Risk assessment for credit portfolios: a coupled Markov Chain model

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Abstract.

Credit portfolios, as for instance Collateralized Debt Obligations (CDO's) consist of credits that are heterogeneous both with respect to their ratings and the involved industry sectors. Estimates for the transition probabilities for different rating classes are well known and documented. We develop a Markov Chain model, which uses the given transition probability matrix as the marginal law, but introduces correlation coefficients within and between industry sectors and between rating classes for the joint law of migration of all components of the portfolio.

We have found a generating function for the one step joint distribution of all assets having non-default credit ratings and a generating function of the loss distribution. The numerical simulations presented here verify that the average number of defaults is not affected by the correlation coefficients, but the percentiles of the number of defaults are heavily dependent upon them. In particular, for strongly correlated assets, the distribution of the number of defaults follows a "cascade" pattern nesting in non-overlapping intervals. As a result, the probability of a large number of defaults is higher for correlated assets than for non-correlated ones.

Key words: credit risk, collateralized debt obligation (CDO), correlation coefficient, coupling, loss distribution, cascade.
JEL classification: G31, G11, C15.

1 Introduction

1.1. Motivation. Credit risk models are used to justify spreads in interest rates and to quantify the inherent risk in credit contracts and credit portfolios. Rating institutions have been accumulating huge data bases for calibration of their statistical models. While the behavior of one single debtor is more or less easy to model, the more important, but also more challenging problem is to find appropriate models for the joint behavior of many debtors in a credit portfolio.

The major motivation for this study are collateral debt obligations (CDO's). They comprise loans and debt instruments which have different credit ratings. Each firm is believed to migrate between credit ratings according to a Markovian transition matrix, for example the Standard and Poor's one-year transition matrix (see below).

These individual evolutions are affected by factors which are common for the whole economy. A realistic model should account for this. On the other hand, the evolution of assets issued by firms belonging to the same industry sector may follow a path specific only to this sector. Microeconomic foundations of this phenomenon are documented in the economic literature. See, for example, a pioneering paper by King (1966) or a review of later findings in Kahle and Walking (1996). Hence, it is necessary to classify assets according to their underlined industry as well and, tracing their migration through credit ratings, it is better to take into account industry specific factors. A detailed discussion of dependence in risk management is given in Embrechts et al. (2002), Frey and McNeil (2003) or Davis and Esparragoza (2004). Nagpal and Bahar (2001), Das et al. (2005), Couders and Renault (2005), and Hamerle et al. (2005) present estimates for default correlation between US corporate obligors. The later paper contains estimates also for some other OECD countries.

The assessment of a risk value (in particular the value-at-risk) to a credit portfolio is a basic problem in liability management. Statistical performance data of single debtors are easily available, and rating institutions help in assessing a rating class to individual debtors. The risk value of the whole portfolio however depends on the common stochastic behavior of all its components and models for the joint performance of several debtors are needed.

1.2. Literature review. Three major model paradigms have been developed in the literature: common factor models (Bluhm et al. 2001), mixed binomial models (Frey and McNeil 2003, Schönbucher and Schubert 2001) and dependent lifetime models (Li 2000). Let us briefly discuss these approaches. The *common factor approach* originates on Merton's firm value model (Merton 1974), developed by Vasicek (1987) and Bluhm et al. (2001). The financial viability of debtor i is described by $v_i = \sqrt{\rho}c + \sqrt{1-\rho}a_i$. Here c is a random variable common to all debtors and a_i is an individual variable, which is independent of c. Debtor i will default in the next period, if $v_i \leq v$, where v is a critical threshold. Typically c and a_i are assumed to be normally distributed. However, Hull and White (2004) use t-distributions for both c and a_i. The *mixed binomial model* assumes that the probability of default for an individual debtor, q, is a random variable. Then, even if conditional on q, the default events of debtors i and j are independent, the unconditional events are dependent, if the distribution of q is non-degenerate. A typical distribution for q is beta, and in multivariate cases Dirichlet. In the *dependent lifetime model* the time until default for debtor i is modelled by an exponential distribution with parameter λ_i, but the distributions for different i are made dependent using a normal copula for the logarithms of the default times.

These models suffer from some drawbacks. They use distributional assumptions, which are difficult to verify. Often there is no explicit correspondence between the correlations (of portfolio components) which may be observed empirically and the numerical parameters determining, typically via copulas, interdependence of assets involved in a model. Also, some of these models are not in accordance with the transition matrices used in rating agencies. For attempts in harmonizing these approaches see Koylouglu and Hickman (1998), and Bluhm et al. (2001).

1.3. The concepts of this paper. Our starting point is the well studied Markov transition behavior of the ratings. While keeping this process as given for each individual debtor, we use a coupling technique to find a reasonable joint evolution of these processes. We do not make distributional assumptions like firm values, default times or copula functions, but confine ourselves to a given set of parameters describing the dependence in terms of correlation coefficients.

We suggest a methodology for generating a portfolio consisting of firms with different credit ratings and underlying industries such that:

1) every individual migration is governed by the same Markovian transition matrix,
2) migrations of firms having the same credit rating are dependent,
3) evolution of firms in different rating classes are also dependent, where these dependencies may vary within and between industry sectors
4) the evolutions of all debtors of the same rating class and in the same sector are exchangeable.

Since the dependence among migrations is introduced through coefficients of correlation, the modeler obtains a conceptually transparent mechanism for incorporating his intuition, experience and data. Moreover, as we deal with coupling Bernoulli random variables, the coefficients of correlation and probabilities of success allow to characterize completely all joint distributions of interest in this case. Lucas (1995), concentrating on default events, also deals with Bernoulli random variables and their correlation. Terming this as the discrete default correlation, Li (2000) introduces the survival time correlation and discusses the relation between these two notions.

For the single-period transition, we obtain a generating function for the joint distribution for all assets involved in the portfolio which do not default as well as a generating function for the total number of defaults. These results can be extended to a multi-period setting, but the corresponding formulae become bulky. Moreover, some people argue, Li (2000) among them, that default correlation is time dependent. More exactly, it appears to be dependent upon economic conditions: it increases during higher volatility periods. See, for example, Das et al. 2005. Thus, for different time instants this calculation may involve a variety of coefficients of correlation and, also, several transition matrices (this possibility is discussed in Duffie and Singleton 2003), rendering the formulae even more complex. As we suggest to use Monte-Carlo simulations to calculate percentiles associated with the number of defaults, time dependent transitions can be easily incorportated in the program we have developed. The runs presented here illustrate the effect of correlations between assets on the distribution of the number of defaults.

2 The model

Consider a diversified portfolio consisting of credits given to different firms. The debtors are non-homogeneous in their credit ratings and they belong to different industry sectors. Assume that there are M non-default rating classes. The ratings are numbered in a descending order so that 1 corresponds to the safest class, while M is the next to the default. Currently $M = 7$ for all the most respected credit ranking agencies. For example, in terms of Standard and Poor's, $1 \longleftrightarrow AAA$, $2 \longleftrightarrow AA$, $3 \longleftrightarrow A$, $4 \longleftrightarrow BBB$, $5 \longleftrightarrow BB$, $6 \longleftrightarrow B$ and $7 \longleftrightarrow CCC$. On the other hand, Nagpal and Bahar (2001), due to scarcity of data concerning defaults, distinguish only two categories – investment grade and non-investment grade. Investment grade rating corresponds to an S&P rating from AAA to BBB, while non-investment grade ratings are BB or lower. We assume that the debtors may be classified into S sectors of industry. For example, Nagpal and Bahar (2001) analyze eleven US industry sectors. Let $\mathcal{N}_{i,s} = \mathcal{N}_{i,s}^{(0)}$ be the initial number of debtors in rating class i from industry sector s. The initial structure of the portfolio is described by the matrix $(\mathcal{N}_{i,s}); i = 1, 2, ..., M; s = 1, 2, ...\mathcal{S}$. In total, there are $\mathcal{N} = \sum_{i=1}^{M} \sum_{s=1}^{\mathcal{S}} \mathcal{N}_{i,s}$ debtors.

The fundamental assumption is that each debtor follows a Markovian rating process with the same distribution, but these processes are coupled in a possibly dependent way. Our construction may be modified to a situation when different industry sectors belonging to the same credit class are not governed by the same transition probabilities. This would imply a sector-wise Credit Metrics approach. We will not do this here, preferring to rely on a given credit ratings transition matrix as a benchmark in the analysis following next.

2.1. The multidimensional Markov Chain. Let $p_{i,j}$ be the probability of transition within one year from the i-th credit rating to the j-th. In particular, $p_{i,M+1}$ is the probability that a debtor who is having i-th credit rating at the beginning of a year defaults by the end of this year. The $M \times (M+1)$ transition matrix $P = (p_{i,j})$ is estimated and reported by rating agencies. For instance, the Standard and Poor's transition matrix P reads

$$P = \begin{pmatrix} 0.9081 & 0.0833 & 0.0068 & 0.0060 & 0.0120 & 0.0000 & 0.0000 & 0.0000 \\ 0.0070 & 0.9065 & 0.0790 & 0.0064 & 0.0006 & 0.0014 & 0.0002 & 0.0000 \\ 0.0009 & 0.0227 & 0.9105 & 0.052 & 0.0074 & 0.0026 & 0.0001 & 0.0006 \\ 0.0002 & 0.0033 & 0.0595 & 0.8693 & 0.0530 & 0.0117 & 0.0012 & 0.0018 \\ 0.0003 & 0.0014 & 0.0067 & 0.0773 & 0.8053 & 0.0884 & 0.0100 & 0.0106 \\ 0.0000 & 0.0011 & 0.0024 & 0.0043 & 0.0648 & 0.8346 & 0.0407 & 0.0520 \\ 0.0022 & 0.0000 & 0.0022 & 0.0130 & 0.0238 & 0.1124 & 0.6486 & 0.1979 \end{pmatrix} \qquad (1)$$

See Credit Metrics (1997) p. 69. This standard matrix will be used in all subsequent examples.

Consider a discrete-time Markov chain $X(t)$, $t \geq 0$ evolving in the state space $\{1,\ 2,\ ...,\ M+1\}$ governed by the probabilities $p_{i,j}$, where the state $M+1$ is absorbing.

We will model the evolution of a portfolio consisting of $\mathcal{N}$ debtors by a multi-dimensional random process $\mathbf{X}(t) = (X_1(t), \dots, X_{\mathcal{N}}(t))$ whose components evolve like $X(t)$. The n-th process $X_n(t)$ starts from a known state $m(n) \in \{1, \dots, M\}$.

If all these individual moves were independent, the common dynamics is completely determined by the transition probabilities governing $X(t)$. However, to capture interrelations of credit ratings' transitions within an industry sector and among different industries, we will introduce some statistical dependency of the components. Of course, there are many ways to do so. In section 2.4 we introduce a simple dependency model based on a coupling technique for Bernoulli random vectors.

2.2. Individual migrations and intersectoral links. Recall that the portfolio includes $\mathcal{N}$ debtors. The debtors are numbered from 1 to $\mathcal{N}$. For the n-th debtor, let $s(n)$ be its industrial sector, and $m(n)$ its current rating. Notice that only the ratings may randomly change over time, the assignment to sectors remains constant.

We will construct $\mathcal{N}$ rating processes $X_1(t), \dots, X_{\mathcal{N}}(t)$, with the following properties:

- Each individual process $X_n(t)$ is a homogeneous Markov process with transition matrix P. In particular, the state $M + 1$ is always absorbing.
- These processes are dependent, the degree of dependency depends on the rating classes and the sectors, the debtors belong to.

Since the processes are homogeneous in time, we may only consider the transition from time $t = 0$ to time $t = 1$ and the index t may be dropped. At time 0, the rating class of the n-th debtor is $m(n)$. The rating class of the same debtor at time 1 will be denoted by X_n. The already defaulted debtors, i.e. those debtors n, for which $m(n) = M + 1$ need not to be considered further, since they stay defaulted.

The construction will be done in several steps. First, consider $\mathcal{N}$ independent random variables $\xi_1, \dots, \xi_{\mathcal{N}}$ such that

$$\Pr\{\xi_n = j\} = p_{m(n),j}.$$

We call the random variables (ξ_n) the *individual factors*. If the rating processes would evolve independently from each other, we would set

$$X_n = \xi_n$$

and the construction would be finished. However, to account for possible dependencies, we introduce additional random variables which will model the business climate as a whole.

Let $\eta_1, \dots, \eta_M$ be further random variables which are independent of the (ξ_n) and have the following marginal distributions

$$\Pr\{\eta_i = j\} = p_{i,j}.$$

These random variables will be chosen to be dependent by the construction described in the next section. The random variables (η_i) are called the *common factors*.

In addition, let $\delta_n, n = 1, \ldots, \mathcal{N}$ be independent Bernoulli random variables, which are independent of all ξ_n and all η_n with the distribution

$$\Pr\{\delta_n = 1\} = 1 - \Pr\{\delta_n = 0\} = q_{m(n),s(n)}$$

where

$$Q = \begin{pmatrix} q_{1,1} & \cdots & q_{1,\mathcal{S}} \\ \vdots & \ddots & \vdots \\ q_{M,1} & \cdots & q_{M,\mathcal{S}} \end{pmatrix}.$$

is a $M \times \mathcal{S}$ matrix of parameters. Let now

$$X_n = \delta_n \xi_n + (1 - \delta_n)\eta_{m(n)}. \tag{2}$$

The equation (2) describes the combination of the two components of the rating risk: The first part is the idiosyncratic risk and the second part is the common (macroeconomic) risk.

One may conclude that as in the common factor models by Vasicek (1987) and Bluhm et al. (2001), our state variable is a "convex combination" of an individual move and a common trend. But technically they are different. In particular, our approach if free of any extra assumption on a firm's value distribution. The only fact concerning distributions we need is the Markovian transition matrix for credit ratings. Thus, our model rests on the principles set by Credit Metrics. On the other hand, the approach taken here resembles the "infectious defaults" modelled by weighted sums of Bernoulli random variables in Davis and Lo (2001). However, we consider correlations of all moves, rather than only default events. Also, based on a credit rating transition matrix, our random variables are not, in general, exchangable.

Introduce the indicator variables

$$\mathcal{I}_{n,j} = \mathbb{1}_{\{X_n = j\}}, \tag{3a}$$

$$\widetilde{\mathcal{I}}_{i,j} = \mathbb{1}_{\{\eta_i = j\}}. \tag{3b}$$

Consider now the rating processes of two debtors n and N, i.e. two processes X_n and X_N, $n \neq N$ and their evolution. Suppose that the first debtor belongs to rating i and sector s, while the second belongs to rating I and sector S, i.e.

$$m(n) = i, \tag{4a}$$

$$m(N) = I, \tag{4b}$$

$$s(n) = s, \tag{4c}$$

$$s(N) = S. \tag{4d}$$

The notational convention is that all quantities associated with the first process are written by lower case symbols, while the same symbols in upper case are used for the second process.

Notice that the marginals are just as required, i.e.

$$\begin{aligned}
\mathbb{E}[\mathcal{I}_{n,j}] &= p_{i,j}, \\
\mathrm{Var}[\mathcal{I}_{n,j}] &= p_{i,j}(1-p_{i,j}), \\
\mathbb{E}[\mathcal{I}_{N,J}] &= p_{I,J}, \\
\mathrm{Var}[\mathcal{I}_{N,J}] &= p_{I,J}(1-p_{I,J}).
\end{aligned}$$

while the joint distribution satisfies by construction (2)

$$\mathbb{E}[\mathcal{I}_{n,j}\mathcal{I}_{N,J}] = (\mathbb{E}[\widetilde{\mathcal{I}}_{i,j}\widetilde{\mathcal{I}}_{I,J}] - p_{i,j}p_{I,J})(1 - q_{i,s} - q_{I,S} + q_{i,s}q_{I,S}) + p_{i,j}p_{I,J}$$

and therefore

$$\mathrm{Corr}[\mathcal{I}_{n,j}, \mathcal{I}_{N,J}] = \mathrm{Corr}[\widetilde{\mathcal{I}}_{i,j}, \widetilde{\mathcal{I}}_{I,J}](1 - q_{i,s})(1 - q_{I,S}). \tag{5}$$

This coefficient of correlation vanishes when $\max(q_{i,s}, q_{I,S})$ is close to 1, while it approaches $\mathrm{Corr}[\widetilde{\mathcal{I}}_{n,j}, \widetilde{\mathcal{I}}_{N,J}]$ if $\max(q_{i,s}, q_{I,S})$ goes to 0.

Thus, the indicators $\mathcal{I}_{n,j}$ and $\mathcal{I}_{N,J}$ are dependent as long as $\max(q_{i,s}, q_{I,S}) < 1$ and $\mathrm{Corr}[\widetilde{\mathcal{I}}_{i,j}, \widetilde{\mathcal{I}}_{I,J}] \neq 0$. This dependence is due to the common factors captured by the dependence of η_i and η_I. But the impacts of these factors are credit class and industry specific. Quantitatively they are measured by the corresponding $q_{i,s}$. In general, the larger is $q_{i,s}$, the less dependent are moves of firms from sector s on the common factors.

2.3. Coupling Bernoulli random variables. Consider again a discrete-time Markov Chain $X(t)$ evolving in the state space $\{1, 2, \ldots, M+1\}$ governed by the probabilities $p_{i,j}$, where the state $M+1$ is absorbing.

Any transition of $X(t)$ may be divided in two phases. In the first phase only the *tendency* is determined. Namely, whether a non-deteriorating move takes place or not. That is, $X(t) \geq X(t+1)$ (a non-deteriorating step), or $X(t) < X(t+1)$, (a deteriorating one). Let $\chi(t)$ be the tendency variable, i.e.

$$\chi(t) = \begin{cases} 1 & \text{if } X(t) \geq X(t+1), \\ 0 & \text{if } X(t) < X(t+1). \end{cases}$$

In other words, $\chi(t)$ is the indicator of a non-deteriorating move at time t. Notice that the tendency variable has a Bernoulli distribution whose probability of a success reads

$$\Pr\{\chi(t) = 1 \mid X(t) = i\} = \sum_{j=1}^{i} p_{i,j} =: p_i^+.$$

In the second phase, the actual move is determined, which conditional on the tendency follows the distribution:

$$\Pr\{X(t+1) = j \mid X(t) = i, \chi(t) = 1\} = \begin{cases} p_{i,j}/p_i^+ & \text{if } j \leq i, \\ 0 & \text{if } j > i; \end{cases} \tag{6a}$$

$$\Pr\{X(t+1) = j \mid X(t) = i, \chi_t = 0\} = \begin{cases} p_{i,j}/(1 - p_i^+) & \text{if } j > i, \\ 0 & \text{if } j \leq i. \end{cases} \tag{6b}$$

Again, we drop the time index t further on.

Let $\boldsymbol{\chi} = (\chi_1, \dots, \chi_M)$ be a vector of tendency variables. Each χ_i is a Bernoulli random variable with success probability p_i^+.

A generic tendency "the market is up" common for the whole economy reveals itself to the i-th credit class through the random variable χ_i. Modeling a common tendency for all credit ranks, the coefficient of correlation, $c_{i,I} = \mathrm{Corr}(\chi_i, \chi_I)$, between χ_i and χ_I has to be non-negative, i.e. correlated Bernoulli variables have to be considered.

To illustrate the possible complications, let us start with the simplest case when $M = 2$. Assume that we are given the following parameters: $p_i^+ \in (0,1)$, $p_I^+ \in (0,1)$ and $c_{i,I} \in [-1,1]$. We have to construct two Bernoulli random variables, χ_i and χ_I, such that p_i^+ and p_I^+ are their probabilities of a success and $c_{i,I} = \mathrm{Corr}(\chi_i, \chi_I)$.

We proceed in the following way. Let $\boldsymbol{\chi} = (\chi_i, \chi_I)$ be a random 2-vector such that

$$\Pr\{\boldsymbol{\chi} = (1,1)\} = \pi_{(1,1)}, \quad \Pr\{\boldsymbol{\chi} = (1,0)\} = \pi_{(1,0)},$$
$$\Pr\{\boldsymbol{\chi} = (0,1)\} = \pi_{(0,1)}, \quad \Pr\{\boldsymbol{\chi} = (0,0)\} = \pi_{(0,0)},$$

with $\pi_{(1,1)} + \pi_{(1,0)} + \pi_{(0,1)} + \pi_{(0,0)} = 1$ due to the normalizing condition. When

$$\pi_{(1,1)} + \pi_{(1,0)} = p_i^+ \text{ and } \pi_{(1,1)} + \pi_{(0,1)} = p_I^+,$$

we get the required probabilities of success. Since $\mathbb{E}\chi_i\chi_I = \pi_{(1,1)}$, we have that

$$\pi_{(1,1)} = c_{i,I}\sqrt{p_i^+(1-p_i^+)p_I^+(1-p_I^+)} + p_i^+ p_I^+.$$

In sum, starting from $\pi_{(1,1)}$, we are able to identify the distribution π of $\boldsymbol{\chi}$, as required. Thus we have produced a pair of Bernoulli random variables with the required properties.

These calculations are not always possible. In fact, $\pi_{(1,1)} \le \min(p_i^+, p_I^+)$. That is,

$$\begin{aligned} c_{i,I}\sqrt{p_i^+(1-p_I^+)p_i^+(1-p_I^+)} &\le p_i^+(1-p_I^+), & \text{(7a)} \\ c_{i,I}\sqrt{p_i^+(1-p_i^+)p_I^+(1-p_I^+)} &\le p_I^+(1-p_i^+), & \text{(7b)} \end{aligned}$$

implying that

$$c_{i,I} \le \bar{c}_{i,I} \text{ with } \bar{c}_{i,I} = \min(a, 1/a), \text{ where } a = \sqrt{\frac{p_I^+(1-p_i^+)}{(1-p_I^+)p_i^+}}. \tag{8}$$

Should one allow negative correlations, then by $\pi_{(1,1)} \ge \max(p_i^+ + p_I^+ - 1, 0)$, a lower bound for $c_{i,I}$ is

$$c_{i,I} \ge -\min(b, 1/b), \text{ where } b = \sqrt{\frac{(1-p_I^+)(1-p_i^+)}{p_I^+ p_i^+}}. \tag{9}$$

In estimated credit rank transition matrices, the values p_i^+ tend to be quite close to 1. In particular, for the above credit rank transition matrix of Standard and Poor's given in (1), $p_1^+ = 0.9081$, $p_2^+ = 0.9135$, $p_3^+ = 0.9341$, $p_4^+ = 0.9323$, $p_5^+ = 0.8910$, $p_6^+ = 0.9072$ and $p_7^+ = 0.8022$. Then the maximum possible coefficient of correlation between the rating migrations of debtors belonging to the first (AAA) and the third (A) credit classes, is $\overline{c}_{1,3} = 0.8165$. In the same way, $\overline{c}_{2,7} = 0.6197$.

Example. Let χ_1 and χ_2 be Bernoulli random variables with success probabilities 0.9 and 0.97 respectively. The minimal/ maximal possible correlation between χ_1 and χ_2 is -0.0585 respectively 0.5276. Here are some examples of the corresponding joint distributions.

$c = -0.0586$	χ_2		
χ_1	0	1	
0	0.00	0.10	0.1
1	0.03	0.87	0.9
	0.03	0.97	

$c = 0$	χ_2		
χ_1	0	1	
0	0.03	0.097	0.1
1	0.027	0.873	0.9
	0.03	0.97	

$c = 0.2$	χ_2		
χ_1	0	1	
0	0.0123	0.0868	0.1
1	0.0168	0.8832	0.9
	0.03	0.97	

$c = 0.5276$	χ_2		
χ_1	0	1	
0	0.03	0.07	0.1
1	0.00	0.90	0.9
	0.03	0.97	

Now let us turn to the general case of constructing a vector of M Bernoulli random variables $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_M)$ with given probabilities of success $p_i^+, i = 1, \ldots, M$ and given coefficients of correlations $c_{i,j} = c_{j,i} \geq 0$ $(c_{i,i} = 1)$.

The necessary conditions for the distribution of $\boldsymbol{\chi}$ are listed next:

(a) The normalizing condition:

$$\sum_{\mathbf{v} \in V} \pi_{\mathbf{v}} = 1,$$

where $P\{\boldsymbol{\chi} = \mathbf{v}\} = \pi_{\mathbf{v}}$, $\mathbf{v} = (v_1, v_2, ..., v_M)$, v_i equals 0 or 1. (The sum is taken over the set V all of 2^M such vectors).

(b) The equations for the marginals:

$$\sum_{\mathbf{v} \in V(i)} \pi_{\mathbf{v}} = p_i^+, i = 1, 2, ..., M,$$

where $V(i)$ contains all $\mathbf{v}$ such that $v_i = 1$.

(c) The conditions for the coefficients of correlation:

$$\sum_{\mathbf{v} \in V(i) \cap V(I)} \pi_{\mathbf{v}} = p_i^+ p_I^+ + c_{i,I} \sqrt{p_i^+(1 - p_i^+) p_I^+(1 - p_I^+)},\ 1 \leq i < I \leq M.$$

The values $c_{i,I} = c_{I,i}$ must satisfy $c_{i,I} \le \overline{c_{i,I}}$ (see (8)) and the symmetric $M \times M$ matrix

$$C = \begin{pmatrix} 1 & c_{1,2} & c_{1,3} & \dots & c_{1,M} \\ c_{2,1} & 1 & c_{2,3} & \dots & c_{2,M} \\ \dots & \dots & \dots & \dots & \dots \\ c_{M-1,1} & c_{M-1,2} & c_{M-1,3} & \dots & c_{M-1,M} \\ c_{M,1} & c_{M,2} & c_{M,3} & \dots & 1 \end{pmatrix}$$

must be non-negative definite.

There are M conditions for marginals and $\frac{M(M-1)}{2}$ coefficients of correlation $c_{i,j}$. Together with the normalizing condition, these give $\frac{M(M+1)}{2} + 1$ equations. When $M \ge 3$, $2^M > \frac{M(M+1)}{2} + 1$ implying that these relations do not determine π the distribution of $\boldsymbol{\chi}$. Indeed, they capture only the joint distributions of pairs of coordinates, but not of triples, quadruples, etc. Consequently, the above system of linear equalities for identifying the distribution π has either no solution or infinitely many solutions. In the latter case, introducing an additional criterion, we may choose one of them. For example, we may regard all above mentioned equalities as linear constraints of a constrained optimization problem with a strictly convex goal function. As a natural particular case, we may use the following quadratic function:

$$\sum_{\mathbf{v} \in V} \left[\pi_{\mathbf{v}} - \prod_{i=1}^{M} (p_i^+)^{v_i} (1 - p_i^+)^{1-v_i} \right]^2 . \tag{10}$$

Minimizing (10) subject to the above linear constraints (a),(b),(c), we search for a distribution of $\boldsymbol{\chi}$ that is the closest in the least squares sense to the one when all coordinates are independent Bernoulli random variables whose probabilities of success are p_i^+. Thus we are faced with quadratic programming problem under linear constraints which may be easily solved numerically. Of course, not all combinations of the parameters are feasible, but a quadratic program solver will detect this. A nice property of such a scheme is that it produces independent random variables when $c_{i,I} = 0$ for all $i \ne I$. We will use this approach in all calculations from here onwards.

The choice (10) reflects the option that one should depart from the independence assumption as little as possible. Alternatively, one might choose another, not independent distribution as reference. For instance, for stress testing the model one could use

$$\sum_{\mathbf{v} \in V} \left[\pi_{\mathbf{v}} - \min_i v_i p_i^+ \right]^2 . \tag{11}$$

Here the upper bounds of the joint probabilities serve as reference.

2.4. Coupling Markov transitions. Having constructed one and only one dependent random vector $\boldsymbol{\chi} = (\chi_1, \dots, \chi_M)$, per stage we construct the common factors η_i in the following way. First, these random variables are chosen to be conditionally independent given $\boldsymbol{\chi}$. Second, we choose

$$\Pr\{\eta_i = j | \chi_i = 1\} = p_{i,j}/p_i^+ \text{ for } j = 1, \dots, i,$$

and

$$\Pr\{\eta_i = j|\chi_i = 0\} = p_{i,j}/(1-p_i^+) \text{ for } j = i+1, i+2, \dots, M+1.$$

For the indicator variables $\widetilde{\mathcal{I}}_{i,j}$ introduced in (3) we have that $\mathrm{Corr}(\widetilde{\mathcal{I}}_{i,j}, \widetilde{\mathcal{I}}_{I,J}) = c_{i,I} d_{j,J}^{(i,I)}$, where

$$d_{j,J}^{(i,I)} = \begin{cases} \sqrt{\frac{p_{i,j}p_{I,J}(1-p_i^+)(1-p_I^+)}{(1-p_{i,j})(1-p_{I,J})p_i^+p_I^+}} & \text{for } j \le i, J \le I; \\ -\sqrt{\frac{p_{i,j}p_{I,J}(1-p_i^+)p_I^+}{(1-p_{i,j})(1-p_{I,J})p_i^+(1-p_I^+)}} & \text{for } j \le i, J > I; \\ -\sqrt{\frac{p_{i,j}p_{I,J}p_i^+(1-p_I^+)}{(1-p_{i,j})(1-p_{I,J})(1-p_i^+)p_I^+}} & \text{for } j > i, J \le I; \\ \sqrt{\frac{p_{i,j}p_{I,J}p_i^+p_J^+}{(1-p_{i,j})(1-p_{I,J})(1-p_i^+)(1-p_I^+)}} & \text{for } j > i, J > I. \end{cases} \tag{12}$$

Consequently, by (2)

$$\mathrm{Corr}[\mathcal{I}_{n,j}, \mathcal{I}_{N,J}] = c_{i,I} d_{j,J}^{(i,I)} (1-q_{i,s})(1-q_{I,S}),$$

given that the relations (4) hold.

That is, moves towards higher credit ratings of any two obligors are non-negatively correlated. The same is true for transitions towards lower credit classes. But the moves in opposite directions, when one debtor moves up while the other shifts down, of any two obligors are non-positively correlated. That is exactly what should take place when the transitions had a common cause like a tendency governing the whole industry. Note that the magnitude of these correlations will be, in general, quite small. In fact, each of the coefficients is a product of four terms. Three of them, $c_{i,I}(1-q_{i,s})$ and $(1-q_{I,S})$, are non-negative and do not exceed one. We may also show that $|d_{j,J}^{(i,I)}| \le 1$. The argument is as follows. Let, for example, $j \le i$ and $J > I$. Then

$$\left[d_{j,J}^{(i,I)}\right]^2 = \frac{p_{i,j}}{p_i^+} \frac{(1-p_i^+)}{(1-p_{i,j})} \frac{p_{I,J}}{(1-p_I^+)} \frac{p_I^+}{(1-p_{I,J})} \le 1.$$

Indeed, $p_{i,j} \le p_i^+ = p_{i,1} + p_{i,2} + \dots + p_{i,i}$, because $j \le i$. Consequently, the first factor here does not exceed one. Also, $p_I^+ = p_{I,1} + p_{I,2} + \dots + p_{I,I} = 1 - p_{I,I+1} + p_{I,I+2} + \dots + p_{I,M+1} \le 1 - p_{I,J}$, since $J > I$. Thus, the fourth factor does not exceed one. Since $1 - p_i^+ \le 1 - p_{i,j}$ and $p_{I,J} \le 1 - p_I^+$ when $p_{i,j} \le p_i^+$ and $p_I^+ \le 1 - p_{I,J}$, respectively, the second and the third factors do not exceed one as well. The same arguments are valid for all cases in (12).

As a numerical example, consider $d_{8,8}^{(i,i)} = \frac{p_{i,8}}{(1-p_{i,8})} \frac{p_i^+}{(1-p_i^+)}$ for the matrix (1). We have: $d_{8,8}^{(1,1)} = 0.0000$, $d_{8,8}^{(2,2)} = 0.0000$, $d_{8,8}^{(3,3)} = 0.0085$, $d_{8,8}^{(4,4)} = 0.0248$, $d_{8,8}^{(5,5)} = 0.0876$, $d_{8,8}^{(6,6)} = 0.5362$, $d_{8,8}^{(7,7)} = 1.000$.

Finally, the *default correlations* $dc(\cdot,\cdot,\cdot,\cdot)$ are defined by

$$\begin{aligned} dc(i,I,s,S) &= \mathrm{Corr}[\widetilde{\mathcal{I}}_{n,M+1}, \widetilde{\mathcal{I}}_{N,M+1}] \\ &= c_{i,I} d_{M+1,M+1}^{(i,I)} (1-q_{i,s})(1-q_{I,S}), \end{aligned} \tag{13}$$

given that the relations (4) hold. These values indicate the correlation between the default event of a member of industry sector s, being currently in class i and the default of an member of sector S being currently in class I.

To calibrate the model, the coefficients of correlation given above may be estimated empirically. All together we have to know $\frac{M(M-1)}{2}$ values $c_{i,I}$ and $M \times \mathcal{S}$ probabilities $q_{i,s}$. Default correlations for assets belonging to the same industry sector and having identical or different ratings are given, for example, in Nagpal and Bahar (2001).

2.5. Portfolio evolution. To trace the migrations in time of the portfolio's components, we have to be able to analyze the joint distribution of the $M \times \mathcal{S}$ dimensional random matrix whose elements are new counts, $\mathcal{N}_{i,s}^{(1)}$. They come to exist at the end of the year when all transitions between credit classes have occurred. For a particular observation of $\mathcal{N}^{(1)}$, taking it as $\mathcal{N}^{(0)}$ now, we may repeat all calculations with independent (in the simplest case) realizations of random variables in question of the current ones. This will give $\mathcal{N}^{(2)}$ conditional to the above observation of $\mathbf{N}^{(1)}$. This process may be repeated as many times as it is necessary. Also we have to keep track of the number of defaults. It is not needed to describe the evolution of the portfolio as such, but it is very important to trace the evolution of its value.

The new counts $\mathcal{N}_{i,s}^{(1)}$ are obtained by the following relations

$$\mathcal{N}_{j,s}^{(1)} = \sum_{n=1}^{\mathcal{N}} \mathbb{1}_{\{s(n)=s\}} \widetilde{\mathcal{I}}_{\{n,j\}}.$$

because the debtors may migrate between credit ratings, but remain in the same industry sector.

The number D of default debtors after one time-step is

$$D = \sum_{n=1}^{\mathcal{N}} \mathcal{I}_{n,M+1}. \tag{14}$$

Risk managers are typically interested in the quantiles of D, say the 99% or 95% quantile, which is the maximum number of defaults, if 1% or 5% of the worst cases are excluded. Quantiles can be found by simulation, but also by looking at generating functions.

3 Generating functions

Let us find first the generating function $G_D(x) = \mathbb{E}[x^D]$ of D, the total number of defaulted debtors after one time step, see (14). This will characterize the loss distribution. We have that $D = \sum_{i=1}^{M} D_i$ with

$$D_i = \sum_{n=1}^{\mathcal{N}} \mathbb{1}_{\{m(n)=i\}} \mathcal{I}_{n,M+1}.$$

When we know $\widetilde{\mathcal{I}}_{i,M+1}$, D_i turns out to be a sum of independent identically distributed Bernoulli random variables, and therefore

$$G_{D_i}(x) = \prod_{s=1}^{\mathcal{S}} [g_{i,s}(x \mid \widetilde{\mathcal{I}}_{i,M+1})]^{\mathcal{N}_{i,s}}.$$

The conditional generating function $g_{i,s}(x \mid \widetilde{\mathcal{I}}_{i,M+1})$ of a generic term in this product reads

$$\begin{aligned} g_{i,s}(x \mid 1) &= x[p_{i,M+1} + (1 - p_{i,M+1})(1 - q_{i,s})] + (1 - p_{i,M+1})q_{i,s}, \\ g_{i,s}(x \mid 0) &= xp_{i,M+1}q_{i,s} + 1 - p_{i,M+1}q_{i,s}. \end{aligned}$$

Given that we know $\widetilde{\mathcal{I}}_{1,M+1}, \dots, \widetilde{\mathcal{I}}_{M,M+1}$, the random variables $D_1, \dots, D_M$ are independent as well. Hence

$$G_D(x) = \sum_{\mathbf{u} \in V} \prod_{i=1}^{M} G_{D_i}(x \mid \ u_i) r_{\mathbf{u}}^{(M+1)}$$

where V is the set of all M vectors with coordinates u_i equal to 0 or 1. The cardinality of V is 2^M. Also

$$\begin{aligned} r_{\mathbf{u} \in V}^{(M+1)} &= \Pr\{\widetilde{\mathcal{I}}_{1,M+1} = u_1, \widetilde{\mathcal{I}}_{2,M+1} = u_2, \dots, \widetilde{\mathcal{I}}_{M,M+1} = u_M\} \\ &= \sum_{\mathbf{v} \in V} \Pr\{\widetilde{\mathcal{I}}_{1,M+1} = u_1, \widetilde{\mathcal{I}}_{2,M+1} = u_2, \dots, \widetilde{\mathcal{I}}_{M,M+1} = u_M \mid (\chi_1, \chi_2, \dots, \chi_M) = \mathbf{v}\} \times \\ &\quad \times \Pr\{(\chi_1, \chi_2, ..., \chi_M) = \mathbf{v}\}. \end{aligned}$$

Taking into account, first, that the events involved in these conditional probabilities are independent if we know $\chi_1, \chi_2, \dots, \chi_M$ and, second, that the distribution of η_i is completely characterized once the value of χ_i is known, we get

$$\begin{aligned} \Pr\{\widetilde{\mathcal{I}}_{1,M+1} &= u_1, \widetilde{\mathcal{I}}_{2,M+1} = u_2, \dots, \widetilde{\mathcal{I}}_{M,M+1} = u_M \mid (\chi_1, \chi_2, ..., \chi_M) = \mathbf{v}\} \\ &= \Pr\{\widetilde{\mathcal{I}}_{1,M+1} = u_1 \mid \chi_1 = v_1\} \times ... \times \Pr\{\widetilde{\mathcal{I}}_{M,M+1} = u_M \mid \chi_M = v_M\}. \end{aligned}$$

Here

$$\Pr\{\widetilde{\mathcal{I}}_{i,M+1} = u_i \mid \chi_i = v_i\} = [1 - \min(v_i, u_i)](1 - p_i^+)^{-1} p_{i,M+1}^{u_i} (1 - p_{i,M+1})^{1-u_i}$$

for $i = 1, 2, \dots, M$. Putting all pieces together we get

$$G_D(x) = \sum_{\mathbf{u} \in V} \prod_{i=1}^{M} \prod_{s=1}^{\mathcal{S}} [g_{i,s}(x \mid u_i)]^{\mathcal{N}_{i,s}} r_{\mathbf{u}}^{(M+1)},$$

where

$$r_{\mathbf{u}}^{(M+1)} = \sum_{\mathbf{v} \in V} \prod_{i=1}^{M} [1 - \min(u_i, v_i)](1 - p_i^+)^{-1} p_{i,M+1}^{u_i} (1 - p_{i,M+1})^{1-u_i} \pi_{\mathbf{v}}.$$

Here $\pi_{\mathbf{v}}$ are the probabilities constructed according to (a),(b), (c).

Now let us turn to the multivariate generating function

$$G(x_{1,1}, x_{1,2}, \ldots, x_{1,\mathcal{S}}, \ldots, x_{M,\mathcal{S}}) = \mathbb{E} \prod_{i=1}^{M} \prod_{s=1}^{\mathcal{S}} x_{i,s}^{\mathcal{N}_{i,s}^{(1)}},$$

of the joint distribution of all $\mathcal{N}_{i,s}^{(1)}$. Let

$$N_{i,s,j} = \sum_{n=1}^{\mathcal{N}} \mathbb{1}_{\{s(n)=s\}} \mathbb{1}_{\{m(n)=i\}} \mathbb{1}_{\{X_n=j\}}$$

be the number of debtors in rating class i and sector s, who changed to the new rating class j.

Given that we know the vector $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_M)$, $N_{i,s,j}$ turn out to be independent identically distributed Bernoulli random variables. Consequently, the conditional generating function of $N_{i,s,j}$ becomes

$$g_{N_{i,s,j}}(x_{i,s} \mid \boldsymbol{\eta}) = [g_{i,s,j}(x_{i,s} \mid \mathbb{1}_{\{\eta_i=j\}}]^{\mathcal{N}_{i,s}}.$$

The conditional generating function $g_{i,s,j}(x_{i,s} \mid \mathbb{1}_{\{\eta_i=j\}})$ reads

$$\begin{aligned} g_{i,s,j}(x_{i,s} \mid 1) &= x_{i,s}[p_{i,j} + (1-p_{i,j})(1-q_{i,s})] + (1-p_{i,j})q_{i,s}, \\ g_{i,s,j}(x_{i,s} \mid 0) &= x_{i,s}p_{i,j}q_{i,s} + 1 - p_{i,j}q_{i,s}. \end{aligned}$$

Given that we know $\boldsymbol{\eta}$ the random variables $N_{i,s,j}$ become independent in i and s. Hence

$$G(x_{1,1}, x_{1,2}, \ldots, x_{1,\mathcal{S}}, \ldots, x_{M,\mathcal{S}}) = \sum_{\mathbf{u}\in V} \prod_{i=1}^{M} g_{i,s,j}(x_{i,s} \mid u_i) r_{\mathbf{u}}^{(j)},$$

where $\mathbf{u}$ stands for an M vector whose coordinates u_i are 0 or 1 and

$$\begin{aligned} r_{\mathbf{u}}^{(j)} &= \Pr\{\widetilde{\mathcal{I}}_{1,j} = u_1, \widetilde{\mathcal{I}}_{2,j} = u_2, \ldots, \widetilde{\mathcal{I}}_{M,j} = u_M\} \\ &= \sum_{\mathbf{v}\in V} \Pr\{\widetilde{\mathcal{I}}_{1,j} = u_1, \ldots, \widetilde{\mathcal{I}}_{M,j} = u_M \mid (\chi_1, \chi_2, ..., \chi_M) = \mathbf{v}\} \cdot \pi_{\mathbf{v}}, \end{aligned}$$

where $\mathbf{v}$ denotes an M vector with coordinates v_i equal to 0 or 1. Taking into account, first, that the events involved in these conditional probabilities are independent if we know $\chi_1, \chi_2, ..., \chi_M$ and, second, that the distribution of $\widetilde{\mathcal{I}}_{i,j}$ is completely determined once the value of χ_i is known, we get

$$\begin{aligned} & \Pr\{\widetilde{\mathcal{I}}_{1,j} = u_1, \widetilde{\mathcal{I}}_{2,j} = u_2, \ldots, \widetilde{\mathcal{I}}_{M,j} = u_M \mid (\chi_1, \chi_2, ..., \chi_M) = \mathbf{v}\} \\ = \; & \Pr\{\widetilde{\mathcal{I}}_{1,j} = u_1 \mid \chi_1 = v_1\} \cdots \Pr\{\widetilde{\mathcal{I}}_{M,j} = u_M \mid \chi_M = v_M\}. \end{aligned}$$

Here

$$Pr\{\widetilde{\mathcal{I}}_{i,j} = u_i \mid \chi_i = v_i\} = \begin{cases} p_{i,j}/p_i^+ & \text{if } j \ge i, v_i = 1, \\ 0 & \text{if } i > j, v_i = 1, \\ 0 & \text{if } j \ge i, v_i = 0, \\ p_{i,j}/(1-p_i^+) & \text{if } i > j, v_i = 0, \end{cases}$$

for $i, j = 1, 2, \ldots, M$.

4 Results of computer simulations

We use the above Standard and Poor's transition matrix P given in (1). It is assumed that there are 4 industry sectors. The time horizon is 3 years. We will consider three different $Q = (q_{i,s})$ matrices and three different $C = (c_{i,I})$ matrices: $Q0$, $Q1$, $Q2$, $C0$, $C1$, $C2$. Here $(Q0)_{i,s} = 1$,

$$(Q1)_{i,s} = \begin{cases} 0.5 \text{ if } & s = 1, \\ 0.6 \text{ if } & s = 2, \\ 0.7 \text{ if } & s = 3, \\ 0.8 \text{ if } & s = 4, \end{cases} \quad (Q2)_{i,s} = \begin{cases} 0.2 \text{ if } & s = 1, \\ 0.3 \text{ if } & s = 2, \\ 0.4 \text{ if } & s = 3, \\ 0.5 \text{ if } & s = 4, \end{cases}$$

$$(C0)_{i,I} = \begin{cases} 1 & \text{if} \quad i = I, \\ 0 & \text{if} \quad i \neq I, \end{cases} \quad (C1)_{i,I} = \begin{cases} 1 & \text{if} \quad i = I, \\ 0.3 & \text{if} \quad i \neq I, \end{cases} \quad (C2)_{i,I} = \begin{cases} 1 & \text{if} \quad i = I, \\ 0.8 & \text{if} \quad i \neq I. \end{cases}$$

The Matrix $Q0$ corresponds to independent industry sectors, in the case of $Q1$ the dependence is weaker than for $Q2$. $C0$ implies firms independently migrating through credit ratings, in the case of $C2$ the dependence is stronger for $C1$.

Given these values, the (discrete) default correlations, $dc(i, i, s, s)$, for two debtors belonging to the same rating class i and industry sector s can be easily calculated. For example, when $Q1$ is coupled with $C1$, $d_{i,i,s,s,} \times 10^2$ read:

INSERT TABLE 1 HERE

(The values for $I = 1, 2$ equal zero because $d_{8,8}^{(1,1)} = d_{8,8}^{(2,2)} = 0.000$.) Their order of magnitude is essentially the same as the empirical findings by Nagpal and Bahar (2001).

Using these matrices, we have run seven models (see below). Using 5000 replications and a staring distribution of 100 debtors in each of the four industry sectors and in each of the rating classes, i.e. 2800 firms in total, we estimated the sample and the 95% quantile of the total number of defaulted debtors:

INSERT TABLE 2 HERE

The mean number of defaults remains constant while there is a clear tendency of increasing of this percentile as the pairwise interdependence of assets increases. More detailed picture of this phenomenon is given by the following figures. Here the distribution of the total number of defaulted firms is visualized by a histogram plot. The mean is indicated by a dotted line and the 95% quantile is indicated by a dashed line.

INSERT FIGURES 1-7 HERE

5 Possible generalizations of the model

The suggested above scheme generates portfolio dynamics where moves in the same direction, defaults, for example, of any two obligors are non-negatively correlated. However some empirical findings, see Nagpal and Bahar (2001) among them, have documented negative correlations between defaults of: (a) debtors belonging to the same industry but having different credit ratings, and (b) obligors from the same industry and the same credit class. In our case, the coefficient of correlation between a default of a debtor from credit class i and a debtor from credit class I, both belonging to industry s, reads

$$dc(i, I, s, s),$$

(see 13). This value may be negative only when $c_{i,I} < 0$. Thus, allowing in our scheme for negative $c_{i,I}$, we may obtain negative correlations between defaults of debtors belonging to the same industry but having different credit ratings, i and I in this particular case. But, since the correlation between defaults of any creditors from ratings i and I contain $c_{i,I}$, all these correlations will be negative as well, regardless of their industry sectors. Consequently, we need a deeper modification of the model than just allowing for negative $c_{i,I}$.

We may introduce one more source of dependence in the model – sector specific correlations. For industry sector s, given a matrix of correlations $c^{(k)}_{I,J}$, we introduce random vectors $\eta^{(s)}_1, \ldots, \eta^{(s)}_M$. They are constructed in the same way as $\eta_1, \ldots, \eta_M$, but with $(c^{(s)}_{i,I})$ instead of $(c_{i,I})$. Consider

$$\zeta_n = \begin{cases} 1 & \text{with probability } \ r_{m(n),s(n)} \\ 0 & \text{with probability } \ 1 - r_{m(n),s(n)}, \end{cases}$$

that also do not depend on all random factors considered so far. Set

$$\begin{aligned} X^*_n &= X_n\zeta_n + \eta^{(s(n)}_{m(n)}(1 - \zeta_n) \\ &= [\delta_n\xi_n + (1 - \delta_n)\eta_{m(n)}]\zeta_n + \eta^{(s(n)}_{m(n)}(1 - \zeta_n) \end{aligned}$$

(compare (2)) for the random vector describing the evolution of n-th firm. With this construction we may have positive correlations between defaults of different credit classes in some industries and negative in the others, by choosing the corresponding $c^{(s)}_{i,I}$ negative. But we still will not be able to have negative correlations between defaults of two debtors belonging to the same industry sector and the same credit class.

Notice however that assuming an exchangeable effect for all debtors of the same industry sector and the same credit class, we get a natural bound on the number of firms in that particular rating/industry group, if we assume negative correlations. Indeed, suppose that $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_N)$ is a vector of Bernoulli variables representing the default indicators, such that

- $\mathbb{E}\gamma_i = p$ (say),
- $\mathrm{Corr}[\gamma_n, \gamma_N] = c, \quad n \neq N$,
- the distribution of $\boldsymbol{\gamma}$ is invariant w.r.t any permutation of the indices.

Then, since $\mathrm{Var}(\sum_{i=1}^N \gamma_i) \geq 0$, one gets that $c \geq -1/(N-1)$. To put it differently, larger negative correlations may only occur for small number of firms in the group under the exchangeability hypothesis.

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$s \setminus i$	3	4	5	6	7
1	0.06	0.19	0.66	4.02	7.50
2	0.03	0.12	0.42	2.57	4.80
3	0.02	0.07	0.24	1.45	2.70
4	0.01	0.03	0.11	0.64	1.20

Table 1.

Model name	Matrix Q	Matrix C	Mean	95% quantile
Model 00	$Q0$	$C0$	254	275
Model 01	$Q1$	$C0$	254	277
Model 10	$Q0$	$C1$	254	440
Model 11	$Q1$	$C1$	254	469
Model 12	$Q1$	$C2$	254	486
Model 21	$Q2$	$C1$	254	621
Model 22	$Q2$	$C2$	254	652

Table 2.

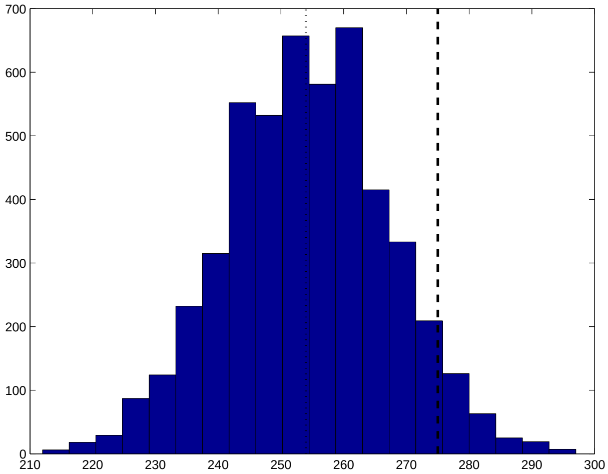

Figure 1. Model 00.

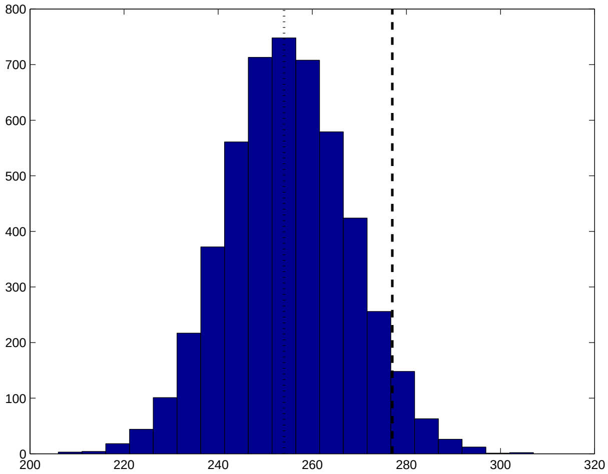

Figure 2. Model 01.

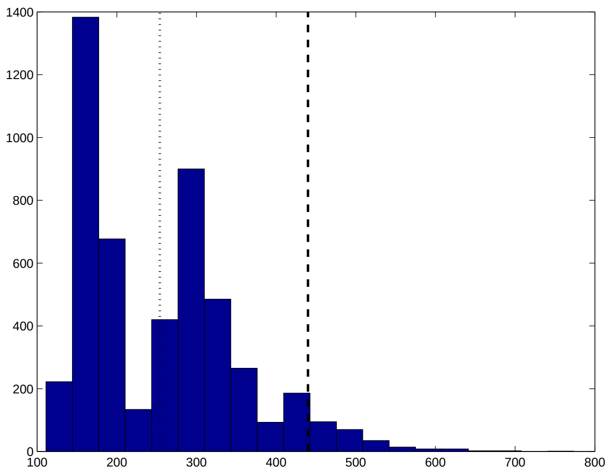

Figure 3. Model 10.

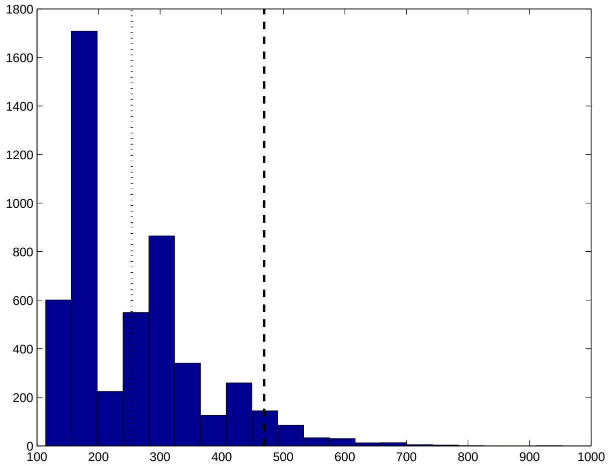

Figure 4. Model 11.

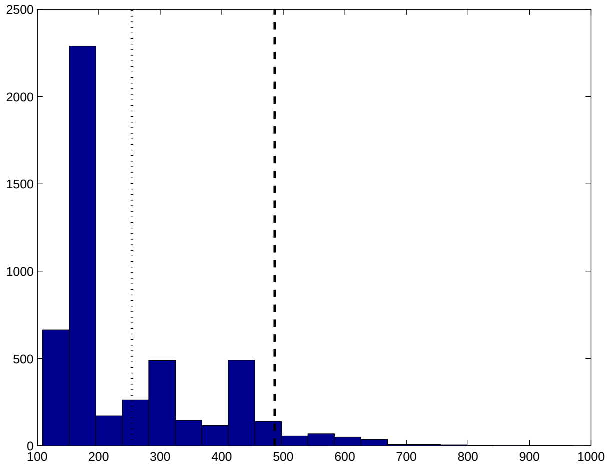

Figure 5. Model 12.

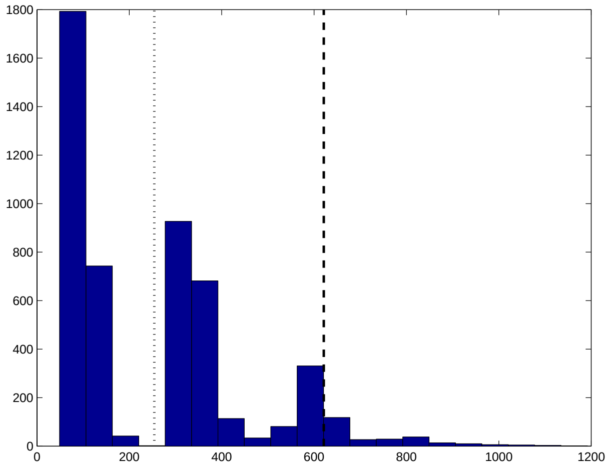

Figure 6. Model 21.

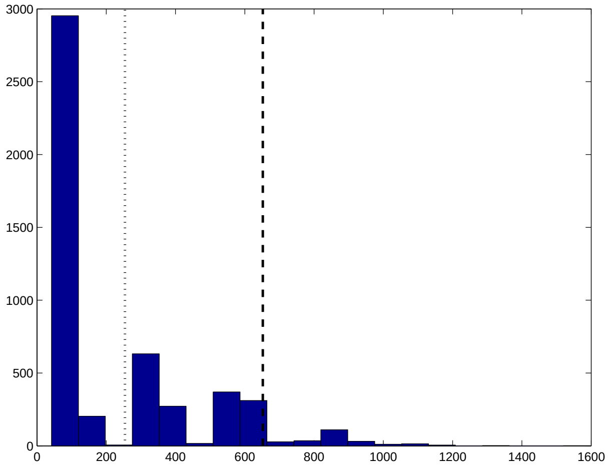

Figure 7. Model 22.