Dual representations of risk measures

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Risk measures

The main result

A collection of risk measures
  The variance and similar pure risk measures
  The standard deviation and similar pure risk measures
  The lower semi-variance and similar pure risk measures
  The lower semi-standard deviation and similar pure risk measures
  Minimal prediction error risk measures
Acceptability functionals - Basic properties

Let $Y$ a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$.

(A1) **Translation equivariance.**

$$A(Y + c) = A(Y) + c$$

for constant $c$.

(A2) **Strictness.**

$$A(Y) \leq \mathbb{E}(Y),$$

where the equality sign holds iff $Y$ is a constant.

(A3) **Concavity.**

$$A(\lambda Y_1 + (1 - \lambda) Y_2) \geq \lambda A(Y_1) + (1 - \lambda) A(Y_2),$$

for $0 \leq \lambda \leq 1$. 
Acceptability functionals - Additional properties

(A4) **Positive homogeneity.**

\[ A(\lambda Y) = \lambda A(Y) \quad \text{for } \lambda > 0. \]

(A5) **monotonicity w.r.t. first order stochastic dominance.**

\[ Y_1 \prec_{FSD} Y_2 \text{ if } \mathbb{E}(U(Y_1)) \leq \mathbb{E}(U(Y_2)) \text{ for all monotonic integrable utility functions } U. \]

\[ A \text{ is monotonic w.r.t the first order stochastic dominance, if } Y_1 \prec_{FSD} Y_2 \text{ implies that } A(Y_1) \leq A(Y_2). \]

(A6) **monotonicity w.r.t. second order stochastic dominance.**

\[ Y_1 \prec_{SSD} Y_2 \text{ if } \mathbb{E}(U(Y_1)) \leq \mathbb{E}(U(Y_2)) \text{ for all monotonic and concave integrable utility functions } U. \]

\[ A \text{ is monotonic w.r.t the second order stochastic dominance, if } Y_1 \prec_{SSD} Y_2 \text{ implies that } A(Y_1) \leq A(Y_2). \]
Coherent risk measures

If \( \rho(Y) = -\mathcal{A}(Y) \), where \( \mathcal{A} \) is an acceptability functional, satisfying (A1) - (A5), then \( \rho \) is said to be coherent.

If \( \mathcal{A} \) is continuous w.r.t. convergence in probability and satisfies (A1) - (A5), then it has a representation

\[
\mathcal{A}(Y) = \inf \{ \mathbb{E}(YZ) : Z \in \mathcal{Z} \}
\]

where \( \mathcal{Z} \) is a set of probability densities containing the constant density 1 (Delbaen, 2000).
Superdifferential representations

Let $\mathcal{A}$ be continuous w.r.t. convergence in probability. It has a superdifferential representation, if

$$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y \cdot Z) + \alpha(Z) : Z \in \mathcal{Z}\},$$

where $\mathcal{Z}$ is a set of (signed) densities. (Föllmer and Schied).

We consider only functionals, which depend on the distribution only, that is which satisfy $\mathcal{A}(Y_1) = \mathcal{A}(Y_2)$, if $Y_1$ and $Y_2$ have the same distribution. A characterization of such functionals was given by Kusuoka (2001).
The main result

Suppose that $\mathcal{A}$ has a superdifferential representation $(\alpha, \mathcal{Z})$. Then

(i) $\mathcal{A}$ depends only on the distribution, if the set $\mathcal{Z}$ is determined by distribution only, i.e. if $Z$ and $\tilde{Z}$ have the same distribution and $Z \in \mathcal{Z}$ then also $\tilde{Z} \in \mathcal{Z}$ and $\alpha(Z) = \alpha(\tilde{Z})$.

(ii) $\mathcal{A}$ is positively homogeneous if $\alpha = 0$.

(iii) $\mathcal{A}$ is monotonic w.r.t. first order stochastic dominance, if $\mathcal{Z}$ contains only nonnegative random variables.

(iv) A positively homogeneous $\mathcal{A}$ is monotonic w.r.t. second order stochastic dominance, if $\mathcal{Z}$ contains only nonnegative densities and in addition is stable w.r.t. conditional expectations, i.e. $Z \in \mathcal{Z}$ implies that $\mathbb{E}(Z|\mathcal{F}) \in \mathcal{Z}$ for any $\sigma$-algebra $\mathcal{F}$. 
Pure risk functionals (Deviation functionals)

$\mathcal{D}$ is called *pure risk functional* or *deviation functional*, if it exhibits the following properties

(D1) **Translation invariance**

$$\mathcal{D}(Y + c) = \mathcal{D}(Y)$$

(D2) **Strictness.**

$$\mathcal{D}(Y) \geq 0,$$

where the equality sign holds iff $Y$ is a constant.

(D3) **Convexity.**

$$\mathcal{D}(\lambda Y_1 + (1 - \lambda) Y_2) \leq \lambda \mathcal{D}(Y_1) + (1 - \lambda) \mathcal{D}(Y_2),$$

for $0 \leq \lambda \leq 1$. 
Besides, the risk functional may also exhibit the following properties

(D4) **Positive homogeneity.**

\[ D(\lambda Y) = \lambda D(Y), \]

for \( \lambda \geq 0 \).

(D5) **Monotonicity w.r.t. convex dominance.**

\( Y_1 \) is dominated w.r.t. convex dominance (in symbol: \( Y_1 \prec_{\text{CXD}} Y_2 \)), if \( \mathbb{E}(U(Y_1)) \leq \mathbb{E}(U(Y_2)) \) for all convex utility functions \( U \). The pure risk measure \( D \) is called monotonic w.r.t the convex dominance, if \( Y_1 \prec_{\text{CXD}} Y_2 \) implies that

\[ D(Y_1) \leq D(Y_2). \]
Subdifferential representations for deviation functionals

Remark: \( \mathcal{D} \) is a deviation functional, if and only if 
\( \mathcal{A}(Y) = \mathbb{E}(Y) - \mathcal{D}(Y) \) is an acceptability functional. 
\( \mathcal{A} \) fulfills (Ai) iff the pertaining \( \mathcal{D} \) fulfills (Di), where \( i = 1,2,3,4 \). 
A representation of \( \mathcal{D} \) of the form 
\[
\mathcal{D}(Y) = \sup\left\{ \mathbb{E}(Y \cdot Z) - \beta(Z) : Z \in \mathcal{Z} \right\}
\]
is called a subdifferential representation of \( \mathcal{D} \).
The main result continued

(v) $\mathcal{D}$ is positively homogeneous if $\beta = 0$.

(vi) $\mathcal{D}$ is monotonic w.r.t. convex dominance, if $\mathcal{Z}$ is stable w.r.t. conditional expectations and $\beta$ is monotonic w.r.t. conditional expectation, i.e. $\beta(Z) \geq \beta(\mathbb{E}(Z|\mathcal{F}))$ for all $\sigma$-algebras $\mathcal{F}$ and all $Z \in \mathcal{Z}$. 
Coupling

(i) The FSD-coupling: If \( Y_1 \prec_{FSD} Y_2 \), then one may construct a pair \( \tilde{Y}_1, \tilde{Y}_2 \) of random variables with the same marginal distributions as \( Y_1, Y_2 \), such that
\[
\tilde{Y}_1 \leq \tilde{Y}_2 \quad \text{a.s.}
\]

(ii) The CXD-coupling: If \( Y_1 \prec_{CXD} Y_2 \), then one may construct a pair \( \tilde{Y}_1, \tilde{Y}_2 \) of random variables with the same marginal distributions as \( Y_1, Y_2 \), such that
\[
\tilde{Y}_1 = \mathbb{E}(\tilde{Y}_2 | \tilde{Y}_1) \quad \text{a.s.}
\]

(iii) The SSD-coupling. If \( Y_1 \prec_{SSD} Y_2 \), then one may construct a pair \( \tilde{Y}_1, \tilde{Y}_2 \) of random variables with the same marginal distributions as \( Y_1, Y_2 \), such that
\[
\tilde{Y}_2 \geq \mathbb{E}(\tilde{Y}_1 | \tilde{Y}_2) \quad \text{a.s.}
\]
The variance $\mathbb{V}ar(Y) = \| Y - \mathbb{E}Y \|^2_2$ and alike

Let $\| V \|_p = \mathbb{E}^{1/p}[\| V \|^p]$. 

(i) $\mathcal{D}(Y) := \| Y - \mathbb{E}Y \|_p^p = \sup \{ \mathbb{E}(Y \cdot Z) - \frac{p^{1-q}}{q} D_q(Z) : \mathbb{E}Z = 0 \}$, where $D_q(Z) = \inf \{ \| Z - a \|_q^q : a \in \mathbb{R} \}$ and $1/p + 1/q = 1$. 

$\beta(Z) = \frac{p^{1-q}}{q} D_q(Z)$, thus $\mathcal{D}$ is convex and monotonic w.r.t. convex dominance (D5).

(ii) $\mathcal{A}(Y) := \mathbb{E}Y - \| Y - \mathbb{E}Y \|_p^p = \inf \{ \mathbb{E}(Y \cdot Z) + \frac{p^{1-q}}{q} D_q(Z) : \mathbb{E}Z = 1 \}$. 

$\mathcal{A}$ is concave, but has none of the properties (A4) - (A6).
The standard deviation $\| Y - \mathbb{E} Y \|_2$ and alike

(i) $\mathcal{D}(Y) := \| Y - \mathbb{E} Y \|_p = \sup \{ \mathbb{E}(Y \cdot Z) : \mathbb{E}(Z) = 0, D_q(Z) \leq 1 \}$, where $1/p + 1/q = 1$. $Z = \{ Z : \mathbb{E}(Z) = 0, D_q(Z) \leq 1 \}$ is stable w.r.t. conditional expectations and hence $\mathcal{D}$ is convex, homogeneous (D4) and monotonic w.r.t. convex dominance (D5).

(ii) $A(Y) := \mathbb{E} Y - \| Y - \mathbb{E} Y \|_p = \inf \{ \mathbb{E}(Y \cdot Z) : \mathbb{E}(Z) = 1, D_q(Z - 1) \leq 1 \}$. $A$ is concave, positively homogeneous (A4), but is not monotonic in general.

(iii) The functional $\mathbb{E} Y - \frac{1}{2} \| Y - \mathbb{E} Y \|_1$ is monotonic w.r.t. SSD, because it has the representation

\[ \inf \{ \mathbb{E}(Y \cdot Z) : \mathbb{E}Z = 1; \exists a : |a| \leq \frac{1}{2}, |1 - Z - a| \leq \frac{1}{2} \text{ a.s.} \}, \]

which implies that all feasible $Z$ are nonnegative.
The lower semi-variance and alike

The lower semi-variance is $\Var^-(Y) = \|[Y - \mathbb{E}Y]^\cdot\|_2^2$

(i) $\mathcal{D}(Y) := \|[Y - \mathbb{E}Y]^\cdot\|_p^p$

$$= \sup\{\mathbb{E}(Y \cdot Z) - \frac{p^{1-q}}{q} \mathbb{E}[(\text{essup } Z - Z)^q] : \mathbb{E}(Z) = 0\}.$$  

(ii) $\mathcal{A}(Y) := \mathbb{E}Y - \|[Y - \mathbb{E}Y]^\cdot\|_p^p$

$$= \inf\{\mathbb{E}(Y \cdot Z) + \frac{p^{1-q}}{q} \mathbb{E}[(Z - \text{essinf } Z)^q] : \mathbb{E}(Z) = 1\}.$$  

$\mathcal{A}$ is concave, but neither positively homogeneous nor monotonic in general.
The lower semi-standard deviation and alike

\[(i) \quad D(Y) = \left\| [Y - \mathbb{E}Y]^- \right\|_p \]
\[= \sup \{ \mathbb{E}(Y \cdot Z) : \mathbb{E}(Z) = 0, Z \leq 1, \|Z - 1\|_q \leq 1 \}. \]

\[(ii) \quad A(Y) = \mathbb{E}Y - \left\| [Y - \mathbb{E}Y]^- \right\|_p \]
\[= \inf \{ \mathbb{E}(Y \cdot Z) : \mathbb{E}(Z) = 1, Z \geq 0, \|Z\|_q \leq 1 \}. \]

Thus \( A \) is positively homogeneous and monotonic w.r.t. SSD.
Minimal prediction error risk measures

Let

\[ \mathcal{D}(Y) := \inf \{ \mathbb{E}[h(Y - a)] : a \in \mathbb{R} \}. \]

Then \( \mathcal{D} \) fulfills (D1) - (D3) and has the representation

(i) \( \mathcal{D}(Y) := \sup \{ \mathbb{E}(YZ) - \mathbb{E}h^*(Z) : \mathbb{E}Z = 0 \} \),
where \( h^* \) is the dual of \( h \).

\[ h^*(v) = \sup \{ uv - h(u) : u \in \mathbb{R} \}. \]

(ii) \( A(Y) := \mathbb{E}Y - \mathcal{D}(Y) = \inf \{ \mathbb{E}(Y \cdot Z) - \mathbb{E}[h^*(1 - Z)] : \mathbb{E}(Z) = 1 \} \).
Examples

**Example. 1** If we take $h(u) = u^+ + \frac{1-\alpha}{\alpha} u^-$, then we get the
$\mathcal{D}(Y) = E(Y) - \text{AV@R}(Y)$, resp. $\mathcal{A}(Y) = \text{AV@R}_\alpha(Y)$, where
$\text{AV@R}_\alpha$ is the average value-at-risk (conditional value-at-risk)

$$
\text{AV@R}_\alpha(Y) = \max \{ a - \frac{1}{\alpha} E([Y - a]^-) : a \in \mathbb{R} \}.
$$

Since the dual of $h$ is

$$
h^*(v) = \begin{cases} 
0 & \frac{\alpha-1}{\alpha} \leq v \leq 1 \\
\infty & \text{otherwise}
\end{cases}
$$

the dual representation of $\text{AV@R}$ is

$$
\text{AV@R}_\alpha(Y) = \min \{ E(Y \cdot Z) : E(Z) = 1, 0 \leq Z \leq 1/\alpha \}.
$$

Thus $\text{AV@R}_\alpha$ is pos. homogeneous and monotonic w.r.t. SSD.
Example 2. If we take $h(u) = u^2$, then we get $\mathcal{D}(Y) = \mathbb{V} \text{ar}(Y)$. Since the dual of $h(u) = u^2$ is

$$h^*(v) = \frac{1}{4} v^2,$$

we arrive at the well known formula

$$\mathbb{V} \text{ar}(Y) = \sup \{ \mathbb{E}(Y \cdot Z) - \frac{1}{4} \mathbb{V} \text{ar}(Z) : \mathbb{E}Z = 0 \}.$$
Example 3. Let us take \( h(u) = u^21_{u \leq 0} + cu1_{u > 0} \). This asymmetric pure risk measure penalizes large negative deviation much higher than large positive deviations. We have that

\[
h^*(v) = \begin{cases} 
\infty & \text{if } v > c \\
0 & \text{if } 0 \leq v \leq c \\
\frac{v^2}{4} & \text{if } v > 0 
\end{cases}
\]

Thus the pertaining pure risk measure is

\[
\mathcal{D}(Y) = \sup \{ \mathbb{E}(YZ) - \frac{1}{4} \mathbb{E}[(Z^-)^2] : \mathbb{E}Z = 0; Z \leq c \}.
\]