very negative value of μ means that the credit constraint provides strong disincentives to capital accumulation.

Finally, we consider the values of $q(t)$. Since $q(t) = k(t)[\delta - \xi Q(t)]$ holds for all stationary equilibria with $Q(t) > 1$ and since both $k(t)$ and $\delta - \xi Q(t)$ are positive and decreasing functions of $Q(t)$, it follows that $q(t)$ is also a decreasing function of $Q(t)$. The minimal value of $q(t)$ is zero and is attained when $\mu = 1$ and $Q(t) = Q_-(1)$ or $Q(t) = Q_+(1)$ depending on whether $\xi \geq \delta/(1+\rho)$ or $\xi < \delta/(1+\rho)$ is satisfied. In all other cases it holds that $q(t) > 0$. Adopting the interpretation of Miao and Wang [7], we therefore see that all but one of the infinitely many stationary equilibria described in parts (b) and (c) of theorem 2 feature stock price bubbles.

We close the present section by analyzing the dynamic stability of the stationary equilibria listed in theorem 2. Since the dynamical system (22)-(25) has one predetermined variable ($k(t)$) and two jump variables ($Q(t)$ and $q(t)$), a stationary equilibrium is locally asymptotically stable and determinate if the Jacobian matrix has exactly one stable eigenvalue and it is locally asymptotically stable and indeterminate if it has more than one stable eigenvalue. Miao and Wang [7] and Sorger [9] have proved that $(k_-(1), Q_-(1), q_-(1))$ is locally asymptotically stable and determinate (one stable eigenvalue) whereas $(k_+(1), Q_+(1), q_+(1))$ is locally asymptotically stable and indeterminate (two stable eigenvalue). These facts give rise to the following conjecture:

Conjecture 1 **(a)** Whenever it exists, the stationary equilibrium $(k_-(\mu), Q_-(\mu), q_-(\mu))$ is locally asymptotically stable and determinate.
(b) Whenever it exists, the stationary equilibrium $(k_+(\mu), Q_+(\mu), q_+(\mu))$ is locally asymptotically stable and indeterminate.

Unfortunately, we have not been able to find a complete proof of this conjecture. However, the conjecture can be verified analytically in certain special cases and there is numerical confirmation for many other cases. If $Q(t) > 1$ holds, which is the case in all the stationary equilibria mentioned in parts (b)-(c) of theorem 2, then the Jacobian matrix is given by

$$M(\mu) = \begin{pmatrix} \xi Q - \delta & \xi k & 1 \\ \alpha(1-\alpha)k^{\alpha-2} & \delta + \rho + \mu\xi(1-2Q) & 0 \\ -(1-\mu)\xi Q(Q-1) & (1-\mu)\xi k(1-2Q) - q & 1+\rho-Q \end{pmatrix}, \tag{28}$$

where we have omitted the time variable from $k(t)$, $Q(t)$, and $q(t)$ for simplicity of presentation. Using the facts that

$$k^{\alpha-2} = \frac{\delta(1+\rho) + \rho(1-\xi)Q}{\alpha k}$$

and

$$q = k(\delta - \xi Q)$$

hold in all stationary equilibria from theorem 2(b-c) one can rewrite the Jacobian matrix as

$$M(\mu) = \begin{pmatrix} \xi Q - \delta & \xi k & 1 \\ \dfrac{(1-\alpha)[\delta(1+\rho) + \rho(1-\xi)Q]}{k} & \delta + \rho + \mu\xi(1-2Q) & 0 \\ -(1-\mu)\xi Q(Q-1) & k[(1-\mu)\xi(1-2Q) - \delta + \xi Q] & 1+\rho-Q \end{pmatrix}.$$

Now consider for example the open-loop case $\mu = 0$ and $Q = Q_-(0) = \delta(1+\rho)/(\delta + \rho\xi)$. Substituting the value of Q into the expression for $M(0)$ and computing the characteristic polynomial $\mathcal{P}(z)$ we obtain

$$\mathcal{P}(z) = D_0 + D_1 z + D_2 z^2 - z^3,$$

where

$$\begin{aligned} D_0 &= -(1-\alpha)\delta(1+\rho)(\delta+\rho) < 0, \\ D_1 &= \frac{(\delta+\rho)[\delta^2 - \rho\xi(1+\rho) - \delta\xi(\alpha + \alpha\rho - \rho)]}{\delta + \rho\xi}, \\ D_2 &= \frac{\rho\xi(1+2\rho) + \delta[\rho + \xi(1+\rho)]}{\delta + \rho\xi} > 0. \end{aligned}$$

Using the results from Strelitz [10] and Weisstein [11] it has been argued in Sorger [9] that $D_0 < 0$ and $D_2 > 0$ together are sufficient for the matrix $M(0)$ to have exactly one stable eigenvalue. This confirms conjecture (a) in the case $\mu = 0$.

Next consider the case where $\mu = \bar{\mu}$ and $Q = Q_-(\bar{\mu}) = Q_+(\bar{\mu}) = [\delta + (\bar{\mu} + \rho)\xi]/(2\bar{\mu}\xi)$. Substituting the Q-value into the expression for $M(\bar{\mu})$ we find that the determinant of $M(\bar{\mu})$ is zero, which supports both conjectures, as it shows that $\bar{\mu}$ is a bifurcation point.

We have numerically checked the above conjecture for many parameter values and found it always to be confirmed.

4 Markov-perfection and recursive equilibrium

In all of the infinitely many relaxed Miao-Wang equilibria identified in the previous section (including the Miao-Wang equilibria corresponding to $\mu = 1$) the constraint function I depends on the individual capital stock of the firm and on the time variable t. The fact that the dependence on t is non-degenerate has the implication that all of these equilibria fail to be Markov-perfect. By that statement we mean that in the case where a shock at some time τ changes the aggregate state of the economy (i.e., the aggregate capital stock), the constraint $i(t) \leq I(k(t), t)$ for $t > \tau$ does no longer serve its intended purpose, namely to limit the firm's investment by the market value of a hypothetical firm owning the fraction ξ of the borrower's capital. To explain this formally, we have to distinguish between the individual capital stock of the representative firm and the aggregate capital stock. We will continue to denote the time path of the individual capital stock by $k : \mathbf{T} \mapsto \mathbb{R}$ and a particular value of it by κ. As for the aggregate capital stock, we will use the notations $K : \mathbf{T} \mapsto \mathbb{R}$ and $\mathcal{K}$, respectively. For simplicity, we present the argument only for Miao-Wang equilibria (i.e., for $\mu = 1$).

The constraint function $I : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ in the previous section was defined by $I(\kappa, t) = \xi Q(t)\kappa + q(t)$, where the triple (K, Q, q) is defined by

$$\begin{aligned}
&\dot{K}(t) \leq I(K(t), t) - \delta K(t),\\
&[Q(t) - 1][I(K(t), t) - \delta K(t) - \dot{K}(t)] = 0,\\
&\dot{Q}(t) = (\delta + \rho)Q(t) - \xi Q(t)[Q(t) - 1] - \alpha K(t)^{\alpha-1},\\
&\dot{q}(t) = \rho q(t) - [Q(t) - 1]q(t),\\
&K(0) = \bar{k},\\
&\lim_{t\to+\infty} e^{-\rho t}[Q(t)K(t) + q(t)] = 0.
\end{aligned}$$

Now suppose that a shock at time τ changes the aggregate capital stock from $K(\tau)$ to $K(\tau)+\Delta$. If the constraint function I is maintained, the market value as of time $t > \tau$ of a firm with κ

units of capital would be $\tilde{Q}(t)\kappa + \tilde{q}(t)$, where $(\tilde{K}, \tilde{Q}, \tilde{q})$ is defined by

$$\begin{aligned}
&\dot{\tilde{K}}(t) \leq I(\tilde{K}(t), t) - \delta\tilde{K}(t),\\
&[\tilde{Q}(t) - 1][I(\tilde{K}(t), t) - \delta\tilde{K}(t) - \dot{\tilde{K}}(t)] = 0,\\
&\dot{\tilde{Q}}(t) = (\delta + \rho)\tilde{Q}(t) - \xi\tilde{Q}(t)[\tilde{Q}(t) - 1] - \alpha\tilde{K}(t)^{\alpha-1},\\
&\dot{\tilde{q}}(t) = \rho\tilde{q}(t) - [\tilde{Q}(t) - 1]\tilde{q}(t),\\
&\tilde{K}(\tau) = K(\tau) + \Delta,\\
&\lim_{t\to+\infty} e^{-\rho t}[\tilde{Q}(t)\tilde{K}(t) + \tilde{q}(t)] = 0.
\end{aligned}$$

It is quite obvious that, whenever Δ is different from 0, the conditions $Q(t) \neq \tilde{Q}(t)$ and $q(t) \neq \tilde{q}(t)$ will generically be true for $t > \tau$. Consequently, it will be the case that

$$I(k(t), t) = \xi Q(t)k(t) + q(t) \neq \xi\tilde{Q}(t)k(t) + \tilde{q}(t) = V(\xi k(t), t).$$

In words, the upper bound on investment at time t for a firm that owns at that time $\kappa = k(t)$ units of capital does not coincide with the value of a firm owning $\xi k(t)$ units of capital. The reason for this result is quite obvious. Since the constraint function depends on time but not on the aggregate state of the economy, an aggregate shock to the economy is not reflected by a corresponding change of the constraint function. In order to obtain a Markov-perfect equilibrium, we therefore have to replace the dependence of I on time t by a dependence on the aggregate state $\mathcal{K}$. This is the purpose of the present section.

We have to develop a recursive equilibrium definition solely in terms of the (individual and aggregate) states of the model. In particular, the credit constraint for the firms should take the form

$$i(t) \leq J(k(t), K(t)),$$

where $k(t)$ and $K(t)$ denote the individual and aggregate capital stock at time t, respectively, and where $J : \mathbf{K}^2 \mapsto \mathbb{R}$ is the new constraint function.

We maintain the assumptions of linear utility (risk neutrality) and a Cobb-Douglas technology. The wage rate as a function of the aggregate capital stock $\mathcal{K}$ is given by $(1 - \alpha)\mathcal{K}^\alpha$ and the operating profit (revenue net of the wage cost) for a firm with κ units capital amounts to $\alpha\mathcal{K}^{\alpha-1}\kappa$. Noting that the real interest rate continues to be equal to ρ at all times, the HJB equation for the representative firm's optimization problem can be written as

$$\begin{aligned}
&\rho W(\kappa, \mathcal{K}) && (29)\\
= \;& \max\{\alpha\mathcal{K}^{\alpha-1}\kappa - \iota + W_1(\kappa, \mathcal{K})(\iota - \delta\kappa) + W_2(\kappa, \mathcal{K})[H(\mathcal{K}) - \delta\mathcal{K}] \,|\, 0 \leq \iota \leq J(\kappa, \mathcal{K})\},
\end{aligned}$$

where $W : \mathbf{K}^2 \mapsto \mathbb{R}$ is the value function of the representative firm and where $H : \mathbf{K} \mapsto \mathbb{R}$ is a function that expresses aggregate investment as a function of the aggregate capital stock.

Definition 4 A recursive equilibrium is a triple (W, h, H) of real-valued functions, where W and h are defined on $\mathbf{K}^2$ and H is defined on $\mathbf{K}$ such that the following conditions hold.
(a) The HJB equation (29) is satisfied and the maximum on the right-hand side of this equation is attained at $\iota = h(\kappa, \mathcal{K})$.
(b) The equilibrium conditions $h(\mathcal{K}, \mathcal{K}) = H(\mathcal{K})$ and $J(\kappa, \mathcal{K}) = W(\xi\kappa, \mathcal{K})$ hold for all $(\kappa, \mathcal{K}) \in \mathbf{K}^2$.
(c) The aggregate state dynamics $\dot{K}(t) = H(K(t)) - \delta K(t)$ has a globally asymptotically stable fixed point in $\mathbf{K}$.

Condition (a) of the above definition essentially says that the optimal investment rate of a firm with κ units of capital equals $h(\kappa, \mathcal{K})$ when the aggregate stock of capital equals $\mathcal{K}$. Condition (b) requires that the aggregate investment rate $H(\mathcal{K})$ is the integral of the individual investment rates $h(\kappa, \mathcal{K})$ over all (identical) firms and that the maximally allowed investment rate $J(\kappa, \mathcal{K})$ equals the value of a firm with $\xi\kappa$ units of capital. Finally, condition (c) is added in order to ensure that the transversality conditions hold.

It will be convenient to define

$$K_A = \left[\frac{\alpha\xi}{\delta(\rho + \xi)}\right]^{1/(1-\alpha)}.$$

Note that

$$K_A = \begin{cases} k_-(1) & \text{if } \xi \geq \dfrac{\delta}{1+\rho}, \\ k_+(1) & \text{if } \xi < \dfrac{\delta}{1+\rho} \end{cases}$$

holds, where $k_-(\mu)$ and $k_+(\mu)$ have been defined in theorem 2. In other words, K_A is the individual or aggregate capital stock in the stationary Miao-Wang equilibrium corresponding to point A in figures 1 and 2.

As in the previous section we look for equilibria with a linear value function of the form

$$W(\kappa, \mathcal{K}) = P(\mathcal{K})\kappa + p(\mathcal{K}), \tag{30}$$

where the functions $P : \mathbf{K} \mapsto \mathbb{R}$ and $p : \mathbf{K} \mapsto \mathbb{R}$ satisfy

$$P(\mathcal{K}) \geq 1 \text{ and } p(\mathcal{K}) \geq 0 \tag{31}$$

for all $\mathcal{K} \in \mathbf{K}$.

Theorem 3 *Assume that $\xi < \delta$ holds and that $\bar{K}$, the upper limit of the state space $\mathbf{K}$, satisfies the condition*

$$K_A < \bar{K} \leq \left(\frac{\alpha}{\delta + \rho}\right)^{1/(1-\alpha)}. \tag{32}$$

There exists a recursive equilibrium (W, h, H) *satisfying (30)-(31) and* $p(\mathcal{K}) = 0$ *for all* $(\kappa, \mathcal{K}) \in \mathbf{K}^2$. *In this equilibrium, the aggregate capital stock satisfies*

$$\dot{K}(t) = H(K(t)) - \delta K(t)$$

for all $t \in \mathbf{T}$ *as well as* $\lim_{t \to +\infty} K(t) = K_A$.

This theorem demonstrates that there exists a recursive equilibrium that generates the same long-run capital stock as the Miao-Wang equilibrium with $\mu = 1$ and $Q(t) = \delta/\xi$. Not only do the long-run capital stocks coincide between the two equilibria but also the steady-state prices. This follows from $P(K_A) = \delta/\xi$ and $p(\mathcal{K}) = 0$ for all $\mathcal{K} \in \mathbf{K}$; see the proof of theorem 3 in the appendix. The crucial difference between the Miao-Wang equilibrium and the recursive one is that the former is a set of time paths whereas the recursive equilibrium consists of functions defined on the state space. In contrast to the (relaxed) Miao-Wang equilibria, the recursive equilibrium is Markov-perfect by construction.

Since $p(\mathcal{K}) = 0$ holds for all $\mathcal{K} \in \mathbf{K}$ in the recursive equilibrium described in theorem 3, this equilibrium does not give rise to a stock price bubble. It would be interesting to see whether there exist also recursive equilibria with bubbles. We believe that this is true and therefore make another conjecture. To formulate it, we define

$$K_B = \left[\frac{\alpha}{(1+\rho)[\delta + \rho(1-\xi)]}\right]^{1/(1-\alpha)}.$$

Note that $K_B = k_-(1)$ holds whenever $\xi < \delta/(1+\rho)$ so that K_B is the capital stock in the stationary Miao-Wang equilibrium corresponding to point B in figure 2.

Conjecture 2 If $\xi < \delta/(1+\rho)$ holds, then there exists a recursive equilibrium (W, h, H) satisfying (30)-(31) for which $p(\mathcal{K}) > 0$ holds for all $\mathcal{K}$ sufficiently close to K_B. The aggregate capital stock generated by this equilibrium satisfies $\lim_{t \to +\infty} K(t) = K_B$.

In order to support this conjecture, we follow essentially the same arguments that have been used to prove theorem 3. In the proof of that theorem we have derived the equilibrium conditions

$$P'(\mathcal{K}) = \frac{(\delta+\rho)P(\mathcal{K}) - \xi P(\mathcal{K})[P(\mathcal{K}) - 1] - \alpha \mathcal{K}^{\alpha-1}}{\xi P(\mathcal{K})\mathcal{K} + p(\mathcal{K}) - \delta \mathcal{K}}, \tag{33}$$

$$p'(\mathcal{K}) = \frac{[1 + \rho - P(\mathcal{K})]p(\mathcal{K})}{\xi P(\mathcal{K})\mathcal{K} + p(\mathcal{K}) - \delta \mathcal{K}}. \tag{34}$$

We define the following two-dimensional manifolds in the three-dimensional $(\mathcal{K}, \mathcal{P}, \wp)$-space:

$$M_{\mathcal{K}} = \{(\mathcal{K}, \mathcal{P}, \wp) \,|\, \xi \mathcal{P}\mathcal{K} + \wp - \delta \mathcal{K} = 0\},$$
$$M_{\mathcal{P}} = \{(\mathcal{K}, \mathcal{P}, \wp) \,|\, (\delta+\rho)\mathcal{P} - \xi \mathcal{P}(\mathcal{P} - 1) - \alpha \mathcal{K}^{\alpha-1} = 0\},$$
$$M_{\wp} = \{(\mathcal{K}, \mathcal{P}, \wp) \,|\, 1 + \rho - \mathcal{P} = 0\}.$$

Their unique intersection is the point $X = (\mathcal{K}, \mathcal{P}, \wp) = (K_B, 1+\rho, [\delta - (1+\rho)\xi]K_B)$. Obviously, it holds at this point that $\mathcal{P} > 1$ and, since $\xi < \delta/(1+\rho)$ has been assumed, also $\wp > 0$. The linearization of (33)-(34) has the Jacobian matrix

$$M = \begin{pmatrix} \xi\mathcal{P} - \delta & \xi\mathcal{K} & 1 \\ \alpha(1-\alpha)\mathcal{K}^{\alpha-2} & \delta + \rho + \xi(1-2\mathcal{P}) & 0 \\ 0 & -\wp & 1+\rho-\mathcal{P} \end{pmatrix},$$

which is structurally the same as $M(1)$ from (28). It is known from Miao and Wang [7] that $M(1)$ has exactly one stable eigenvalue if $(\mathcal{K}, \mathcal{P}, \wp) = (K_B, 1+\rho, [\delta - (1+\rho)\xi]K_B)$. The stable/unstable manifold theorem implies therefore that there exists a one-dimensional stable manifold passing through the intersection point X. This manifold defines real-valued functions $\hat{\mathcal{P}}$ and $\hat{\wp}$ which are defined (at least) locally around the intersection point K_B and which satisfy equations (33)-(34). Using these functions, one can compute the value function W from (30), the function h from $h(\kappa, \mathcal{K}) = W(\xi\kappa, \mathcal{K})$, and the function H from $H(\mathcal{K}) = h(\mathcal{K}, \mathcal{K})$. By construction, these functions satisfy the equilibrium conditions stated in definition 4 locally around K_B. What remains to be shown in order to turn the conjecture into a theorem is that this local solution can be extended to the entire state space.

5 Concluding remarks

In this paper we have argued that the equilibrium definition in Miao and Wang [7] involves an implicit assumption about the formulation of the crucial endogenous credit constraint. By dropping this implicit assumption, the equilibrium set is considerably enlarged. The underlying reason for such a strong form of equilibrium indeterminacy is the possibility of representing the constraint in infinitely many different ways as a function of time and states (closed-loop representations). The phenomenon is known in the dynamic games literature by the name of informational non-uniqueness; see Başar [3]. It would also arise in other models with endogenous credit constraints provided that the constraint is formulated in a closed-loop form, such as in the paper by Kiyotaki and Moore [6]. As demonstrated in section 4 one can avoid informational non-uniqueness by applying a recursive equilibrium definition. This has the additional advantage of leading to Markov-perfect equilibria.

Appendix

Proof of theorem 1

(a) From definition 3 it follows that $(c, s, b, k, \ell, i, r, w, \pi, v, V)$ must be an I_μ-equilibrium. Because of (12)-(15) and (19) this implies that $\ell(t) = 1$, $b(t) = 0$, $s(t) = 1$, and $v(t) = Q(t)k(t) + q(t)$ hold for all $t \in \mathbf{T}$. This, in turn, shows that the representative household's wealth at time t is equal to $b(t) + v(t)s(t) = Q(t)k(t) + q(t)$. Since the triple (c, s, b) solves the representative household's optimization problem, the Euler equation and the transversality condition must hold. Due to risk neutrality the Euler equation boils down to $r(t) = \rho$ for all $t \in \mathbf{T}$. Because of this result and the fact that the household's wealth at time t is equal to $Q(t)k(t) + q(t)$, the transversality condition is given by (27).

Let us no turn to the representative firm's optimization problem. Condition (26) follows from (8). The Hamilton-Jacobi-Bellman (HJB) equation for the firm's problem is given by

$$\begin{aligned} & \rho V(\kappa, t) - V_2(\kappa, t) \\ = \; & \max \left\{ \kappa^\alpha \lambda^{1-\alpha} - w(t)\lambda - \iota + V_1(\kappa, t)(\iota - \delta\kappa) \,|\, 0 \leq \lambda,\, 0 \leq \iota \leq I_\mu(\kappa, t) \right\}, \end{aligned}$$

where we have already used $r(t) = \rho$. Noting that $V_1(\kappa, t) = Q(t) \geq 1$ holds according to (19)-(20), the necessary and sufficient first-order conditions for the maximization on the right-hand side of this equation are

$$\lambda = \left[\frac{1-\alpha}{w(t)} \right]^{1/\alpha} \kappa \tag{35}$$

and

$$I_\mu(\kappa, t) \geq \iota \text{ and } [Q(t) - 1][I_\mu(\kappa, t) - \iota] = 0. \tag{36}$$

Substituting these relations back into the HJB equation we obtain

$$\rho V(\kappa, t) - V_2(\kappa, t) = \alpha \left[\frac{1-\alpha}{w(t)} \right]^{(1-\alpha)/\alpha} \kappa + [Q(t) - 1] I_\mu(\kappa, t) - \delta Q(t)\kappa. \tag{37}$$

For $\kappa = k(t)$ we obtain from (7), (12), (21), and (35)-(36) that (22)-(23) and

$$\ell(t) = \left[\frac{1-\alpha}{w(t)} \right]^{1/\alpha} k(t) = 1$$

must hold for all $t \in \mathbf{T}$. The latter condition can be solved as $w(t) = (1-\alpha)k(t)^\alpha$. Substituting this result and (19)-(21) into (37) we obtain

$$\begin{aligned} & \rho[Q(t)\kappa + q(t)] - \dot{Q}(t)\kappa - \dot{q}(t) \\ = \; & \alpha k(t)^{\alpha-1}\kappa + [Q(t) - 1] \left\{ \xi Q(t)[(1-\mu)k(t) + \mu\kappa] + q(t) \right\} - \delta Q(t)\kappa. \end{aligned}$$

Since this equation has to hold for all $(\kappa, t) \in \mathbf{K} \times \mathbf{T}$ it is straightforward to derive (24)-(25). This completes the proof of part (a).

(b) It has already been mentioned that condition (21) ensures that $I_\mu(k(t), t) = V(\xi k(t), t)$ holds for all $t \in \mathbf{T}$ so that condition (b) of definition 3 is satisfied. It remains to be shown that there exist functions $(c, s, b, \ell, i, r, w, \pi, v)$ such that $(c, s, b, k, \ell, i, r, w, \pi, v, V)$ is an I_μ-equilibrium. To this end we define the functions c, s, b, ℓ, i, r, w, π, and v by

$$\begin{aligned}
&s(t) = 1\,,\ b(t) = 0\\
&r(t) = \rho\,,\ w(t) = (1-\alpha)k(t)^\alpha\,,\\
&\ell(t) = 1\,,\ i(t) = \dot{k}(t) + \delta k(t)\,,\\
&v(t) = Q(t)k(t) + q(t)\,,\ \pi(t) = \rho v(t) - \dot{v}(t)\,,\\
&c(t) = \pi(t) + w(t).
\end{aligned}$$

Obviously, (c, s, b) is a feasible solution for the representative household's optimization problem. The Euler equation of this problem holds due to $r(t) = \rho$, and the transversality condition due to $v(t) = Q(t)k(t) + q(t)$ and (27). Finally, because of $\pi(t) = \rho v(t) - \dot{v}(t)$ the two assets have the same return at all times $t \in \mathbf{T}$ so that the household is indifferent regarding its portfolio. This shows that condition (a) of definition 1 holds.

To verify condition (b) of definition 1 one needs to show that the HJB equation holds and that $(\lambda, \iota) = (\ell(t), i(t))$ maximizes its right-hand side. This follows by noting that the first-order optimality conditions for the representative firm's optimization problem presented in the proof of part (a) above are necessary and sufficient.

The market clearing conditions (12)-(15) hold because of (19) and the specification of the functions $(c, s, b, \ell, i, r, w, \pi)$, and condition (16) follows from (19) and (20). This proves that condition (c) of definition 1 is satisfied as well. The proof of the theorem is now complete.

Proof of lemma 1

(a) The inequality $\bar{\mu} \geq 1$ is equivalent to

$$\delta(1+2\rho) - \xi(1+\rho) \geq 2\sqrt{\delta\rho(1+\rho)(\delta-\xi)}.$$

Because of $\delta \geq \xi$, the left-hand side is non-negative. Taking squares on both sides is therefore an equivalence transformation which leads after simplifications to $[\delta - (1+\rho)\xi]^2 \geq 0$. This proves part (a).

(b) The discriminant in the definitions of $Q_-(\mu)$ and $Q_+(\mu)$ is non-negative whenever $\mu \leq \bar{\mu}$. This proves that $Q_-(\mu)$ is a real number for all $\mu \leq \bar{\mu}$. The continuity of the function Q_- on

$(-\infty, \bar{\mu}]$ is obvious for all $\mu \neq 0$ and it follows from the rule of de l'Hopital for $\mu = 0$. Let us denote the square root appearing in the definition of $Q_-(\mu)$ by S. Then we can write the derivative of $Q_-(\mu)$ as

$$\frac{1}{2\mu^2\xi}\left\{S - \delta - (\mu+\rho)\xi + \mu\xi\left[1 + \frac{\delta(1+2\rho) - (\mu+\rho)\xi}{S}\right]\right\}.$$

This expression is positive if and only if

$$S[S - \delta - (\mu+\rho)\xi] + \mu\xi[S + \delta(1+2\rho) - (\mu+\rho)\xi]$$

is positive. Using the fact that $S^2 = [\delta + (\mu+\rho)\xi]^2 - 4\delta\mu\xi(1+\rho)$, this condition can be written as

$$S < \frac{(\delta+\rho\xi)^2 - \mu\xi[\delta(1+2\rho) - \rho\xi]}{\delta+\rho\xi}. \tag{38}$$

Note that the assumption $\delta \geq \xi$ implies that the term $\delta(1+2\rho) - \rho\xi$ is positive. Combining this with $\mu \leq \bar{\mu}$ it follows that the right-hand side of (38) is positive provided that $(\delta+\rho\xi)^2 - \bar{\mu}\xi[\delta(1+2\rho) - \rho\xi]$ is positive which, according to the definition of $\bar{\mu}$, is the case if and only if

$$[\delta(1+2\rho) - \rho\xi]^2 - (\delta+\rho\xi)^2 < 2[\delta(1+2\rho) - \rho\xi]\sqrt{\delta\rho(1+\rho)(\delta-\xi)}$$

holds. Straightforward algebraic manipulations show that this inequality is indeed true. Hence, we can take squares on both sides of (38) without changing the inequality. Doing that shows that (38) holds and it follows that Q_- is strictly increasing on $(-\infty, \bar{\mu}]$.

As μ approaches $-\infty$ the square root appearing in the definition of $Q_-(\mu)$ behaves asymptotically as $-\mu\xi$. Hence, $Q_-(\mu)$ behaves asymptotically as $(\mu\xi + \mu\xi)/(2\mu\xi) = 1$. The formula for $Q_-(1)$ can easily be verified by substitution of $\mu = 1$ into the definition of $Q_-(\mu)$.

(c) Analogously to case (b) we see that $Q_+(\mu)$ is real for all $\mu \leq \bar{\mu}$ except possibly for $\mu = 0$. It is also clear that the function Q_+ is continuous on the interval $(0, \bar{\mu}]$. The derivative of $Q_+(\mu)$ is

$$-\frac{1}{2\mu^2\xi}\left[S + \delta + (\mu+\rho)\xi - \mu\xi\left(1 - \frac{\delta(1+2\rho) - (\mu+\rho)\xi}{S}\right)\right].$$

Following the same steps as in case (b) we see that this expression is negative if and only if

$$S > -\frac{(\delta+\rho\xi)^2 - \mu\xi[\delta(1+2\rho) - \rho\xi]}{\delta+\rho\xi}.$$

We have shown in the proof of part (b) that the right-hand side of this inequality is negative. Since S is positive, the inequality holds and it follows that the function Q_+ is strictly decreasing on $(0, \bar{\mu}]$. It is obvious from its definition that $Q_+(\mu)$ approaches $+\infty$ as μ approaches 0 from above and that the value $Q_+(1)$ is as stated in the lemma.

(d) This statement follows immediately from the observation that $S = 0$ holds for $\mu = \bar{\mu}$.

Proof of theorem 2

(a) It is easy to verify that $(k(t), Q(t), q(t)) = (k_*, 1, 0)$ satisfies conditions (20) and (22)-(25) independently of the value of μ. Moreover, if $Q(t) = 1$ holds for all $t \in \mathbf{T}$, then it follows from (24) and (25) that $q(t) = 0$ and $k(t) = k_*$ must hold. This proves that there cannot be any other stationary equilibrium satisfying $Q(t) = 1$.

Now suppose that $Q(t) > 1$ is satisfied in a stationary equilibrium. Then it follows from (22)-(23) that (22) holds with equality. Because of stationarity, (22) implies that $q(t) = k(t)[\delta - \xi Q(t)]$. Substituting this result into (25) one obtains $k(t) = 0$ or

$$\mu\xi Q(t)^2 - [\delta + (\mu + \rho)\xi]Q(t) + \delta(1 + \rho) = 0. \tag{39}$$

We can rule out $k(t) = 0$, because this is inconsistent with (24). The above quadratic equation for $Q(t)$ has the solutions $Q_-(\mu)$ and $Q_+(\mu)$. Combining the observations $q(t) = k(t)[\delta - \xi Q(t)]$, $k(t) > 0$, and $Q(t) > 1$ with the equilibrium condition $q(t) \geq 0$ we see that $\delta > \xi$ must be satisfied.

(b) Assume that $\xi < \delta$ holds. From lemma 1 it follows that $1 < Q_-(\mu) \leq \delta/\xi$ holds for all $\mu \leq 1$. Furthermore, we have

$$(\delta + \rho)Q_-(\mu) - \mu\xi Q_-(\mu)[Q_-(\mu) - 1] = \delta(1 + \rho) + \rho(1 - \xi)Q_-(\mu) > 0,$$

where we have used the fact that $Q_-(\mu)$ is a root of equation (39). Hence, by defining $k(t) = k_-(\mu)$ and $q(t) = q_-(\mu)$ for all $t \in \mathbf{T}$ it follows that conditions (22)-(25) of theorem 1 are satisfied.

(c) This case can be proved analogously to case (b) by noting that lemma 1 implies that for all $\mu \in (1, \bar{\mu}]$ we have

$$1 < Q_-(\mu) \leq Q_+(\mu) < \frac{\delta}{\xi},$$

where the weak inequality holds strictly unless $\mu = \bar{\mu}$.

Proof of theorem 3

If (30) and $P(\mathcal{K}) > 1$ hold for all $(\kappa, \mathcal{K}) \in \mathbf{K}^2$, then we have $W_1(\kappa, \mathcal{K}) = P(\mathcal{K}) > 1$ and it follows that the maximum on the right-hand side of the HJB equation (29) is attained at $\iota = J(\kappa, \mathcal{K})$. Hence, we define

$$h(\kappa, \mathcal{K}) = J(\kappa, \mathcal{K}) = W(\xi\kappa, \mathcal{K}) = \xi P(\mathcal{K})\kappa + p(\mathcal{K})$$

and

$$H(\mathcal{K}) = h(\mathcal{K}, \mathcal{K}) = \xi P(\mathcal{K})\mathcal{K} + p(\mathcal{K})$$

for all $(\kappa, \mathcal{K}) \in \mathbf{K}^2$. Obviously, condition (b) of definition 4 is satisfied. Moreover, by substituting these relations and (30) into (29), the HJB equation turns into

$$\begin{aligned} & \rho P(\mathcal{K})\kappa + \rho p(\mathcal{K}) \\ = \; & \alpha \mathcal{K}^{\alpha-1}\kappa + [P(\mathcal{K}) - 1][\xi P(\mathcal{K})\kappa + p(\mathcal{K})] - \delta P(\mathcal{K})\kappa + [P'(\mathcal{K})\kappa + p'(\mathcal{K})][\xi P(\mathcal{K})\mathcal{K} + p(\mathcal{K})]. \end{aligned}$$

This equation holds for all $(\kappa, \mathcal{K}) \in \mathbf{K}^2$ if and only if P and p satisfy the system of two differential equations

$$P'(\mathcal{K}) = \frac{(\delta + \rho)P(\mathcal{K}) - \xi P(\mathcal{K})[P(\mathcal{K}) - 1] - \alpha \mathcal{K}^{\alpha-1}}{\xi P(\mathcal{K})\mathcal{K} + p(\mathcal{K}) - \delta \mathcal{K}}, \tag{40}$$

$$p'(\mathcal{K}) = \frac{[1 + \rho - P(\mathcal{K})]p(\mathcal{K})}{\xi P(\mathcal{K})\mathcal{K} + p(\mathcal{K}) - \delta \mathcal{K}}. \tag{41}$$

It is therefore sufficient to prove that there exist functions P and p satisfying $P(\mathcal{K}) > 1$ and $p(\mathcal{K}) \geq 0$ for all $\mathcal{K} \in \mathbf{K}$ such that (40)-(41) hold for all $(\kappa, \mathcal{K}) \in \mathbf{K}^2$. Defining $p(\mathcal{K}) = 0$ for all $\mathcal{K} \in \mathbf{K}$, equation (41) and $p(\mathcal{K}) \geq 0$ are trivially satisfied and equation (40) simplifies to

$$P'(\mathcal{K}) = \frac{(\delta + \rho)P(\mathcal{K}) - \xi P(\mathcal{K})[P(\mathcal{K}) - 1] - \alpha \mathcal{K}^{\alpha-1}}{\xi P(\mathcal{K})\mathcal{K} - \delta \mathcal{K}}.$$

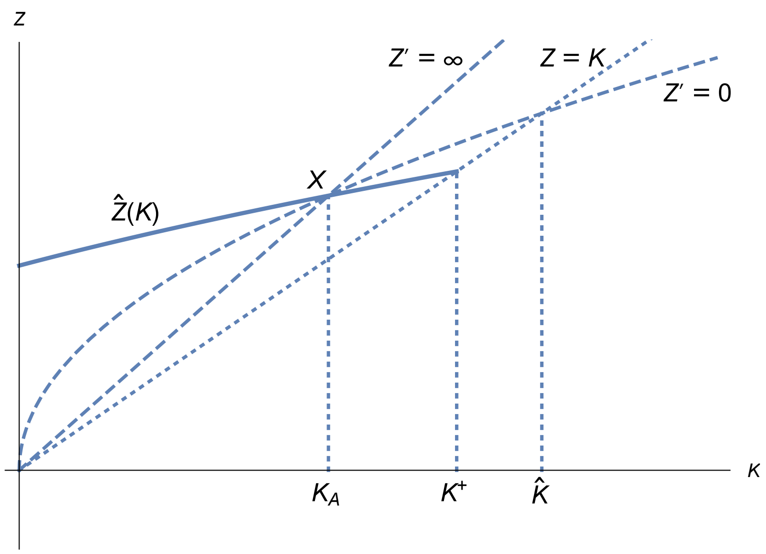

Figure 3: The phase diagram of equation (42).

It will be convenient to define the function $Z : \mathbf{K} \mapsto \mathbb{R}$ by $Z(\mathcal{K}) = P(\mathcal{K})\mathcal{K}$. With this definition, the above equation is equivalent to

$$Z'(\mathcal{K}) = \frac{(\rho + \xi)Z(\mathcal{K}) - \alpha \mathcal{K}^{\alpha}}{\xi Z(\mathcal{K}) - \delta \mathcal{K}} \tag{42}$$

and the condition $P(\mathcal{K}) > 1$ can be written as

$$Z(\mathcal{K}) \geq \mathcal{K}. \tag{43}$$

We analyze the system (42)-(43) in a $(\mathcal{K}, Z)$-phase diagram; see figure 3. The isoclines $Z'(\mathcal{K}) = 0$ and $Z'(\mathcal{K}) = \infty$ are given by

$$Z = \frac{\alpha \mathcal{K}^\alpha}{\rho + \xi}$$

and

$$Z = \frac{\delta \mathcal{K}}{\xi},$$

respectively. The isoclines intersect at the origin and at the point $X = (\hat{K}, \delta\hat{K}/\xi)$, where $\hat{K}$ is defined in the theorem. The Jacobian of (42) is

$$\begin{pmatrix} -\delta & \xi \\ -\alpha^2 \mathcal{K}^{\alpha-1} & \rho + \xi \end{pmatrix},$$

which can be evaluated at the fixed point X as

$$\begin{pmatrix} -\delta & \xi \\ -\dfrac{\alpha\delta(\rho+\xi)}{\xi} & \rho + \xi \end{pmatrix}.$$

The determinant of this matrix is $-(1-\alpha)\delta(\rho+\xi) < 0$, which proves that the fixed point X is saddle point. The eigenvectors are $(1, y_+)^\top$ and $(1, y_-)^\top$, where

$$y_{+,-} = \frac{2\alpha\delta(\rho+\xi)}{\xi\left[\delta + \rho + \xi \pm \sqrt{\delta^2 + 2(1-2\alpha)\delta(\rho+\xi) + (\rho+\xi)^2}\right]}.$$

It is straightforward to show that

$$0 < y_+ < \frac{\alpha\delta}{\xi} < \frac{\delta}{\xi} < y_-.$$

Since the slope of the isocline $Z'(\mathcal{K}) = 0$ at the fixed point X is equal to $\alpha\delta/\xi$ and that of the isocline $Z'(\mathcal{K}) = \infty$ is equal to δ/ξ everywhere, we conclude that there exists a solution of (42) that passes through X (the saddle path tangent to the eigenvector $(1, y_+)^\top$) and which is positively sloped at X but flatter than the isocline $Z'(\mathcal{K}) = 0$. Let us refer to this solution as $\hat{Z} : \mathbf{K} \mapsto \mathbb{R}$.[6]

Since $\xi < \delta$ has been assumed it follows that the isocline $Z'(\mathcal{K}) = \infty$ lies above the line $Z = \mathcal{K}$. This implies in particular that the fixed point X is located in the interior of the area defined

[6] The existence of the solution $\hat{Z}$ locally around $\hat{K}$ follows from the stable/unstable manifold theorem. Its global existence can be inferred from the phase diagram.

by (43). The two isoclines partition the phase plane into the four areas; see figure 3. The figure depicts the graph of the function $\hat{Z}$ as a bold curve. Now consider the graph of $\hat{Z}$ to the right of the fixed point X. This graph must remain below the isocline $Z'(\mathcal{K}) = 0$. Since the latter intersects the line $Z = \mathcal{K}$ at $\hat{K} = [\alpha/(\rho+\xi)]^{1/(1-\alpha)}$, it follows that the graph of $\hat{Z}$ must intersect the line $Z = \mathcal{K}$ at some point $K^+ \in (K_A, \hat{K})$. At that point it must therefore hold that $\hat{Z}'(K^+) < 1$. Since $\hat{Z}$ is a solution of the differential equation (42) and $\hat{Z}(K^+) = K^+$, it follows that

$$\hat{Z}'(K^+) = \frac{(\rho+\xi)K^+ - \alpha(K^+)^\alpha}{\xi K^+ - \delta K^+} < 1.$$

This inequality implies that

$$K^+ > \left(\frac{\alpha}{\delta+\rho}\right)^{1/(1-\alpha)} > K_A.$$

Assumption (32) in the theorem therefore ensures that $\hat{Z}(\mathcal{K}) > 1$ holds for all $\mathcal{K} \in \mathbf{K}$. Defining $P(\mathcal{K}) = \hat{Z}(\mathcal{K})\mathcal{K}$ completes the proof of the first statement in the theorem.

Since aggregate investment at time t is equal to $H(K(t))$ it follows immediately that the aggregate capital stock evolves according to $\dot{K}(t) = H(K(t)) - \delta K(t)$. By the definition of the functions H, Z, and p we have $H(\mathcal{K}) = \xi\hat{Z}(\mathcal{K})$ so that the aggregate state dynamics is $\dot{K}(t) = \xi\hat{Z}(K(t)) - \delta K(t)$. This differential equation has a unique fixed point at $\mathcal{K} = K_A$, and this fixed point is globally asymptotically stable. This completes the proof of the theorem.

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