Risk assessment for credit portfolios: a coupled Markov chain model

Yuri M. Kaniovski, Georg Ch. Pflug

Finance and Decisions ’05
29. April 2005
Credit portfolios

Credit risk models

Coupled Markov Chains

Value-at-risk
A portfolio of $m$ collateralized debt obligations:

<table>
<thead>
<tr>
<th>debtor</th>
<th>rating</th>
<th>sector</th>
<th>volume</th>
<th>duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1</td>
<td>AA</td>
<td>S1</td>
<td>50</td>
<td>5 yrs</td>
</tr>
<tr>
<td>D2</td>
<td>B</td>
<td>S1</td>
<td>20</td>
<td>3 yrs</td>
</tr>
<tr>
<td>D3</td>
<td>AAA</td>
<td>S2</td>
<td>30</td>
<td>4 yrs</td>
</tr>
<tr>
<td>D4</td>
<td>A</td>
<td>S2</td>
<td>10</td>
<td>5 yrs</td>
</tr>
<tr>
<td>D5</td>
<td>BB</td>
<td>S2</td>
<td>15</td>
<td>4 yrs</td>
</tr>
<tr>
<td>D6</td>
<td>A</td>
<td>S3</td>
<td>10</td>
<td>3 yrs</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
The state of the portfolio

<table>
<thead>
<tr>
<th>ratings</th>
<th>sectors</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S1</td>
<td>S2</td>
<td>S3</td>
<td>S4</td>
<td>S5</td>
</tr>
<tr>
<td>AAA</td>
<td>12</td>
<td>22</td>
<td>24</td>
<td>13</td>
<td>30</td>
</tr>
<tr>
<td>AA</td>
<td>26</td>
<td>13</td>
<td>11</td>
<td>9</td>
<td>14</td>
</tr>
<tr>
<td>A</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>BBB</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>BB</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>CCC</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>total</td>
<td>56</td>
<td>57</td>
<td>54</td>
<td>46</td>
<td>65</td>
</tr>
</tbody>
</table>
The state of the portfolio after one year

<table>
<thead>
<tr>
<th>ratings</th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
<th>S5</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>10</td>
<td>20</td>
<td>25</td>
<td>10</td>
<td>31</td>
</tr>
<tr>
<td>AA</td>
<td>25</td>
<td>13</td>
<td>9</td>
<td>9</td>
<td>15</td>
</tr>
<tr>
<td>A</td>
<td>10</td>
<td>8</td>
<td>7</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>BBB</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>BB</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>CCC</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>total</td>
<td>56</td>
<td>57</td>
<td>54</td>
<td>46</td>
<td>65</td>
</tr>
</tbody>
</table>
Univariate credit risk models

Let $D_1, \ldots, D_N$ be the default indicator within the observation period, i.e.

$$D_n = \begin{cases} 0 & \text{if debtor } n \text{ is alive at maturity} \\ 1 & \text{otherwise} \end{cases}$$

We are interested in the distribution of $\sum_{n=1}^{N} D_n$ and especially in its quantiles. The following models have been developed

- Lifetime Models
- Firm Value Models
- Mixed Binomial Models
- Queuing Models
- Markov Chain Models
Lifetime Models

Li (2001). Every debtor \( n \) has a ”lifetime” \( \tau_n \). The distribution function \( F_{i(n)} \) of \( \tau_n \) depends only on its rating class \( i(n) \):

\[
P\{ \tau_n \leq t \} = F_{i(n)}(t).
\]

The contract defaults, if the lifetime of the debtor is smaller than the maturity of the contract \( T \), i.e.

\[
D_n = 1_{\{\tau_n \leq T\}}.
\]

For the whole portfolio \( (\sum_{n=1}^{N} D_n) \), one may consider

- Optimistic extreme: All lifetimes and hence all default indicators are independent
- Pessimistic extreme: All lifetimes within the same rating class are identical, but independent between rating classes
- Copula approach: The lifetimes within an between rating classes are coupled through given copulas, i.e.

\[
D_n = 1_{\{U_n \leq F_{i(n)}^{-1}(t)\}},
\]

where \( U_n \) are normally coupled uniform random variables.
Firm value models

Bluhm (2001), Vasicek (1987), based on Merton (1974). There is a firm value $V_n$ such that $D_n = 1\{V_n \leq K_n\}$, with $X_n$ being normally distributed. For a portfolio, one assumes a common factor $Y$ and individual factors $Z_n$ (all i.i.d. standard normal distributed) such that $V_n = \rho_n Y + \sqrt{1 - \rho_n^2} Z_n$. If all $\rho_n$ are identical, then one may calculate

$$P\{\sum_{n=1}^{N} D_n = k\} = \binom{N}{K} \int_{-\infty}^{\infty} \Phi\left(\frac{K - \rho u}{\sqrt{1 - \rho^2}}\right)^k (1 - \Phi\left(\frac{K - \rho u}{\sqrt{1 - \rho^2}}\right))^{m-k} \phi(u) \, du$$

with $\Phi, \phi$ being standard normal distribution and density.

Extensions of this model consider a time series $V_n(t)$ and define the lifetime as $\tau_n = \inf\{t : V_n(t) \leq K_n\}$. 
Mixed binomial models

Frey and McNeil (2003). The default probability of an individual $p_i$ depends on a random variable $Y$ and is therefore random itself. While the default events are independent given this $Y$, the unconditional events show a dependency.

$$P\left\{ \sum_{n=1}^{N} D_n = k \right\}$$

$$= \binom{N}{K} \int_{-\infty}^{\infty} p(y)^k (1 - p(y))^{N-k} f(y) \, dy$$

Variants are the *gamma-poisson* and the *beta-binomial* model.
M. Davis (2004). Leaky bucket analysis: The movement of firms between non-default rating classes is modeled like a Markov queuing process. It resembles water flowing between several buckets. However, from time to time, a firm disappears and this resembles that the fact that the total water is not constant, but may decrease due to leakage.

In mathematical terms, the transitions of each obligor follow a Poisson process, but the transition probabilities depend on an external process $X_t$ describing the macroeconomic situation.
Coupled Markov Processes

We start with the observation that the rating $X_n(t)$ of every debtor $n$ follows a Markov process with the same transition. However these processes are coupled through the fact that there are

- Macroeconomic effects, which affect the whole economy
- Sectorwise effects, which affect only some sectors of the economy (Kahle and Walking, 1996)

The Coupled Markov chain approach incorporates macroeconomic as well as sectoral effects while maintaining the marginal rating transition probabilities. Sectoral effects are important to see diversification effects.
Let $X_n(t)$ be the rating process of the $n$-th debtor. We have that $D_n = 1_{\{X_n(T) = \text{default}\}}$. The marginal distribution of $X_n(t)$ follows a Markov Chain with given transition matrix $P$ (see e.g. Credit Metrics (1997) p. 69)

\[
\begin{pmatrix}
0.9081 & 0.0833 & 0.0068 & 0.0060 & 0.0120 & 0.0000 & 0.0000 & 0.0000 \\
0.0070 & 0.9065 & 0.0790 & 0.0064 & 0.0006 & 0.0014 & 0.0002 & 0.0000 \\
0.0009 & 0.0227 & 0.9105 & 0.052 & 0.0074 & 0.0026 & 0.0001 & 0.0006 \\
0.0002 & 0.0033 & 0.0595 & 0.8693 & 0.0530 & 0.0117 & 0.0012 & 0.0018 \\
0.0003 & 0.0014 & 0.0067 & 0.0773 & 0.8053 & 0.0884 & 0.0100 & 0.0106 \\
0.0000 & 0.0011 & 0.0024 & 0.0043 & 0.0648 & 0.8346 & 0.0407 & 0.0520 \\
0.0022 & 0.0000 & 0.0022 & 0.0130 & 0.0238 & 0.1124 & 0.6486 & 0.1979
\end{pmatrix}
\]

The joint distribution of all $X_n(t)$ must be modeled to accommodate correlation within and between sectors.
Coupled Markov Chains I

\[ X_1(0) \rightarrow X_1(1) \rightarrow \ldots \rightarrow X_1(T) \]
\[ X_2(0) \rightarrow X_2(1) \rightarrow \ldots \rightarrow X_2(T) \]
\[ \vdots \rightarrow \vdots \rightarrow \vdots \rightarrow \vdots \]
\[ X_N(0) \rightarrow X_N(1) \rightarrow \ldots \rightarrow X_N(T) \]
Coupled Markov Chains II

Moves for members of the same rating class and sector:
To compromise between independent and dependent moves, we assume that a member of a rating class $i$ and sector $l$ makes an independent move with probability $q_{i,l}$ and follows a common move with probability $1 - q_{i,l}$.

Let $X_{i}^{*}$ be a rating process which follows the transition $P$ and which exists for every rating class $i$ and let $\tilde{X}_{n}$ an individual independent rating process for debtor $n$. Then the final move is according to

$$X_{n} = \begin{cases} \tilde{X}_{n} & \text{with probability } q_{i,l} \\ X_{n}^{*} & \text{with probability } 1 - q_{i,l} \end{cases}$$
## Concordant and discordant moves

<table>
<thead>
<tr>
<th></th>
<th>Process 1</th>
<th>Process 2</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>p_BA \cdot p_CA</td>
<td>p_BA \cdot p_CB</td>
<td>p_BA \cdot p_CC</td>
<td>p_BA \cdot p_CD</td>
<td>p_BA</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>p_BB \cdot p_CA</td>
<td>p_BB \cdot p_CB</td>
<td>p_BB \cdot p_CC</td>
<td>p_BB \cdot p_CD</td>
<td>p_BB</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>p_BC \cdot p_CA</td>
<td>p_BC \cdot p_CB</td>
<td>p_BC \cdot p_CC</td>
<td>p_BC \cdot p_CD</td>
<td>p_BC</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>p_BD \cdot p_CD</td>
<td>p_BD \cdot p_CB</td>
<td>p_BD \cdot p_CC</td>
<td>p_BD \cdot p_CD</td>
<td>p_BD</td>
<td></td>
</tr>
<tr>
<td></td>
<td>p_CA</td>
<td>p_CB</td>
<td>p_CC</td>
<td>p_CD</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Let $\chi_n$ be an indicator variable indicating the events:

$$
\chi_n = \begin{cases} 
1 & \text{if debtor } n \text{ does not deteriorate in rating} \\
0 & \text{if a downmove in rating occurs for debtor } n 
\end{cases}
$$

The probability $p_i^+$ of $\chi_n = 1$ depends only on the rating $i$.

$p_{AAA}^+ = 0.9081$, $p_{AA}^+ = 0.9135$, $p_A^+ = 0.9341$, $p_{BBB}^+ = 0.9323$,
$p_{BB}^+ = 0.8910$, $p_B^+ = 0.9072$ and $p_{CCC}^+ = 0.8022$.

We dissect the move in rating in two steps

- The determination of $\chi_n$, i.e. whether a downmove occurs or not.
- The conditional distribution of the next rating, given $\chi_n$.

We couple the Bernoulli indicator variables $\chi$ over rating classes and sectors, but leave the conditional distributions of the move, given $\chi$ independent for every debtor and identical to the ones given by the transition matrix.
Coupling Bernoulli random variables

If $\chi_i$ and $\chi_j$ are two Bernoulli random variables, they can be coupled in order to lead to a correlation $\rho_{i,j}$ as follows.

<table>
<thead>
<tr>
<th>class i</th>
<th>$\chi_j = 1$</th>
<th>class j</th>
<th>$\chi_j = 0$</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_i = 1$</td>
<td>$p_i^+ p_j^+ + \rho_{i,j} s_{i,j}$</td>
<td>$p_i^+ - p_i^+ p_j^+ + \rho_{i,j} s_{i,j}$</td>
<td>$p_i^+$</td>
<td></td>
</tr>
<tr>
<td>$\chi_i = 0$</td>
<td>$(1 - p_i^+) p_j^+ + \rho_{i,j} s_{i,j}$</td>
<td>$1 - (1 - p_i^+) p_j^+ + \rho_{i,j} s_{i,j}$</td>
<td>$1 - p_i^+$</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td>$p_j^+$</td>
<td>$1 - p_j^+$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where $s_{i,j} = \sqrt{p_i^+(1 - p_i^+) p_j^+(1 - p_j^+)}$. The correlation matrix

$$C = \begin{pmatrix}
1 & \rho_{1,2} & \rho_{1,3} & \ldots & \rho_{1,m} \\
\rho_{2,1} & 1 & \rho_{2,3} & \ldots & \rho_{2,m} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\rho_{m-1,1} & \rho_{m-1,2} & \rho_{m-1,3} & \ldots & \rho_{m-1,m} \\
\rho_{m,1} & \rho_{m,2} & \rho_{m,3} & \ldots & 1
\end{pmatrix}$$

must be non-negative definite.
How to evaluate the quantiles of the loss distribution?

We want to evaluate the quantiles (Value-at-risk) of the distribution of the number of losses. To do so we may

▶ use the generating function of number of losses

▶ determine the total transition matrix: Notice that the number of states is enormous
50 debtors in 8 rating classes gives 264 million states
Therefore we use the lumping technique for huge Markov chains: lumping of states with small probability bringing down the above number to 6150 states

▶ Simulation
Simulations

We have 8 rating classes and introduce 4 industry sectors. The sector communalities $q_{i,l}$ are collected in a matrix $Q$.

\[
(Q0)_{i,j} = 0 \quad (Q1)_{i,j} = \begin{cases} 
0.5 \text{ if } j = 1, \\
0.6 \text{ if } j = 2, \\
0.7 \text{ if } j = 3, \\
0.8 \text{ if } j = 4,
\end{cases}
\]

The correlation matrix is $C$.

\[
(C0)_{i,j} = \begin{cases} 
1 \text{ if } i = j, \\
0 \text{ if } i \neq j,
\end{cases} \quad (C1)_{i,j} = \begin{cases} 
1 \text{ if } i = j, \\
0.3 \text{ if } i \neq j,
\end{cases} \quad (C2)_{i,j} = \begin{cases} 
1 \text{ if } i = j, \\
0.8 \text{ if } i \neq j.
\end{cases}
\]
The Model 00 reflects the total independence of every individual rating process. We start with 2800 debtors and consider a three years period. The total number of defaults has expectation 245, irrespective of the dependency of the 2800 stochastic rating processes.

<table>
<thead>
<tr>
<th>Model name</th>
<th>Matrix $Q$</th>
<th>Matrix $C$</th>
<th>Mean</th>
<th>95% quantile</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 00</td>
<td>$Q_0$</td>
<td>$C_0$</td>
<td>254</td>
<td>275</td>
</tr>
<tr>
<td>Model 01</td>
<td>$Q_1$</td>
<td>$C_0$</td>
<td>254</td>
<td>277</td>
</tr>
<tr>
<td>Model 10</td>
<td>$Q_0$</td>
<td>$C_1$</td>
<td>254</td>
<td>440</td>
</tr>
<tr>
<td>Model 11</td>
<td>$Q_1$</td>
<td>$C_1$</td>
<td>254</td>
<td>469</td>
</tr>
<tr>
<td>Model 12</td>
<td>$Q_1$</td>
<td>$C_2$</td>
<td>254</td>
<td>486</td>
</tr>
<tr>
<td>Model 21</td>
<td>$Q_2$</td>
<td>$C_1$</td>
<td>254</td>
<td>621</td>
</tr>
<tr>
<td>Model 22</td>
<td>$Q_2$</td>
<td>$C_2$</td>
<td>254</td>
<td>652</td>
</tr>
</tbody>
</table>
Credit portfolios
Credit risk models
Coupled Markov Chains
Value-at-risk

Risk assessment for credit portfolios
Credit portfolios
Credit risk models
Coupled Markov Chains
Value-at-risk

Model 11

Model 12

Yuri M. Kaniovski, Georg Ch. Pflug
Risk assessment for credit portfolios