STOCHASTIC MODELS IN FINANCE

Begleitmaterial zur Vorlesung von

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1 A Glossary of Important Terms

- The term *Asset* (Aktiva) describes the collection of all values a company holds, such as buildings, machinery, but also securities and other financial contracts. Assets appear on the active side of the company’s balance sheet. Besides the cited tangibles, sometimes also intangibles, like the company’s reputation are counted as assets.

- A *security* (Wertpapier) is a certificate proving the owner has invested in some organization’s equity or debt. Securities are standardized contracts, allowing the holder to sell its rights without informing the other party in the marketplace (A shareholder may sell the share without informing the company, a bondholder may sell the bond without informing the debtor). The opposite of securities are OTC-contracts (over the counter contracts), which are nonstandardized and where the two parties are fixed for the duration of the contract.

- A *bond* (Anleihe) is a fixed income security, which pays to its holder regularly the coupon/interest and at maturity time the face value. We distinguish between (1) government bonds (the government is the issuer) and (2) corporate bonds (a company is the issuer).

  Bonds are subject to interest risk (the market interest rate may change) and the default risk. If the bond issuer is not able to meet the payment obligations, default occurs and some legal actions are taken. Typically, on default only a fraction (the recovery rate) of the face value is paid.

  Bonds are priced at the bond market. The pricing depends on modelling the market interest rate process and the default risk. The default risk is estimated by rating agencies (e.g. Moody’s and Standard&Poor) and expressed in rating classes (AAA,AA,A,BBB,BB,B,CCC,D). D means default.

- A *stock* (Aktie) is a security, which gives the holder ownership rights of the company. Stockholders select by voting managers and control the firm’s activity. Stocks are priced through the bid-ask mechanism of a stock exchange and are subject to the value risk. Also called *shares*.

- *equity* (Stammkapital) is the shareholder’s stake in the company.

- *hybrid instruments*, such as convertibles, mandatory convertibles and warrants are combinations of bonds and stock securities.

- A *convertible bond* (Wandelanleihe) is a corporate bond, which gives the owner the option to exchange it at a predetermined date to a predetermined number of stocks\(^1\).

\(^{1}\)“predetermined” may mean that the number of stocks or the price are numerically fixed at the issue
• A warrant is a corporate bond, which gives the owner the option to buy at a predetermined date a predetermined number of stocks at a predetermined price.

• Derivatives are contracts, which perform in close relation to another basic contract, the underlying. Examples are: Options (underlying is the stock to be bought or sold), futures (underlying is the today’s commodity price), swaps (underlying is the interest rate or exchange rate process), credit derivatives (underlying is the basic credit portfolio).

• An option gives its holder the right to buy or sell stocks at a predetermined price. A European option may be exercised only at the predetermined date. An American option may be exercised at any time before. A (plain vanilla) call option gives the right to buy a given number of stocks at a predetermined price (the strike price). A (plain vanilla) put option gives the right to sell a given number of stocks at a predetermined price. Other than plain vanilla options are called exotic options (e.g. a barrier option, which is activated if the price of the underlying crosses some predetermined level. A barrier option comes in the forms of a trigger option: the right starts, or a knockout option: the right ends). A Margrabe option also called exchange option is the right to exchange one type of stock by another type of stock.

• A forward is an agreement to buy or sell a commodity or a financial asset at a specified future date for a fixed price. It is a completed contract and the commodity or financial asset will be delivered, unlike an option, which may be exercised or not. Forwards are OTC contracts.

• A future is the undertaking to buy or sell a standard quantity of a financial asset or commodity at a future date and at a fixed price. Futures resemble forwards, but are standardized contracts (i.e. every future contract has standardized terms that dictate the size, the unit of price quotation, the delivery date and contract months) and must be traded on a recognized exchange.

• Swaps are exchanges of the cash flows between two counterparties designed to offset interest rates or currency risks and to match their assets to their liabilities. The parties to a swap do not exchange principal, or the underlying fixed amount of debt, but just cash-flow, or the interest payments.

• Liquidity of the market is the property that standard transactions are possible (at least at the trading times)
• **completeness** of the option market means that contracts with all possible strike prices and maturities are available and traded.

## 2 The time value of money: deterministic interest rates

The basic instrument to deal with is a financial contract. A *financial contract*, agreed with to parties A and B obliges these parties to transfer amounts \( c_t \) of money (the cash-flows of the contract) at certain times \( t \). At the time of contracting, neither the exact amounts or the exact times must be determined, however the exact formula how to get to these data should be fixed (e.g. a reference to the stock market or to some indices such as the "Sekundärmarkttrendite").

Distinguish financial contracts from *commodity contracts*, where one part pays money while the other delivers some commodity at agreed times, quantities and costs, again either determined in advance or based on some formula involving unknowns at the time of contracting. Examples of commodity contracts are energy contracts such as energy futures or swing options.

Contracts in this sense are: bank accounts, credits and loans, shares (stocks), bonds (governmental fixed income securities) corporate bonds (private fixed income securities), derivatives (futures, forward contracts, options, swaps, etc.)

![A cash-flow structure](image)

\[
A \text{ cash-flow structure } c = (c_1, \ldots, c_T)
\]

Money today is better than the same amount tomorrow, or in some other future time
(Why?). If the market determines that $K$ unit today equals $K(1 + r)$ units in one year, we call $r$ the (one-year) interest rate.

For bank accounts, if a capital is hold for more than one year, typically a capitalization of the accrued interests occur at the beginning of each calendar year. Thus $K$ units on the first of January 2000 result is a capital of $K(1 + r)^t$ units on the first of January 2000 + $t$.

Notice the Bernoulli inequality:

$$ (1 + r)^t > 1 + rt $$

for $r > 0, t > 0$.

If the capital $K(t)$ varies over the year, the accrued interest in the year is

$$ r \cdot \int_0^1 K(t) \, dt $$

(unit is one year) or in discrete time

$$ r \cdot \frac{1}{365} \sum_{d=1}^{365} K(d) $$

(unit is one day) i.e. the interest is calculated on the basis of the average capital during the year. Thus, a capital of one unit in January and zero in the other month produces the same interest as a capital of one unit in December and zero in the other month, although in the first case the bank had the capital much earlier and should pay more interest (but does not).

For some contracts, capitalization occurs several times per year (quarterly, monthly). If capitalization takes place $n$ times per year, after one year the capital $K$ is worth $K(1 + r/n)^n$, which, for $n \to \infty$ tends to $\exp(r)$.

**Example.** $r = 0.05$

<p>| | |</p>
<table>
<thead>
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<td>$1 + r$</td>
<td>1.05</td>
</tr>
<tr>
<td>$(1 + r/2)^2$</td>
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</tr>
<tr>
<td>$(1 + r/10)^{10}$</td>
<td>1.05114</td>
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<tr>
<td>$(1 + r/360)^{360}$</td>
<td>1.051267</td>
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<td>$\exp(r)$</td>
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### 2.1 Pricing of interest rate dependent contracts

For $t = 1, \ldots, T$, let $r_t$ be the yearly interest rates for not default prone zero coupon bonds maturing at time $t$. The stock exchange fixes by its bid-ask mechanisms $\pi_t$, the today’s price of a default free zero-coupon bond with maturity $t$ and principal value 1.
From this, we calculate the interest rate for maturity $t$: $r_{0,t} := \pi_t^{-1/t} - 1$, i.e.

$$\pi_t = (1 + r_{0,t})^{-t}.$$ 

The collection of interest rates is the yield curve: $(r_{0,1}, \ldots, r_{0,T})$.

We begin with looking at financial contracts, which pay fixed, predetermined sums at fixed, predetermined times, for simplicity, the times $1, 2, \ldots$. Suppose that a contract consists in exchanging the amount $c_t$ at time $t$, for $t = 1, \ldots, T$. The values $c_t$ may be positive (the contract holder gets money) or negative (the contract holder pays money). The vector $c = (c_1, \ldots, c_T)$ is called the cash-flow vector produced by the contract. The today’s fair price of this contract is denoted by $\pi(c_1, \ldots, c_T)$. We will require that the pricing operator $\pi$ is a linear mapping from $\mathbb{R}^T$ to $\mathbb{R}$, however bearing in mind that later on this requirement will be weakened to certain nonlinear pricing rules.

**Pricing rule A:** The price equals the discounted cash-flow

Introduce the notation $\pi_t := \pi(0, \ldots, 0, 1, 0, \ldots, 0) = (1 + r_{0,t})^{-t}$ for a contract paying 1 at time $t$ and nothing at other times. Because of the linearity of the pricing operator

$$\pi(c_1, \ldots, c_T) = \sum_{t=1}^{T} c_t (1 + r_{0,t})^{-t} = \sum_{t=1}^{T} c_t \pi_t.$$ 

### 2.2 Sensitivity, elasticity and duration

Recall the notions of sensitivity and elasticity of functions: The sensitivity of a function $f$ is its derivative

$$f'(x) = \frac{\partial}{\partial x} f(x).$$
The elasticity is
\[ \frac{\partial \log f(x)}{\partial \log x} = \frac{x \cdot f'(x)}{f(x)}. \]

If the yield curve is flat, i.e. \( r_{0,t} \equiv r \), then the price of the contract \((c_1, \ldots, c_T)\) is a function of only one parameter \( r \). Introduce

- **the (interest rate) sensitivity**

\[
S(c_1, \ldots, c_T; r) = -\frac{\partial \pi(c_1, \ldots, c_T)}{\partial (1+r)} = -\frac{\partial \pi(c_1, \ldots, c_T)}{\partial r} = \sum_{t=1}^{T} c_t \frac{\partial \pi_t}{\partial r} = \sum_{t=1}^{T} c_t (1+r)^{-t-1}
\]

- **(interest rate) elasticity**

\[
D(c_1, \ldots, c_T; r) = -\frac{\partial \log \pi(c_1, \ldots, c_T)}{\partial \log(1+r)} = -\frac{\partial}{\partial \log(1+r)} \log\left[ \sum_{t=1}^{T} c_t (1+r)^{-t} \right]
= \frac{\sum_{t=1}^{T} c_t (1+r)^{-t}}{\sum_{t=1}^{T} c_t (1+r)^{-t}} = \frac{S(c_1, \ldots, c_T; r) \cdot (1+r)}{\pi(c_1, \ldots, c_T)}.
\]

Since
\[
D(0, \ldots, 0, 1, 0, \ldots, 0; r) = -\frac{\partial \log \pi_t}{\partial \log(1+r)} = t.
\]

this elasticity is also called the "duration", or more precisely, the M\(\text{a}\)Cauley duration. Notice that, in contrast to the sensitivity, the duration is not linear in \( c_t \).

Suppose that our portfolio consists of assets and liabilities. We may view assets and liabilities as just types of general financial contracts. Both produce future cash-flows, however with different signs. Typically, an asset will produce nonnegative cashflows in the future, whereas a liability will require nonpositive cashflows. An asset has a positive and a liability a negative today’s price.

Let the assets be numbered \( 1, \ldots, M_A \) and the liabilities be numbered \( M_A + 1, \ldots, M = M_A + M_L \). Each of the \( M \) contracts produces cash flows \( c_t^{(i)}, i = 1, \ldots, M \). Suppose that we hold \( x_i, i = 1, M \) pieces of each contract. Then the cash flows produced at time \( t \) are \( \sum_{i=1}^{M} c_t^{(i)} \). The price of the total AL-portfolio is
\[
\pi = \sum_{i=1}^{M} x_i \pi(c_1^{(i)}, \ldots, c_T^{(i)}),
\]
the interest sensitivity of the portfolio is

\[ S = \sum_{i=1}^{M} x_i \sum_{t=1}^{T} c_{t}^{(i)} t (1 + r)^{-t-1}, \]

the duration is

\[ D = \frac{\sum_{i=1}^{M} x_i \sum_{t=1}^{T} c_{t}^{(i)} t (1 + r)^{-t}}{\sum_{i=1}^{M} x_i \sum_{t=1}^{T} c_{t}^{(i)} (1 + r)^{-t}} = S \cdot (1 + r)/\pi. \]

If a portfolio of assets and liabilities is composed in such a manner that the interest sensitivity is zero, we say that the portfolio is immunized w.r.t interest rate changes. Notice that \( S \) is zero if and only if \( D \) is zero. Thus the duration of an immunized portfolio is zero. This is equivalent to say that a small change in interest rates will not influence the value of the whole portfolio.

**Exercise.** Suppose that we may buy two types of assets with cashflows \( c_{t}^{(1)} = 5 \), \( c_{t}^{(2)} = 10 \) and \( c_{t}^{(2)} = 6 \), \( c_{t}^{(2)} = 6 \). On the other hand, our liability has cashflows \( c_{t}^{(3)} = -29 \), \( c_{t}^{(3)} = -41 \). The interest rate is 5%. Find an immunized portfolio.

### 2.3 Forward interest rates and instant interest rates

Suppose that the yield curve \( r_{0,t} \) is given. For \( s < t \), the **forward interest rates** \( r_{s,t} \) are defined through the relation

\[
(1 + r_{s,t})^s \cdot (1 + r_{t,s})^{t-s} = (1 + r_{0,t})^t.
\]

We call \( \rho_t = \lim_{\delta \to 0} r_{t,t+\delta} \) the **instant interest rate** (overnight rate, spotrate), if it exists.

Let \( T - t = n\delta \). By

\[
(1 + r_{t,T})^{T-t} = (1 + r_{t,t+\delta})^{\delta} \cdot (1 + r_{t+\delta,t+2\delta})^{\delta} \cdots (1 + r_{t+(n-1)\delta,t+n\delta})^{\delta}
\]

we get by taking logarithms that

\[
\log(1 + r_{t,T}) = \frac{1}{T-t} \delta \sum_{i=0}^{n-1} \log(1 + r_{t+i\delta,t+(i+1)\delta}).
\]

and passing to the limit \( n \to \infty \)

\[
\log(1 + r_{t,T}) = \frac{1}{T-t} \int_t^T \log(1 + \rho_u) \, du.
\]
and
\[ 1 + r_{t,T} = \exp \left( \frac{1}{T-t} \int_t^T \log(1 + \rho_u) \, du \right) = \left[ \exp \left( \int_t^T \log(1 + \rho_u) \, du \right) \right]^{1/(T-t)}. \]

Introducing the product integral
\[ \prod_a^b f(u) \, du = \exp \left[ \int_a^b f(u) \, du \right] \]
one gets that
\[ 1 + r_{t,T} = \left[ \prod_t^T \log(1 + \rho_u) \, du \right]^{1/(T-t)}. \]

Notice that the product integral satisfies for \( a < b < c \)
\[ \prod_a^b f(u) \, du \cdot \prod_b^c f(u) \, du = \prod_a^c f(u) \, du. \]

### 3 Stochastic interest rate models and other price models

Let \( \dot{R}_{s,t} \) be a stochastic process to model the forward interest rates \( r_{s,t} \) that maintains the following minimum consistency criteria
\[ (1 + \dot{R}_{s,t})^{-s} (1 + \dot{R}_{t,u})^{u-t} = (1 + \dot{R}_{s,u})^{u-s} \quad (1) \]
The models can be discrete or continuous in time and/or space. The usual way to model random forward interest rates is to model the logarithmic spotrates \( X(u) = \log(1 + \rho_u) \).

The process \( R_{s,t} \) depends on the process \( X(t) \) by
\[ R_{t,T} = \exp \left[ \int_t^T X(u) \, du \right]^{1/(T-t)} - 1. \]

The process \( B_{t,T} \), which serves as stochastic discount,
\[ B_{t,T} = (1 + R_{t,T})^{-(T-t)} = \exp \left[ - \int_t^T X(u) \, du \right] \]
is called a stochastic deflator.

Denote by \( \pi_T(t) \) the price of a zero coupon bond with face value 1 at time \( t \) with maturity \( T \). There are several pricing rules, i.e. rules how to get a price from the stochastic spotrate process.
• The local expectation rule
\[ \pi_T(t) = \mathbb{E}[\exp(-\int_t^T X(u) \, du)] = \mathbb{E}[B_{t,T}] \]

• The unbiased expectation rule
\[ \pi_T(t) = \exp[-\int_t^T \mathbb{E}(X(u)) \, du] \]

• The return-to-maturity rule
\[ \pi_T(t) = \left[ \mathbb{E}[\exp(\int_t^T X(u) \, du)] \right]^{-1} \]

• The yield-to-maturity rule
\[ \pi_T(t) = \left[ \mathbb{E}[\exp(-\frac{1}{T-t} \int_t^T X(u) \, du)] \right]^{T-t} \]

Typically, the prices \( \pi_T(0) \) are observed and these prices are used for calibrating the process \( X(t) \).

### 3.1 Discrete time processes

In discrete time, the spot rate process is \( (X_s); s = 0, 1, \ldots \) and

\[ (1 + R_{s,t})^{t-s} = (1 + X_s) \cdot (1 + X_{s+1}) \cdot \cdots (1 + X_{t-1}). \]

We typically consider processes, which are driven by a mean zero driving process \( Z_t \). First order processes are of the form

\[ X_{t+1} = F(X_t, Z_t). \]

#### 3.1.1 The additive model (Random walk model)

The simplest model is the Bernoulli random walk model. Here, the driving process is \( Z_t = 2Y_t - 1 \), where \( Y_t \sim B(1, 1/2) \). The process is

\[ X_{t+1} = X_t + u \cdot (2Y_t - 1) \]

with some starting value \( X_0 = x_0 \). It is easy to see that \( \mathbb{E}(X_t) = x_0 \) and \( \text{Var}(X_t) = u^2 \cdot t \). The process is not stationary.
3.1.2 The multiplicative model (Black-Derman-Toy or Lattice model)

The additive model has the disadvantage that the process may fall negative. The multiplicative model avoids this.

\[ X_{t+1} = X_t \cdot v^{2Y_{t-1}} \]

Notice that \( \log X_t \) follows the recursion

\[ \log X_{t+1} = \log X_t + (\log v) \cdot (2Y_t - 1). \]

3.1.3 The autoregressive process with mean reversion

\[ X_{t+1} = X_t + a(\mu - X_t) + Z_t \]

where \((Z_t)\) is a zero mean i.i.d. process.

3.2 Continuous time processes

3.2.1 The Wiener process - Brownian motion

Let \( X_t \) be a symmetric random walk, i.e.

\[ X_{t+1} = X_t + (2 \cdot Y_t - 1), \]

where \( Y_t \) is a sequence of independent Bernoulli \( B(1, 1/2) \) random variables. Let \( W_n(t) = \frac{1}{\sqrt{n}} X_{[nt]} \), where \([nt]\) is the largest integer not exceeding \( nt \). Notice that because of construction, the process \( W_n \) has the following properties

- \( \mathbb{E}(W_n(t)) = 0 \), \( \mathbb{V}ar(W_n(t)) = \frac{[nt]}{n} \)
- \( W_n \) is a martingale: \( \mathbb{E}(W_n(t)|W_n(s)) = W_n(s) \quad s < t \)
- \( W_n \) has independent increments
• \( W_n \) has stationary increments

As \( n \to \infty \), the process \( W_n \) converges to a limiting process, called the Wiener process \( W(t) \).

The Wiener process \( W(t) \) has following properties:

• \( W(t) \sim N(0, t) \), in particular \( \mathbb{E}(W(t)) = 0 \), \( \text{Var}(W(t)) = t \)
• \( \text{Cov}(W(s), W(t)) = \min(s, t) \)
• \( W \) is a martingale: \( \mathbb{E}(W(t)|W(s)) = W(s) \) \( s < t \)
• \( W \) has independent increments
• \( W \) has stationary increments

Let us prove the last two assertions. By the martingale property \( \mathbb{E}(W(t)|W(s)) = W(s) \) for \( s < t \) we have that

\[
\text{Cov}(W(t) - W(s), W(s)) = \mathbb{E}[(W(t) - W(s)) \cdot W(s)] = \mathbb{E}\{\mathbb{E}[(W(t) - W(s)) \cdot W(s)|W(s)]\}
\]
\[
= \mathbb{E}[W(s)\mathbb{E}(W(t) - W(s)|W(s))] = 0.
\]

Hence

\[
\text{Cov}(W(t), W(s)) = \text{Cov}(W(t) - W(s), W(s)) + \text{Var}(W(s)) = s
\]

and

\[
\text{Var}(W(t) - W(s)) = \text{Var}(W(t)) + \text{Var}(W(s)) - 2\text{Cov}(W(t), W(s))
\]
\[
= t + s - 2s = t - s \quad \text{(since } s < t \text{)}
\]

Assuming \( s < t < u < w \) it holds that

\[
\text{Cov}(W(v) - W(u), W(t) - W(s)) = \text{Cov}(W(v), W(t)) - \text{Cov}(W(v), W(s)) - \text{Cov}(W(u), W(t)) + \text{Cov}(W(u), W(s))
\]
\[
= t - s - t + s = 0
\]

i.e. the Wiener process has uncorrelated, hence independent increments.

The Wiener process is the basis for stochastic calculus: The notation \( dW(t) \) denotes an infinitesimal increment of \( W \). The infinitesimal increment \( dW(t) \) is a stochastic version of the the deterministic increment \( dt \). Here are some formulas for \( dt \):
\[ dt^2 = 2t \, dt \]
\[ (dt)^2 = 0 \]

However, for \( dW \) we have
\[ [dW(t)]^2 = dt \]

Proof: Let \( t_1, \ldots, t_n \) a partition of \([0, T]\).

\[
\mathbb{E}\left[ \sum_{i=1}^{n} (W(t_i) - W(t_{i-1}))^2 \right] = \sum_{i=1}^{n} \text{Var}(W(t_i) - W(t_{i-1})) = \sum_{i=1}^{n} t_i - t_{i-1} = T
\]

\[
\text{Var}\left( \sum_{i=1}^{n} (W(t_i) - W(t_{i-1})) \right) = \sum_{i=1}^{n} \text{Var}(W(t_i) - W(t_{i-1}))^2 = 2 \sum_{i=1}^{n} (t_i - t_{i-1})^2 \to 0
\]
as the partition gets finer and finer. Here we used the fact that for a normal variable \( V \sim N(0, \sigma^2) \),
\[ \mathbb{E}(V^2) = \sigma^2 \text{ and } \text{Var}(V^2) = \mathbb{E}(V^4) - \sigma^4 = 3\sigma^4 - \sigma^4 = 2\sigma^4 \].

Therefore
\[
\sum_{i=1}^{n} [W(t_i) - W(t_{i-1})]^2 \to \int_{0}^{T} (dW(t))^2
\]
\[
\sum_{i=1}^{n} [W(t_i) - W(t_{i-1})]^2 \to \int_{0}^{T} dt = T,
\]
hence \((dW(t))^2 = dt\).

### 3.3 Stochastic differential equations (SDEs)

An autonomous stochastic differential equation is defined by a drift function \( f \) and a diffusion function \( \sigma \). It reads
\[
dX(t) = f(X(t)) \, dt + \sigma(X(t)) \, dW(t) \tag{2}
\]

These processes are generated as follows. Divide the interval \([0,1]\) in \( n \) parts (equally sized), the equation (2) can be considered as a limit of the following equation for \( n \) tending to \( \infty \)

\[
X_n(t_i) = X_n(t_{i-1}) + f(X(t_{i-1}))(t_i - t_{i-1}) + \sigma(X_n(t_{i-1}))(W(t_i) - W(t_{i-1})) \sim N(0, t_i - t_{i-1})
\]
The random increment
\[ \Delta X_n(t_i) = X_n(t_i) - X_n(t_{i-1}) \]
has the following property
\[
\begin{align*}
\mathbb{E}[\Delta X_n(t_i)|X_n(t_i)] &= f(X_n(t_{i-1}))(t_i - t_{i-1}) + o(t_i - t_{i-1}) \\
\mathbb{V}ar[\Delta X_n(t_i)|X_n(t_i)] &= \sigma^2(X_n(t_{i-1}))(t_i - t_{i-1}) + o(t_i - t_{i-1}).
\end{align*}
\]

**Proposition.** The process (2) allows a stationary solution iff there are constants \( c_1 \) and \( c_2 \) such that \( g(x) := m(x)[c_1 S(x) + c_2] \) is a density, where
\[
\begin{align*}
S(x) &= \int_0^x s(u) \, du \\
s(x) &= \exp(-2 \int_0^x \frac{f(u)}{\sigma^2(u)} \, du) \\
m(x) &= \frac{1}{s(x)\sigma^2(x)}.
\end{align*}
\]
The stationary density is \( g(x) \).

### 3.3.1 The geometric brownian motion (GBM)

The geometric brownian motion is
\[
Y(t) = \exp(\sigma W(t) - t\sigma^2/2)
\]
where \( W(t) \) is a Wiener process.

Using the fact that \( \mathbb{E}(e^Z) = e^{\mu + \sigma^2/2} \) for \( Z \sim N(\mu, \sigma^2) \) we get that
\[
\begin{align*}
\mathbb{E}[Y(t)] &= \mathbb{E}[\exp(\sigma W(t) - t\sigma^2/2)] \\
&= \exp(t\sigma^2/2) \cdot \exp(-t\sigma^2/2) \\
&= 1
\end{align*}
\]
and \( \mathbb{V}ar(Y(t)) = \exp(t\sigma^2/2) - 1 \).

The geometric brownian motion is not stationary, its variance increases.

**Proposition.** The Ito formula.

Let \( dX(t) = f_1(X(t)) \, dt + \sigma_1(X(t)) \, dW(t) \) and let \( Y(t) = h(X(t), t) \). Then \( Y(t) \) satisfies the following stochastic equation
\[
dY(t) = [h_t(X(t), t) + h_x(X(t), t)f_1(X(t)) + \frac{1}{2} h_{xx}(X(t), t)\sigma_1^2(X(t))] \, dt + h_x(X(t), t)\sigma_1(X(t)) \, dW(t).
\]
Here $h_t = \frac{\partial}{\partial t} h(x,t)$, $h_x = \frac{\partial}{\partial x} h(x,t)$, $h_{xx} = \frac{\partial^2}{\partial x^2} h(x,t)$.

**Proposition.** Using Ito’s formula we may show that the GBM satisfies

$$dY(t) = \frac{dY(t)}{Y(t)} = \sigma dW(t).$$

**Proof.** Since $Y(t) = \exp(\sigma W(t) - t\sigma^2/2)$ we use Ito’s formula for $f_1 = 0$, $\sigma_1 = 1$, $h(x,t) = \exp(\sigma x - t\sigma^2/2)$, $h_1 = -h \cdot \sigma^2/2$, $h_x = \sigma \cdot h$, $h_{xx} = \sigma^2 \cdot h$, and get

$$dY(t) = \frac{dh(W(t),t)}{dh(W(t),t)} = \left(-Y(t)\sigma^2/2 + Y(t)\sigma^2/2\right) dt + \sigma Y(t) \, dW(t) = \sigma Y(t) \, dW(t)$$

Since there is no drift, the GBM is a martingale.

In contrast, the GBM with drift is

$$Y(t) = \exp(\sigma W(t) - t\sigma^2/2 + \mu t).$$

It fulfills the SDE

$$dY(t) = \mu Y(t) \, dt + Y(t)\sigma \, dW(t)$$

or - equivalently -

$$dY(t) = \mu dt + \sigma dW(t) = \frac{dY(t)}{Y(t)} = \mu \, dt + \sigma \, dW(t).$$

### 3.3.2 The Vasicek model

$$dX(t) = a(\mu - X(t)) \, dt + \sigma \, dW(t)$$

where $a$ is the mean reversion force, $\mu$ is the long term mean and $\sigma^2$ is the constant volatility. This model allows negative interest rates. This process has stationary distribution $N(\mu, \frac{\sigma^2}{2a})$.

### 3.3.3 The Cox-Ingersoll-Ross (CIR) model

$$dX(t) = a(\mu - X(t)) \, dt + \sigma \sqrt{X(t)} \, dW(t)$$

where $a$ is the mean reverting force, $\mu$ is the long term mean and $\sigma$ is the volatility parameter. Such a model disallows negative values for interest rates. This process has stationary distribution $\text{Gamma}\left(\frac{2\mu}{\sigma^2} - 1, \frac{\sigma^2}{2a}\right)$. 
3.3.4 The mean reverting GBM

\[
dX(t) = a(\mu - X(t)) \, dt + \sigma X(t) \, dW(t).
\]

This process has a stationary distribution, which is of Pareto type.

3.3.5 Pricing of zero coupon bonds according to the local expectation rule

Suppose that the spot interest rate follows the SDE

\[
dX(t) = f(X(t)) \, dt + \sigma(X(t)) \, dW(t)
\]

Let \( \pi(r, t, T) \) be the price of a zero coupon bond at time \( t \), which matures at time \( T \), given that at time \( t \) the spot rate is \( r \), i.e. \( \pi(r, t, T) = \mathbb{E}[\exp(- \int_t^T X(u) \, du) | X(t) = r] \). Then \( \pi(r, t, T) \) follows the following partial differential equation

\[
\pi(r, t, T) \cdot r = \pi_t(r, t, T) + f(r) \pi_r(r, t, T) + \frac{1}{2} \sigma^2(r) \pi_{rr}(r, t, T)
\]

with boundary condition

\[
\pi(r, T, T) = 1.
\]

Proof.

\[
\pi(r, t, T) = \mathbb{E}[\exp(- \int_t^{t+h} X(u) \, du) \cdot \exp(- \int_t^T X(u) \, du) | X(t) = r]
\]

\[
= \exp(-X(t) \cdot h + o(h)) \cdot \mathbb{E}[\exp(- \int_{t+h}^T X(u) \, du) | X(t) = r]
\]

Since \( X(\cdot) \) is Markovian, we have

\[
\pi(r, t, T)[\exp(X(t) \cdot h + o(h)) - 1] = \mathbb{E}[\pi(X(t+h), t+h, T) | X(t) = r] - \pi(r, t, T)
\]

and therefore noticing that \( X(t) = r \),

\[
\pi(r, t, T)(hr + o(h))
\]

\[
= \mathbb{E}[h \pi_t(r, t, T) + (X(t+h) - r) \pi_r(r, t, T) + \frac{1}{2}(X(t+h) - r)^2 \pi_{rr}(r, t, T) | X(t) = r]
\]

\[
= h \pi_t(r, t, T) + \pi_r(r, t, T) \mathbb{E}(X(t+h) - r) + \frac{1}{2} \pi_{rr}(r, t, T) \mathbb{E}(X(t+h) - r)^2]
\]

\[
= h \pi_t(r, t, T) + \pi_r(r, t, T)hf(r) + \frac{1}{2} \pi_{rr}(r, t, T)h \sigma^2(r)
\]
Therefore, dividing by $h$ and taking the limit w.r.t $h \to 0$, we get the final result

$$
\pi(r, t, T) \cdot r = \pi_t(r, t, T) + f(r)\pi_r(r, t, T) + \frac{1}{2}\sigma^2(r)\pi_{rr}(r, t, T).
$$

**Examples.**

1. If the spot interest rate follows the SDE

$$
dX(t) = c\, dt + b\, dW(t)
$$

then

$$
\pi(r, t, T) = \exp(-r\tau - \frac{c}{2}\tau^2 + \frac{b^2}{6}\tau^3)
$$

with $\tau = T - t$.

2. If the spot interest rate is a Vasicek process

$$
dX(t) = a(\mu - X(t))\, dt + \sigma\, dW(t)
$$

then

$$
\pi(r, t, T) = \exp\{-\tau y + \frac{1}{a}(e^{-a\tau} - 1)(r - y) - \frac{\sigma^2}{4a^3}(e^{-a\tau} - 1)^2\}
$$

with

$$
y = \mu - \frac{\sigma^2}{2a^2}.
$$

**Proposition. The Girsanov Theorem: Change of measure for diffusion processes.**

Assume that under a probability measure $P$, the process $X$ is a diffusion process with drift

$$
dX_t = \mu_t\, dt + \sigma_t\, d\bar{W}_t
$$

Then one may construct a probability measure $Q$ on the same probability space such that under $Q$, the process $X$ has the same diffusion, but no drift, i.e.

$$
dX_t = \sigma_t\, dW_t
$$

with $W_t$ being a Wiener process under $Q$. The density of $Q$ with respect to $P$ is given by the Girsanov formula

$$
\frac{dQ}{dP} = \exp\left(-\int \left(\frac{\mu_t}{\sigma_t}\right)\, d\bar{W}_t - \frac{1}{2}\int \left(\frac{\mu_t}{\sigma_t}\right)^2\, dt\right).
$$
where $\bar{W}$ is a Wiener process under $P$.

**Proof.** We start with a normal density with mean $\mu$ and covariance matrix $\Sigma$:

$$f(\bar{x}, \mu, \Sigma) = \frac{1}{(2\pi)^{n/2} \det(\Sigma)} \exp \left( -\frac{1}{2} (\bar{x} - \mu)^T \Sigma^{-1} (\bar{x} - \mu) \right).$$

If $P$ has density $f(\bar{x}, \mu, \Sigma)$ and $Q$ has density $f(\bar{x}, 0, \Sigma)$, then

$$\frac{dP}{dQ} = \frac{f(\bar{x}, \mu, \Sigma)}{f(\bar{x}, 0, \Sigma)} = \exp \left( -\frac{1}{2} (\bar{x} - \mu)^T \Sigma^{-1} (\bar{x} - \mu) + \frac{1}{2} \bar{x}^T \Sigma^{-1} \bar{x} \right)$$

$$= \exp \left( \bar{x}^T \Sigma^{-1} \mu - \frac{1}{2} \mu^T \Sigma^{-1} \mu \right)$$

We show the formula only for deterministic $\mu_t$ and $\sigma_t$. For the given processes

$$dX_t = \mu_t \, dt + \sigma_t \, d\bar{W}_t \quad \text{under } P$$

$$dX_t = \sigma_t \, dW_t \quad \text{under } Q$$

we choose a partition $(t_1, \ldots, t_n)$ and set

$$D_{t_i} = X_{t_{i+1}} - X_{t_i} = \sigma_{t_i} \cdot (W_{t_{i+1}} - W_{t_i}).$$

Notice that

$$\mathbb{E}[D_{t_i}] = \mu_{t_i}(t_{i+1} - t_i)$$

$$\text{Var}[D_{t_i}] = \sigma_{t_i}^2 (t_{i+1} - t_i)$$

Then, under $Q$

$$f_\mu(D_1, \ldots, D_n) = f_0(D_1, \ldots, D_n) \exp \left( \sum \frac{\mu_{t_i} (t_{i+1} - t_i)}{\sigma_{t_i}^2 (t_{i+1} - t_i)} (X_{t_{i+1}} - X_{t_i}) - \frac{1}{2} \sum \frac{\mu_{t_i}^2 (t_{i+1} - t_i)^2}{\sigma_{t_i}^2 (t_{i+1} - t_i)} \right)$$

which converges as $n \to \infty$ to

$$\exp \left( \int \frac{\mu_t}{\sigma_t^2} \, dX_t - \frac{1}{2} \int \frac{\mu_t^2}{\sigma_t^2} \, dt \right) = \exp \left( \int_{0}^{T} \frac{\mu_t}{\sigma_t} \, dW_t - \frac{1}{2} \int_{0}^{T} \left( \frac{\mu_t}{\sigma_t} \right)^2 \, dt \right)$$

We need however $dQ/dP$ under $P$, which is:

$$\frac{dQ}{dP} = \exp \left( - \int \frac{\mu_t}{\sigma_t} \, dW_t + \frac{1}{2} \int \left( \frac{\mu_t}{\sigma_t} \right)^2 \, dt \right)$$

Since

$$dX_t = \sigma_t \, dW_t \quad \text{under } Q$$

$$dX_t = \mu_t \, dt + \sigma_t \, d\bar{W}_t \quad \text{under } P$$

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We have that
\[ \sigma_t dW_t = \mu_t dt + \sigma_t d\overline{W}_t \]
where $\overline{W}$ is a Wiener process under $P$. Therefore, under $P$,
\[ \frac{dQ}{dP} = \exp \left( - \int \frac{\mu_t}{\sigma_t} dW_t + \frac{1}{2} \int \left( \frac{\mu_t}{\sigma_t} \right)^2 dt \right) = \exp \left( - \int \left( \frac{\mu_t}{\sigma_t^2} \right) \sigma_t dW_t + \frac{1}{2} \int \left( \frac{\mu_t}{\sigma_t} \right)^2 dt \right) = \exp \left( - \int \left( \frac{\mu_t}{\sigma_t} \right) d\overline{W}_t - \int \left( \frac{\mu_t}{\sigma_t^2} \right) \mu_t dt + \frac{1}{2} \int \left( \frac{\mu_t}{\sigma_t} \right)^2 dt \right) = \exp \left( - \int \left( \frac{\mu_t}{\sigma_t} \right) dW_t - \frac{1}{2} \int \left( \frac{\mu_t}{\sigma_t} \right)^2 dt \right). \]

4 Pricing Rules for financial contracts

To find prices for financial contracts in the primary market (the market of goods) as well as in the secondary market (the market of financial contracts) is done by a tatonnement (ask and bid matching) procedure on stock exchange or options and future exchange markets or by a bargaining procedure for over-the-counter contracts. What role can mathematics and statistics play in pricing?

By statistical methods, one may analyze the historical path of prices and fit some models for prediction. This is however retrospective and gives only a little help to understand future bargaining results.

There are however situations, in which one may use mathematical methods to determine prices. This is the situation, where a contract $A$ has to be priced, but this contract can be brought into close relation to contracts $B_1, \ldots, B_k$, for which prices are already known. In such a situation, there is no freedom in pricing $A$, but its price has to be consistent with the prices $B_1, \ldots, B_k$. This is completely analogous to linear algebra: if a point $A$ in the linear span of the points $B_1, \ldots, B_k$, then the value of any linear operator on $A$ is determined by its values on $B_1, \ldots, B_k$.

It may be evident (though also sometimes debatable) that a contract for selling 5 kilos of apples and 3 kilos of pears can be priced knowing the price for the contracts of 1 kilo apples and 1 kilo pears respectively. The first contract is in the span of the second two. However, in stochastic price models, the situation is not as simple as that.
Pricing rules in stochastic markets depend on models. Thus, *given a model* $\mathcal{M}$, we may infer from the prices of $B_1, \ldots, B_k$ the price of $A$, if $A$ is in the span of $B_1, \ldots, B_k$. Now suppose that the parties, irrespective of whether they have done the mathematical calculations or not, agree on a different price for $A$ than our model $\mathcal{M}$ would predict. This does typically not embarrass the financial modeler. He would simply modify the model $\mathcal{M}$ to a new model $\mathcal{M}'$ in such a way that it would predict the observed prices of $A$ correctly. Thus the role of contract $A$, priced by the market, was to improve the model. The new model can now be used to price a further new contract $C$, again the parties may agree on the price or bargain a different price – an endless game.

From this description one may see that mathematical pricing is a to be seen in an endless correction loop between calculation of prices and estimation of models.

We review here the state-of-the-art pricing rules:

### 4.1 Deterministic pricing

#### Rule A: price = discounted cash-flow

$$\pi(c_1, \ldots, c_T) = \sum_{t=1}^{T} c_t \pi(0, \ldots, 0, 1, 0, \ldots, 0) = \sum_{t=1}^{T} c_t \pi_t.$$ 
(This is a linear pricing rule)

### 4.2 Stochastic pricing

If either the cash-flows $C_t$ or the interest rates $R_{0,t}$ or both are stochastic, then we may use the simple expectation rule

#### Rule B: price = expected, discounted cash-flow

$$\pi(C_1, \ldots, C_T) = \mathbb{E}\left[\sum_{t=1}^{T} C_t (1 + R_{0,t})^{-t}\right].$$ 
This again a linear pricing rule.
4.3 Pricing through stochastic optimization

1. The price of an underlying contract is determined by stock exchange ask-bid pricing mechanism.

2. What is the correct price of a derivative contract (the payment of the derivative contract is a function of the value of the underlying contract)

| Rule C: The price of a derivative contract is the minimal initial capital needed to replicated (or superreplicate) the cash-flow of the derivative contract by implementing an appropriate trading strategy. |

Let $S_t$ be the stochastic price of the underlying at time $t$ and $C_T$ the cash-flows from the derivative contract at maturity $T$ ($C_T = f(S_T)$), where $f$ is typically nonlinear.

Example: European call option with maturity $T$ and exercise value $K$:

$$C_t = \begin{cases} 0 & \text{for } 0 < t < T \\ \max(S_T - K, 0) & \text{for } t = T \end{cases}$$

The price $\pi$ of this option is the minmal solution of the following optimization problem:

$$\begin{align*}
\text{Minimize (in } x_t, y_t) : & \ w \\
\text{subject to} : & \ x_0 + y_0 S_0 \leq w \quad \text{the initial capital condition} \\
& \ x_t(1 + R_t) + y_t S_t + 1 \geq x_{t+1} + y_{t+1} S_{t+1} \quad t = 0, \ldots, T - 1 \\
& \ x_T + y_T S_T \geq C_T \quad \text{the (super)replication condition} \\
\end{align*}$$

An implicit constraint is that the processes $x_t$ and $y_t$ are measurable w.r.t. the $\sigma$-algebra $\mathcal{F}_t$ generated by $(R_1, \ldots, R_t; S_1, \ldots, S_t)$.

$x_t$ is the amount to be invested in the bond (random interest rate $R_t$) and $y_t$ is the number of shares hold of the underlying.

Remarks:

- The problem (3) does not contain any probabilities. Only the possible values of the processes $(R_t, S_t)$ and the generated $\sigma$-algebras enter the calculations.
• The pricing rule is homogeneous, i.e.
\[ \pi(\lambda C_T) = \lambda \pi(C_T) \]
and subadditive, i.e.
\[ \pi(C_T^{(1)} + C_T^{(2)}) \leq \pi(C_T^{(1)}) + \pi(C_T^{(2)}) \].

The pricing rule is linear, iff the dual feasible set is a singleton (unique martingale measure).

4.3.1 Tree models

A stochastic process \( S_t \) is called a tree process, if the conditional distribution of \( S_1, S_2, \ldots, S_{t-1} \) given \( S_t \) is degenerated.

A tree process taking only finitely many values may be represented by a finite tree.

\[ \mathcal{N} = \{0, 1, 2, \ldots, N\} \] is the node set
0 is the root
\( T \) is set of terminal nodes (level \( T \))
n– is the predecessor of the node \( n \)
n+ is the set of successors of node \( n \)
The following processes "sit" on the nodes of the tree:

the interest rate process $R_n$, $n \in \mathcal{N} \setminus \mathcal{T}$
the process of underlying's value $S_n$, $n \in \mathcal{N}$

For European options, the cash-flows $C_n$ are only defined for the terminal nodes ($n \in \mathcal{T}$). But the approach allows for much more general derivative contracts, which make payments at all times.

The value of the derivative contract is:

$$
\min \left\{ w : x_0 + y_0 S_0 \leq w, \right. \\
\left. x_n - (1 + R_{n-}) + y_n S_n \geq x_n + y_n S_n \text{ for all } n \in \mathcal{N} \setminus \mathcal{T} \text{ except the root} \right. \\
\left. x_n - (1 + R_{n-}) + y_n S_n \geq C_n \text{ for all terminal nodes } n \in \mathcal{T} \right\}
$$

This is a linear program (LP). Recall that to every linear program of the form

$$
\min c^T x \\
A \cdot x \geq b
$$

there corresponds a dual program

$$
\min b^T \lambda \\
A^T \cdot \lambda = c \\
\lambda \geq 0
$$

with the same optimal value.

In order to dualise (4) we have to introduce for every node a nonnegative dual variable $\lambda_n$. The dual program is

$$
\max \sum_{n \in \mathcal{T}} \lambda_n C_n \\
\lambda_0 = 1 \\
\lambda_n = (1 + R_n) \sum_{m \in n+} \lambda_m \\
\lambda_n S_n = \sum_{m \in n+} \lambda_m S_m \\
\lambda_n \geq 0
$$

Let

$$
\gamma_n = \lambda_n \cdot (1 + R_{n-}) \cdot (1 + R_{n-}) \cdots \cdot (1 + R_0)
$$

and

$$
Z_n = S_n \cdot (1 + R_{n-})^{-1} \cdot (1 + R_{n-})^{-1} \cdots \cdot (1 + R_0)^{-1}
$$

The dual program expressed in the new variables is
\[
\max \sum_{n \in T} \gamma_n \cdot (1 + R_{n-})^{-1} \cdot (1 + R_{n-})^{-1} \cdots (1 + R_0)^{-1} C_n
\]
\[
\gamma_0 = 1
\]
\[
1 = \sum_{m \in \mathbb{N}^+} \gamma_m
\]
\[
Z_n = \sum_{m \in \mathbb{N}^+} \gamma_m Z_m
\]
\[
\gamma_n \geq 0
\] (4)

The \( \gamma_n \) may be interpreted as conditional probabilities of the node \( n \) given its predecessor. If one sets the total node probabilities as \( p_n = \gamma_n \cdot \gamma_{n-} \cdot \gamma_{n--} \cdots 1 \), then \( (Z_n) \) will be a martingale w.r.t \( (p_n) \).

**Definition.** A stochastic process \( Z_t \) is a martingale w.r.t. the filtration \( \mathcal{F}_t \), if for all \( t \) we have that
\[
\mathbb{E}(Z_t | \mathcal{F}_{t-1}) = Z_{t-1}.
\]

Martingales are sometimes also referred to as fair games. For the tree model, the martingale condition reads:
\[
Z_n = \frac{\sum_{m \in \mathbb{N}^+} p_m Z_m}{\sum_{m \in \mathbb{N}^+} p_m}.
\]

**Pricing rule D:** (dual of rule C):

| Rule D (dual pricing rule): If the price of a derivative contract is finite, its value equals the maximum of the expected, discounted cash-flow (rule B), where the maximum is taken over all equivalent probability distributions, which make the discounted underlying process a martingale. |

**Consequence.** The pricing operator is homogeneous and subadditive. It is additive iff there is just one equivalent martingale measure (the dual feasibility set contains only one point). Economists call this the complete market situation.

Examples for stock price models which lead to the complete market situation:

- Binary trees
- The geometric Brownian motion. This process is the basis for the Black-Scholes formula (see later)

**The call-put parity.**

Let \( \pi(S_0, T, C_T) \) be the today’s price of a derivative contract with maturity \( T \) and cash-flow \( C_T \).
Recall that the payment function of a European call option is $[S_T - K]^+ = \max(S_T - K, 0)$ and of a European put option is $[S_T - K]^− = \min(S_T - K, 0)$. Since $[S_T - K]^+ - [S_T - K]^− = S_T - K$, we have - if the dual problem is unique and hence the pricing rule is linear -

\[
\pi(S_0, T, [S_T - K]^+) - \pi(S_0, T, [S_T - K]^−) = \pi(S_0, T, S_T) - \pi(S_0, T, K) = S_0 - Ke^{-rT}.
\]

i.e.

- price of the call option with maturity $T$ and strike price $K$
- price of the put option with maturity $T$ and strike price $K = S_0 - Ke^{-rT}$.

This relation is called the call-put parity.

### 4.4 Option pricing for GBM: the Black-Scholes formula

If the binomial lattice model is refined in such way that both the time intervals and the logarithmic changes tend to zero, then the model tends to the geometric Brownian motion. This continuous time model is the basis of many calculations in financial mathematics, mostly because of its simplicity and the fact that it allows a unique martingale measure.

Assume therefore that the stock prices $S_t$ follow a GBM with drift

\[
S_t = S_0 \exp(\sigma W_t - t\sigma^2/2 + \mu t)
\]

where $W_t$ is the Wiener process. The constant $\sigma^2$ is called the volatility and $\mu$ is called the drift of $S_t$.

The riskless bond process is modeled by

\[
B_t = \exp(rt); \quad B_0 = 1.
\]

The process $B_t$ is considered as numeraire and discounting $S_t$ with $B_t$ gives the discounted stock process $Z_t$:

\[
Z_t = \exp(-rt)S_t.
\]

The process $Z_t$ satisfies a stochastic differential equation (SDE)

\[
dZ_t = Z_t\sigma dW_t + Z_t(\mu - r + \frac{1}{2}\sigma^2)dt. \tag{5}
\]

A European call option has payment function $[S_T - K]^+$. We may however consider a general payment function $f(S_T)$. What is the correct today’s price $\pi(S_0, 0)$ of this contract?
Let $\Pi(S, t)$ be the price of this contract at time $t$, given that the price of the underlying at time $t$ is $S$. We know that at time of maturity

$$\Pi(S_T, T) = f(S(T)).$$

To calculate the today’s price $\Pi(S_0, 0)$, we may use the pricing rule C. This however requires the solution of a stochastic optimization problem in continuous time and space. Alternatively, one may also use the method of differential equations and the martingale method (pricing rule D).

1. **The method of differential equations.** We consider the *delta hedge*, i.e. a portfolio consisting of one unit of the derivative contract and $-\Delta$ units of the underlying (the stocks) and call its value $V(S, t)$. The amount of $\Delta$ has to be fixed later.

$$V(S, t) = \Pi(S, t) - \Delta \cdot S,$$

i.e.

$$dV = d\Pi - \Delta dS$$

Since

$$dS = \mu S \, dt + \sigma S \, dW$$

one gets that

$$dV = \frac{\partial \Pi}{\partial t} \, dt + \frac{\partial \Pi}{\partial s} \, dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2} \, dt - \Delta dS.$$  

We choose now $\Delta = \frac{\partial \Pi}{\partial S}$ with the effect that only the deterministic part remains, i.e.

$$dV = \left( \frac{\partial \Pi}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2} \right) \, dt$$

Since there is only one riskless interest rate on the market, one must have

$$dV = r V \, dt$$

i.e. that

$$\left( \frac{\partial \Pi}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 \Pi}{\partial S^2} \right) \, dt = r \left( \Pi - \frac{\partial \Pi}{\partial S} S \right) \, dt$$

leading to the *Black-Scholes differential equation*

$$\frac{\partial \Pi}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 \Pi}{\partial S^2} + r S \frac{\partial \Pi}{\partial S} - r \Pi = 0.$$  

Together with the boundary condition

$$\Pi(S, T) = f(S)$$

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the solution of the pricing problem may be found.
Let us indicate the solution for the special case of an European option
\[ f(S) = [S - K]^+. \]

It took the two authors Black and Scholes several years to find the analytical solution, but once the solution is found it is easy to check its correctness:

\[
\Pi(S, t) = S \Phi \left( \frac{\log \frac{S}{K} + \left( r + \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T - t}} \right) - K e^{-r(T-t)} \Phi \left( \frac{\log \frac{S}{K} + \left( r - \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T - t}} \right)
\]

Here
\[ \Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-v^2/2} dv. \]

Since
\[ \lim_{b \to \infty} \Phi \left( \frac{a + b}{\sqrt{b}} \right) = \begin{cases} 1 & a > 0 \\ 0 & a < 0 \end{cases} \]

one sees that for \( t = T \)
\[ \Pi(S, T) = S \cdot 1_{[S > K]} - k \cdot 1_{[S > K]} = [S - K]^+ \]
i.e. this solution fulfills the boundary condition. The price of the option at time 0 is
\[
\pi(S_0, T, [S_T - K]^+) = \Pi(S_0, 0) = S_0 \Phi \left( \frac{1}{\sigma \sqrt{T}} \log \left( \frac{S_0}{K} \right) + (r + \sigma^2/2)(T - t) \right) - K e^{-rT} \Phi \left( \frac{1}{\sigma \sqrt{T}} \log \left( \frac{S_0}{K} \right) + (r - \sigma^2/2)(T - t) \right). \tag{6}
\]
This price may be obtained also by solving the optimal replication problem. One may therefore ask about the portfolio strategy, which leads to the optimal value (6). It turns out, that the number of units of the stock to be held at time \( t \) is given by
\[
\frac{\partial \Pi(s, t)}{\partial s} \bigg|_{s=S_t} = \Phi \left( \frac{1}{\sigma \sqrt{T-t}} \log \left( \frac{S_t}{K} \right) + (r + \sigma^2/2)(T - t) \right) \tag{7}
\]
and the value kept in stock is
\[
K e^{-r(T-t)} \Phi \left( \frac{1}{\sigma \sqrt{T-t}} \log \left( \frac{S_t}{K} \right) + (r - \sigma^2/2)(T - t) \right). \tag{8}
\]
Since the stock units equal the derivative of the option price w.r.t the actual stock value and since this derivative is called the $\delta$, the strategy (7)-(8) is known under the name of the delta-hedge.

2. **Martingale method.** We know by Girsanov’s formula that we may find the (unique) probability $Q$ such that under $Q$ the discounted stock process

$$dZ_t = (\mu - r)Z_t dt + Z_t \sigma dW_t$$

is a martingale

$$dZ_t = Z_t \sigma d\tilde{W}_t$$

From the pricing rule D, we know that the correct price at time $t$ is

$$e^{-r(T-t)} E_Q(f(S_T) | S_t)$$

and in particular for $t = 0$

$$\pi(S_0, T, f(S_T)) = e^{-rT} E_Q[f(S_T)].$$

Since

$$S_t = e^{rt} Z_t$$

with

$$dZ_t = \sigma Z_t d\tilde{W}_t$$

under $Q$ one gets that

$$dS_t = e^{rt} Z_t \sigma d\tilde{W}_t + re^{rt} Z_t dt$$

$$= S_t \sigma d\tilde{W}_t + r dt$$

The solution of this SDE is

$$S_t = S_0 \cdot \exp(\sigma \tilde{W}_t - \frac{1}{2} \sigma^2 t) \cdot \exp(rt)].$$

Therefore the distribution of $S_T$ is

$$S_T \sim S_0 \cdot \exp \left[ N((r - \frac{\sigma^2}{2})T, \sigma^2 T) \right].$$

Again, we may specialize the result to the European option

$$f(S_T) = [S_T - K]^+.$$
and get by integration the Black-Scholes formula, the same result as in (6)

\[
\pi(S_0, T, [S_T - K]^+) = e^{-rT} E_Q([S_T - K]^+)
\]

\[
= S_0 \Phi\left(\frac{\log \left(\frac{S_0}{K}\right) + \left(\frac{r}{2}\right) T}{\sigma \sqrt{T}}\right)
\]

\[
- K e^{-rT} \Phi\left(\frac{\log \left(\frac{S_0}{K}\right) + \left(\frac{r}{2}\right) T}{\sigma \sqrt{T}}\right).
\]

The moneyness is defined as the \(Q\)-probability that the option pays money, i.e.

\[
Q\{S_T > K\} = P\{S_0 \cdot \exp\left(\sigma \sqrt{T} \cdot Z + \left(\frac{r}{2}\right)\right) > K\} =
\]

\[
= P\{\sigma \sqrt{T} \cdot Z + \left(\frac{r}{2}\right) > \log \frac{K}{S_0}\} =
\]

\[
= P\{Z > \frac{\log \frac{K}{S_0} - \left(\frac{r}{2}\right)}{\sigma \sqrt{T}}\} =
\]

\[
= P\{Z < \frac{\log \frac{S_0}{K} - \left(\frac{r}{2}\right)}{\sigma \sqrt{T}}\} =
\]

\[
= \Phi\left(\frac{\log \frac{S_0}{K} + \left(\frac{r}{2}\right)}{\sigma \sqrt{T}}\right)
\]

Here \(Z\) is a standard normal variable.

**Exercise.** Explain why the drift \(\mu\) does not appear in the Black-Scholes formula.

### 4.4.1 Sensitivity analysis and the greeks

Let \(\pi(S, T)\) be the today’s \((t = 0)\) price of a derivative, which matures at time \(T\), if the today’s price of the underlying is \(S\).

Define the following quantities, called the greeks:

\[
\delta = \frac{\partial}{\partial S} \pi(S, T),
\]

\[
\gamma = \frac{\partial^2}{\partial S^2} \pi(S, T),
\]

\[
\theta = \frac{\partial}{\partial T} \pi(S, T).
\]
For instance, for a European call option the Black-Scholes formula gives

\[
\delta = \Phi \left( \frac{1}{\sigma \sqrt{t}} \left[ \log \left( \frac{S}{K} \right) + (r + \sigma^2/2)t \right] \right).
\]

\[
\gamma = \exp \left( -\frac{1}{2} \left( \frac{1}{\sigma \sqrt{t}} \left[ \log \left( \frac{S}{K} \right) + (r + \sigma^2/2)t \right] \right)^2 \right) \cdot \frac{1}{S \sigma \sqrt{2\pi t}}.
\]

\(\theta\) is more complicated.

4.5 The fundamental theorem of asset pricing

The fundamental theorem of asset pricing states that the absence of arbitrage opportunities is equivalent to the existence of a (not necessarily unique) equivalent martingale measure. We will derive and comment this theorem.

Suppose that a \(M\)-dimensional integrable stochastic price (column) vector process \(S_t, t = 1, \ldots, T\) defined on some probability space \((\Omega, \mathcal{A}, P)\) is given. Each component of \(S_t\) describes the price of one specific asset (stock, bonds etc.). It is assumed that each asset may be traded without transaction costs and with negative holdings (short selling). We assume that \(S_t\) is adapted to the filtration \((\mathcal{F}_t)\). \(S_0\) is the today’s price vector and is deterministic and \(\mathcal{F}_0\) is the trivial \(\sigma\)-algebra.

An arbitragist starts with no or negative initial capital and gets nonnegative, not identically zero capital at time \(T\). Suppose that \(y_t\) is the (row) vector of holdings he has at time \(t\). Then \(y_t\) an arbitrage strategy \(y_t\) must be \(\mathcal{F}_t\) measurable and must a.s. satisfy

(i) \(y_0^T S_0 \leq 0\) start with no capital

(ii) \(y_{t-1}^T S_t \geq y_t S_t\) self financing condition, \(t=1, \ldots, T-1\)

(iii) \(y_{T-1}^T S_T \geq 0\) nonnegative wealth at maturity

Here is the fundamental theorem:

In order that a stochastic model for asset prices be reasonable, i.e. excludes absurd arbitrage possibilities, it is necessary and sufficient that there exists at least one new measure \(Q\), which makes the renormalized process \(Z_t = S_t / S_t^{(k)}\) a martingale. Here \(S_t^{(k)}\) may be any positive component of the process \(S_t\), which serves as numeraire.
Let us prove the theorem for tree structures. Suppose that $S_n$ is a vector of $M$ processes on a tree and $y_n$ are the $M$-vectors of holdings.

For every node $n$, the self financing condition is that $y_n^T S_n \leq y_n - S_n$ for all nonterminal nodes. The program to detect arbitrage is

$$\max \sum_{n \in T} y_n^T S_n$$

subject to

$y_0^T S_0 \leq 0$

$y_n^T S_n \geq y_n^T S_n$ \quad $n \neq 0$

$y_n^T S_n \geq 0$ \quad $n \in T$

(9)

Notice that this program has either maximal value of 0 or is unbounded, since every solution with $\sum_{n \in T} y_n^T S_n > 0$ can be multiplied by an arbitrary large positive constant to give another, better solution. The dual of (9) is

$$\max 0$$

$$\lambda_n S_n = \sum_{m \in n^+} \lambda_m S_m \quad n \in \mathcal{N} \setminus T$$

$$\lambda_n \geq 0$$

Notice that the dual has objective function 0. Therefore it may either be feasible (in this case the optimal value is 0) or infeasible.

If there are no arbitrage possibilities, then the primal is bounded and the dual is feasible. Hence the nonexistence of arbitrage is equivalent to the existence of constants $\lambda_n \geq 0$ satisfying

$$\lambda_n S_n = \sum_{m \in n^+} \lambda_m S_m \quad n \in \mathcal{N} \setminus T.$$

Notice that the constants $\lambda$ are only determined up to a constant.

Choose now any strictly positive component $s_n^{(k)}$ of $S_n$. Let $Z_n = S_n / s_n^{(k)}$ and $p_n = \lambda_n S_n / \lambda_0 s_0^{(k)}$. Then in the no-arbitrage case we have that for the $k$-th component

$$p_n = \sum_{m \in n^+} p_m$$

(10)

and for the other components

$$Z_n = \frac{1}{p_n} \sum_{m \in n^+} Z_m \cdot p_m$$

(11)

Let us check quickly that the existence of a martingale measure is sufficient to prevent arbitrage. Suppose that there is a trading strategy, which leads to $y_n S_n \geq 0; n \in T$ and $\sum_{n \in T} \lambda_n y_n^T S_n > 0$. Because of the martingale property however, $0 < \sum_{n \in T} \lambda_n y_n^T S_n = \sum_{n \in T} p_n y_n^T Z_n = y_0^T Z_0 = y_0^T S_0 / s_0^{(k)}$. Thus only a positive initial capital may lead to a nonnegative, not identically zero final wealth.
5 Portfolio optimization

Financial decisions have two dimensions: A value dimension, which is measured by a location parameter of the profit distribution and a risk dimension, which is measured by a dispersion parameter. Value is expressed by expectation (or some other location parameter). The dispersion is measured by a translation-invariant functional $D$.

The optimal portfolio problem consists in finding the composition of a portfolio, which leads to a compromise between high expected return and low risk. The price of the portfolio is limited by the available budget.

Let $\xi = (\xi^{(1)}, \ldots, \xi^{(M)})^\top$ be the vector of possible returns per unit of price of each of $M$ assets for one holding period and $x = (x_1, \ldots, x_M)^\top$ is the vector of asset holdings (again in unit of price). If a total budget of $B$ is invested today, i.e. $B = x_1 + \cdots + x_M$, then the random value at the end of the holding period is $Y_x = \sum_{m=1}^{M} x_m \xi^{(m)}$.

The standard decision problem is to minimize the risk of the outcome among all feasible decisions $x \in \mathcal{X}$ under a the constraint that the the expected return must be larger than $\mu$.

\[
\begin{array}{l}
\text{Minimize} \quad D[Y_x] \\
\mathbb{E}[Y_x] \geq \mu \\
x \in \mathcal{X}
\end{array}
\] (12)

If the risk functional is the variance or the standard deviation, the pertaining model is the Markowitz model.

Denote by $r$ the expected return vector $r = \mathbb{E}\xi$ and by $C$ the covariance matrix $C = \mathbb{E}[(\xi - r) \cdot (\xi - r)^\top]$ of the asset data. Since the standard deviation is the square root of the variance, it does not matter, whether the variance or the standard deviation is considered as the objective function. In the following, the variance is minimized, but the standard deviation is shown in the risk/return diagrams.

The model is

\[
\begin{array}{l}
\text{Minimize} \quad x^\top C x \\
\text{subject to} \\
r^\top x \geq \mu \quad \text{minimal expected return } \mu \\
1_M^\top x \leq 1 \quad \text{budget constraint} \\
x \geq 0 \quad \text{nonnegativity}
\end{array}
\] (13)

This is a quadratic program with linear constraints. The number of variables is $M$ (all nonnegative), the number of constraints is 2. The Markowitz model has become very popular, mostly due to the fact that is is simple and its complexity does not increase with the sample size. In fact, both for theoretical models and for discrete or sampled models,
all one has to do is to calculate the covariance matrix and the mean returns first and use these parameters in the optimization model.

The relation between the lower bound return $\mu$ and the minimal variance is called the efficient frontier. Figure 5 shows the efficient frontier in the upper part of the picture. Risk (in this case the variance) is shown in the $x$-axis and return in the $y$-axis. The risk/return values of the 6 assets are indicated as numbers. The lower part shows the composition of the optimal portfolios in the same risk scale as the efficient frontier above.

Figure 1: A variance efficient frontier. Portfolio weights must be nonnegative.

A variant of this model is the CAPM model (capital asset pricing model), which drops (unrealistically) the nonnegativity constraints and sets all inequalities to equalities in (??)

$$
\begin{align*}
\text{Minimize} & \quad x^\top Cx \\
\text{subject to} & \quad r^\top x = \mu \quad \text{expected return } \mu \\
& \quad 1^\top Mx = 1 \quad \text{budget constraint}
\end{align*}
$$

For this model, we can find the explicit solution.

**Proposition.** Assume that $C$ is invertible. Then the solution of (14) is affine-linear in $\mu$ and given by

$$
x^*_\mu = \mu \left[ \frac{c}{ac-b^2} C^{-1} r - \frac{b}{ac-b^2} C^{-1} 1_M \right] + \left[ \frac{a}{ac-b^2} C^{-1} 1_M - \frac{b}{ac-b^2} C^{-1} r \right].
$$
where
\[
\begin{align*}
a &= r^\top C^{-1} r \\
b &= r^\top C^{-1} 1_M \\
c &= 1^\top C^{-1} 1_M.
\end{align*}
\]

The moments of the random return \( Y_\mu = x_\mu^\top \cdot \xi \) are
\[
\begin{align*}
\mathbb{E}(Y_\mu) &= \mu \quad \text{by construction} \\
\mathbb{V}ar(Y_\mu) &= \frac{\mu^2 c - 2\mu b + a}{ac - b^2}.
\end{align*}
\]
i.e. \( \mathbb{V}ar(Y_\mu) \) is a quadratic function in \( \mu \).

**Proof.** Introducing the Lagrange multipliers \( \lambda \) and \( \gamma \) the Lagrange function is
\[
\frac{1}{2} x^\top C x - \lambda [x^\top r - \mu] - \gamma [x^\top 1 - 1]
\]
and the necessary conditions are given by the following equations
\[
\begin{align*}
C x - \lambda r - \gamma 1_M &= 0 \\
r^\top x &= \mu \\
1_M^\top x &= 1
\end{align*}
\]
Thus
\[
x = \lambda C^{-1} r + \gamma C^{-1} 1_M
\]
and one may calculate \( \lambda \) and \( \gamma \) from the equations
\[
\begin{align*}
x^\top r &= \lambda r^\top C^{-1} r + \gamma r^\top C^{-1} 1_M = \mu \\
1_M^\top r &= \lambda 1_M^\top C^{-1} r + \gamma 1_M^\top C^{-1} 1_M = 1
\end{align*}
\]
One gets
\[
\begin{align*}
\lambda &= \frac{\mu c - b}{ac - b^2} \\
\gamma &= \frac{a - \mu b}{ac - b^2}
\end{align*}
\]
with \( a, b, c \) given by (15) which leads to the asserted equation for the optimal portfolio \( x_\mu^* \). \( \Box \)
5.0.1 The linear relation between $\text{Cov}(Y_\mu, \xi_m)$ and $r_m$

The return of the optimal portfolio is $Y_\mu = x_\mu^\top \cdot \xi$. The covariance with the return of the $m$-th asset is $\text{Cov}(x_\mu^\top \xi, \xi_m) = \sum_j c_{mj} x_j = e_m^\top C x_\mu^*$, where $e_m$ is the $m$-th unit vector. Thus for all $m$

$$
\text{Cov}(x_\mu^\top \xi, \xi_m) = e_m^\top C x_\mu^*(\mu) = r_m\left[\frac{\mu c - b}{ac - b^2}\right] + \left[\frac{a}{ac - b^2} - \frac{b}{ac - b^2}r_m\right]
$$

i.e. all points $(r_m, \text{Cov}(Y_\mu, \xi_m))$ for $m = 1, \ldots, M$ lie all on one straight line.

5.0.2 Introducing a risk-free asset

We add now asset 0 to the set of possible assets, which has return $r_0$ and is risk free (i.e. has zero variance). Suppose we form a new portfolio (including the risk-free asset) out of an old $x$ (which does not include the risk free asset) in such a way that we take $\delta$ of risk free and a portion $(1 - \delta)$ of $x$. This new portfolio has expected return $\delta r_0 + (1 - \delta)x^\top r$ and standard deviation $(1 - \delta)\sqrt{x^\top C x}$. Geometrically, in the return-risk plane, it lies on a straight line segment connecting the points $(0, r_0)$ and $(\sqrt{x^\top C x}, x^\top r)$. Thus by adding all these line segments to the feasible set, we get the new feasible set of risk/return combinations.

Denote the efficient frontier function without risk-free asset by $\mu \mapsto \sigma(\mu)$, where $\sigma(\mu) = \sqrt{x_\mu^\top \tilde{C} x_\mu^*}$ (where $\tilde{C}$ is the covariance matrix of the risk-prone assets).
We extend the optimization problem in the following way: The decision vector, the return vector and the covariance matrix are augmented for the risk free return

\[ x = \begin{pmatrix} x_0 \\ \tilde{x} \end{pmatrix}, \quad r = \begin{pmatrix} r_0 \\ \tilde{r} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{C} \end{pmatrix}, \quad \xi = \begin{pmatrix} r_0 \\ \tilde{\xi} \end{pmatrix}. \]

The augmented decision problem is

\[
\begin{align*}
\text{Minimize} \quad & \frac{1}{2} \tilde{x}^T \tilde{C} \tilde{x} \\
\text{subject to} \quad & \tilde{r}^T \tilde{x} + r_0 x_0 = \mu \\
& \mathbf{1}_M^T \tilde{x} + x_0 = 1
\end{align*}
\]

where \( \mathbf{1}_M \) is the vector of ones of the same length as the number of risky assets.

**Proposition. Two fund Theorem.** For the extended Markowitz model which includes a risk-free asset, the optimal solution is

\[
x^* = \frac{1}{d} \begin{pmatrix} \tilde{r} - \mu \mathbf{1}_M \end{pmatrix} \tilde{C}^{-1} \left( \tilde{r} - r_0 \mathbf{1} \right)
\]

where

\[
d = (\tilde{r} - r_0 \mathbf{1}_M)^T \tilde{C}^{-1} (\tilde{r} - r_0 \mathbf{1}_M).
\]

All efficient portfolios are affine combinations of the risk-free portfolio

\[
\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\]

and the *market portfolio* \( x_+ \), where

\[
x_+ = \begin{pmatrix} 0 \\ \tilde{C}^{-1}(\tilde{r} - r_0 \mathbf{1}_M) \\ \mathbf{1}_M^T \tilde{C}^{-1}(\tilde{r} - r_0 \mathbf{1}_M) \end{pmatrix}
\]

**Proof.** Introducing the Lagrange multipliers, one finds that the necessary conditions are given by the following equations

\[
\begin{align*}
\tilde{C} \tilde{x} - \lambda \tilde{r} - \gamma \mathbf{1}_M &= 0 \\
\lambda r_0 + \gamma &= 0 \\
\tilde{r}^T \tilde{x} + r_0 x_0 &= \mu \\
\mathbf{1}_M^T \tilde{x} + x_0 &= 1
\end{align*}
\]
Setting $\gamma = -\lambda r_0$ and assuming that $\tilde{C}$ is invertible, one gets
\[
\tilde{x} = \lambda \tilde{C}^{-1}(\tilde{r} - r_0 \mathbb{1}).
\]

Using the other equations, the result is
\[
x^*(\mu) = \left( x^*_0(\mu) \tilde{x}^*(\mu) \right), \quad \tilde{x}^*(\mu) = \frac{\mu - r_0}{(\tilde{r} - r_0 \mathbb{1}_M)^\top \tilde{C}^{-1}(\tilde{r} - r_0 \mathbb{1}_M)} \tilde{C}^{-1}(\tilde{r} - r_0 \mathbb{1}_M), \quad x^*_0(\mu) = 1 - \mathbb{1}_M^\top \tilde{x}^*(\mu).
\]

The return $Y_{\mu}$ of the optimal portfolio $x^*_\mu$ is of course equal to $\mu$, its variance is
\[
\text{Var}(Y_{\mu}) = x^*_\mu^\top C x^*_\mu = \frac{(\mu - r_0)^2}{(\tilde{r} - r_0 \mathbb{1}_M)^\top \tilde{C}^{-1}(\tilde{r} - r_0 \mathbb{1}_M)}.
\]

The standard deviation is a linear function in $\mu$:
\[
\text{Std}(Y_{\mu}) = \sqrt{x^*(\mu)^\top C x^*(\mu)} = \frac{\mu - r_0}{\sqrt{(\tilde{r} - r_0 \mathbb{1}_M)^\top \tilde{C}^{-1}(\tilde{r} - r_0 \mathbb{1}_M)}}.
\]

Denote by $Y_+ = x^+_1 \cdot \xi$ the return of the market portfolio. It has expectation
\[
\mu_+ := \mathbb{E}(Y_+) = r^\top x_+ = \tilde{r}^\top \tilde{C}^{-1}(\tilde{r} - r_0 \mathbb{1}_M) \mathbb{1}_M^\top \tilde{C}^{-1}(\tilde{r} - r_0 \mathbb{1}_M).
\]

Its variance is
\[
\sigma_+^2 := \text{Var}(Y_+) = \frac{(\tilde{r} - r_0 \mathbb{1}_M)^\top \tilde{C}^{-1}(\tilde{r} - r_0 \mathbb{1}_M)}{[\mathbb{1}_M^\top \tilde{C}^{-1}(\tilde{r} - r_0 \mathbb{1}_M)]^2}.
\]

If $x$ is any portfolio, then
\[
\text{Cov}(x^\top \cdot \xi, x^+ \cdot \xi) = x^\top C x^+ = x^\top \begin{pmatrix} 0 & (\tilde{r} - r_0 \mathbb{1}_M) \\ \mathbb{1}_M \tilde{C}^{-1}(\tilde{r} - r_0 \mathbb{1}_M) \end{pmatrix} = \begin{pmatrix} 0 & \tilde{x}^\top (\tilde{r} - r_0 \mathbb{1}_M) \\ \mathbb{1}_M \tilde{C}^{-1}(\tilde{r} - r_0 \mathbb{1}_M) \end{pmatrix}.
\]

Thus $\text{Cov}(x^\top \cdot \xi, x^+ \cdot \xi)$ a linear function in $x$. Setting $Y_x$ the return of portfolio $x$ and using the fact that $\mathbb{E}(Y_x) = x^\top \cdot r = x_0 r_0 + \tilde{x}^\top \cdot \tilde{r}$, i.e. $\tilde{x}^\top (\tilde{r} - r_0 \mathbb{1}_M) = \mathbb{E}(Y_x) - x_0 r_0 - r_0 (1 - x_0)$ one gets a linear relation between $\text{Cov}(Y_x, Y_+)$ and $\mathbb{E}(Y_x)$
\[
\text{Cov}(x^\top \cdot \xi, x^+_1 \cdot \xi) = \frac{\tilde{x}^\top (\tilde{r} - r_0 \mathbb{1}_M)}{\mathbb{1}_M \tilde{C}^{-1}(\tilde{r} - r_0 \mathbb{1}_M)} = \frac{\mathbb{E}(Y_x) - x_0 r_0 - (1 - x_0) r_0}{\mathbb{1}_M \tilde{C}^{-1}(\tilde{r} - r_0 \mathbb{1}_M)} = \frac{\mathbb{E}(Y_x) - r_0}{\mathbb{1}_M \tilde{C}^{-1}(\tilde{r} - r_0 \mathbb{1}_M)}.
\]
Now, using the fact that
\[
\frac{\mu_+ - r_0}{\sigma^2_+} = \frac{\tilde{r}^T \tilde{C}^{-1}(\tilde{r} - r_0 \mathbf{1}) - r_0 \mathbf{1}^T \tilde{C}^{-1}(\tilde{r} - r_0 \mathbf{1})}{\mathbf{1}^T \tilde{C}^{-1}(\tilde{r} - r_0 \mathbf{1})} = \frac{\mathbf{1}^T \tilde{C}^{-1}(\tilde{r} - r_0 \mathbf{1})^2}{(\tilde{r} - r_0 \mathbf{1})^T \tilde{C}^{-1}(\tilde{r} - r_0 \mathbf{1})}
\]
we get the relation
\[
E(Y_x) = r_0 + \frac{\mu_+ - r_0}{\sigma^2_+} \mathbf{1}^T \mathbf{x}_+^T \cdot \mathbf{X}_m \cdot \mathbf{x}_+ \quad (17)
\]
or, denoting by \( \beta(x) = \frac{\mathbf{1}^T \mathbf{x}_+^T \cdot \mathbf{X}_m \cdot \mathbf{x}_+}{\sigma^2_+} \) the regression coefficient of the return of \( Y_x \) w.r.t. to the return \( Y_+ \) of the market portfolio, one gets
\[
E(Y_x) = r_0 + \beta(x) \cdot (\mu_+ - r_0). \quad (18)
\]
The quantity \( \beta(x) \) is called the \textit{beta coefficient} of the portfolio \( x \). The expected return \( r_0 + \beta(x)(\mu_+ - r_0) \) is called the \textit{alpha coefficient} of the portfolio.

Thus we arrive at the following proposition.

**Proposition.** For any portfolio \( x \)

(i) \( E(Y_x) = r_0 + \beta(x) \cdot (\mu_+ - r_0) \).

(ii) \( \text{Var}(Y_x) \geq \left( \frac{E(Y_x) - r_0}{\mu_+ - r_0} \right)^2 \sigma^2_+ \). Equality holds only if \( \text{Corr}(Y_x, Y_+) = 1 \).

**Proof.** Only the second assertion has to be proved. For any portfolio \( x \), the portfolio \((1 - \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda x_+ \) with \( \lambda = \frac{E(Y_x) - r_0}{\mu_+ - r_0} \) has the same expectation as \( Y_x \) and lies on the efficient frontier. Therefore \( \text{Var}(Y_x) \geq \left( \frac{E(Y_x) - r_0}{\mu_+ - r_0} \right)^2 \sigma^2_+ \). From (i) one gets \( \left( \frac{E(Y_x) - r_0}{\mu_+ - r_0} \right)^2 = \frac{\text{Cov}^2(Y_x, Y_+)}{\sigma^2_+} \leq \frac{\text{Var}(Y_x)}{\sigma^2_+} \). Thus equality can hold only if \( \text{Cov}^2(Y_x, Y_+) = \text{Var}(Y_x) \cdot \text{Var}(Y_+) \), i.e. if \( \text{Corr}(Y_x, Y_+) = 1 \).

**References**


Figure 3: A variance efficient frontier including a risk free asset.