

# Mach, Thirring & Lense, Gödel – getting dizzy in space-time<sup>\*</sup>

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## Abstract:

Contrary to Newton's concept of inertia, general relativity predicts an influence of rotating matter on the structure of space and time. Anticipated by Ernst Mach, effects of this type have first been derived by Hans Thirring and Josef Lense. Almost ninety years later, we face their experimental verification. An even more dramatic scenario is provided by Kurt Gödel's cosmological model, in which nearby observers rotate with respect to each other and are free to travel to their own past.

Powerpoint presentation of the talk: <http://homepage.univie.ac.at/franz.embacher/Rel/Goedel/>  
(containing some additional technical material and details)

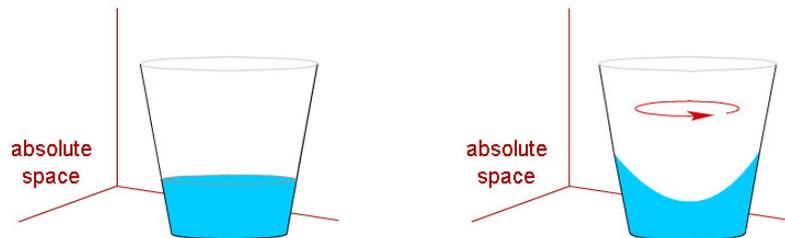
## Absolute space: Isaac Newton (1687)

According to Isaac Newton, inertia is a phenomenon that relates the motion of bodies (and their resistance against the action of a force) to absolute space. The rotation of a body with respect to absolute

space causes the appearance of extra (Coriolis and centrifugal) forces. This is illustrated in Newton's famous "bucket experiment":

When a bucket filled

with water is rotating around its symmetry axis, the surface of the water takes a curved (parabolically shaped) form. Thus, rotation is conceived by Newton as an *absolute* concept.

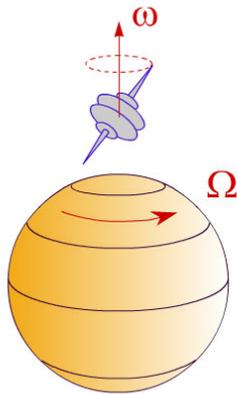


## Inertia as a relative concept: Ernst Mach (1883)

Almost two hundred years later (during which the concept of absolute space had a variable fate) the major theoretical attack against absolute space was due to the Austrian-Czech physicist Ernst Mach. In his book *Die Mechanik in ihrer Entwicklung -- historisch kritisch dargestellt*<sup>1</sup> (*The Science of Mechanics: A Critical and Historical Account of Its Development*<sup>2</sup>), Mach refused absolute concepts of space and time whatsoever. He conceived inertia (and hence the extra forces arising in Newton's bucket experiment) as a consequence of the rotation relative to the bulk of matter in the universe. This is known as

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“Mach’s principle”. It implies that a rotation of the bucket together with *all* the matter in the universe around some axis is unobservable! Thus, any motion (whether rotational or translational) is *relative*. Thus, Mach was able to state a prediction: When a bucket is surrounded by a “several miles thick” wall that co-rotates along with it relative to the rest of the universe (the “distant matter”), then the water surface should curve a little bit, the reason being that the “non-rotating state” (the local inertial frame, i.e. a state of motion in which one



does not feel any rotation) emerges due to some average over the distribution and motion of all the matter in the universe. According to this argument, a related type of effect would act on a Foucault pendulum placed at the north pole of the earth. Since the “non-rotating state” at the north pole emerges as an average over the rotating earth *and* the rest of all masses, the plane of the pendulum should in fact be *dragged* along with the earth’s rotation at a tiny amount.

Phenomena like these became later on known as *Machian effects*. In modern terms, buckets and pendulums are replaced by (torque-free) gyroscopes that are dragged along with the rotation of massive bodies. When the earth’s angular frequency is denoted by  $\Omega$ , the axis of a torque-free gyroscope placed near the earth should undergo a small precession. For later purpose, we denote the angular frequency of this precession by  $\omega$ . In lack of a mathematical theory of inertia that would replace Newtonian gravity, Mach was not able to quantify predictions of this type. He just could state that quantities like  $\omega$  should be *very* small. Today,  $\omega$  is called the Thirring-Lense frequency.

## General relativity: Albert Einstein (1915)

Thirty years later, Albert Einstein published his general theory of relativity. It contained Newton’s theory of gravity as a limiting case for small masses and slow motions. Let us highlight the three main points of his theory that are relevant for our topic.

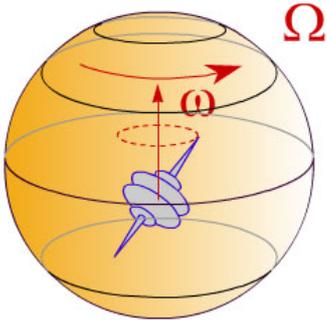
- Gravity is identified with the geometry of space-time. (Mathematically, it is described in terms of a metric).
- Matter curves space-time. (This is the content of the famous field equations of Einstein’s theory).
- The free motion of a (small) body in a given gravitational field is such that its proper time between the beginning and the end of its journey is maximal. (Mathematically, its motion is described by the so-called geodesic equation).

Although general relativity is still regarded as the best description of gravity available, the full richness of its predictions was worked out only by and by during the following decades.

On the long way to this theory, Einstein was led by the ideas of Ernst Mach, in particular by the concept of inertia as depending only on matter and not on some pre-existing entity. Hence, as soon as the theory had been brought into its final shape, the question arose: does it predict Machian effects?

# Machian effects: Hans Thirring and Josef Lense (1918)

Shortly after the theory of general relativity was published, the Vienna physicist Hans Thirring did the first step in order to decide whether it accounts for Machian effects. He considered a uniformly rotating massive spherical shell (with empty space inside) and computed the gravitational field generated by such an object. Using a number of simplifying assumptions (the shell is infinitely thin, its angular frequency  $\Omega$  and its mass  $M$  being small enough so as to allow for the application of the so-called linear – weak field – approximation of the theory), he managed to solve Einstein’s field equations. The geometry inside the shell turned out to be that of flat space, but rotating with respect to the exterior region<sup>3</sup>. From the point of view of Newton’s theory, this was completely unexpected: General relativity predicts Machian effects!

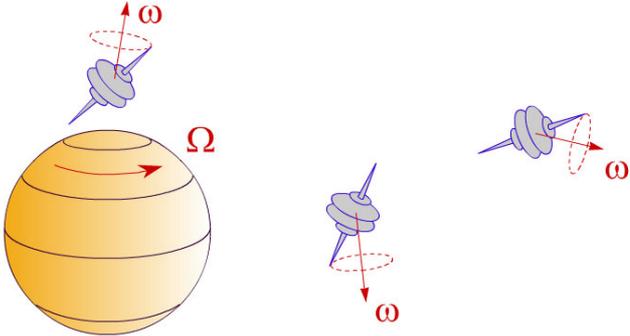


In contrast to Mach, Thirring had a mathematical theory at hand which lead to quantitative predictions. His result is so simple that we can show it here:

$$\frac{\omega}{\Omega} = \frac{4 GM}{3 c^2 R} \equiv \frac{2 R_s}{3 R}, \tag{1}$$

where  $G$  is the gravitational constant,  $c$  is the velocity of light,  $R$  is the radius of the shell and  $R_s$  is its Schwarzschild radius. If, for example, the radius and the mass of the shell equal those of the earth, the factor at which the interior region (and hence any gyroscope within) is dragged along by the shell’s rotation, is essentially the ratio  $R_s / R$ . Taking into account that the Schwarzschild radius of the earth is 0.886 cm, this is indeed a very small quantity! Its value is  $9 \times 10^{-10}$ .

Let us at this point rise the question how gyroscopes behave in the region outside the shell or outside a rotating spherical object like a star or a planet with mass  $M$ , radius  $R$  and angular frequency  $\Omega$ . (Thirring did not perform this analysis for the sphere, but it is easily carried out using analogous methods). As one may expect, the magnitude of the precession as well as its direction depends on the location of the gyroscope. In

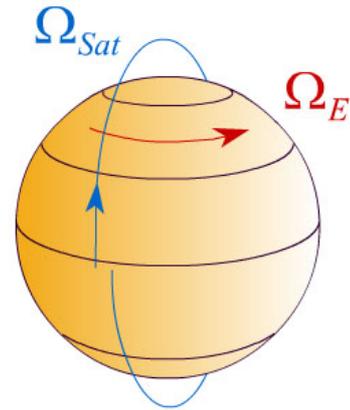


the equatorial plane, the dragging frequency (Thirring-Lense frequency)  $\omega$  is determined by

$$\frac{\omega}{\Omega} \approx -\frac{GM}{c^2 R} \left(\frac{R}{r}\right)^3, \tag{2}$$

where  $r$  is the distance of the gyroscope from the center of the object. A prefactor (of the order 1) that depends on whether a thin shell or a massive ball is considered has been omitted. The minus sign indicates that the dragging occurs with opposite orientation as compared to that of the central body. However, it also shows that we have to be careful when interpreting this type of effect too close to Mach’s original reasoning in terms of some

“average” between distant matter and a rotating object. The  $1/r^3$ -behaviour illustrates that the magnitude of Machian effects decreases faster than the gravitational ( $1/r^2$ ) force. It shows that Machian effects are best viewed in close vicinity of rotating objects.



Together with the Vienna mathematician Josef Lense, Thirring found a further “Machian” manifestation: Thirring and Lense posed the question whether the rotation of a central mass such as the sun or a planet affects the motion of satellites. In absence of rotation, the conservation of angular momentum confines any satellite motion around a spherical object to a plane. However, it turned out that, in case the central body rotates, the orbits of a satellite moving over the poles are not closed! After one orbit, the satellite does not exactly hit the point where it started but is slightly displaced. In a sense, one may imagine that the orbital plane is dragged by the rotation of the central body. When the angular frequencies of the central object and of the satellite orbit are denoted by  $\Omega_E$  and  $\Omega_{Sat}$ , respectively, and the satellites undergoes a circular polar orbit of radius  $r$ , the amount of displacement is given by <sup>4</sup>

$$d = \frac{4\pi}{5} \frac{\Omega_E}{\Omega_{Sat}} \frac{R_s R^2}{r^2}. \quad (3)$$

In case of a satellite on a close polar orbit around the earth, this implies  $d = 0.13$  cm, which in turn corresponds to an angular velocity of the orbital plane of 0.26 arc seconds per year! By the time of Thirring and Lense, such an effect was considered far too small to be observed.

## Does the Thirring-Lense effect exist in nature?

Forty years after the first quantitative prediction of Machian effects, people began to think about possibilities for experimental verification. In recent times, two attempts have been made:

- In a joint project of the NASA with the Italian space agency ASI (Agenzia Spaziale Italiana) position data of two satellites <sup>5</sup> (LAGEOS und LAGEOS2) were evaluated. After taking into account all known (non Machian) reasons that cause satellite orbits not to be closed (in particular the deviation of the earth’s gravitational field from spherical symmetry), it was announced that a net effect of  $99\% \pm 5\%$  of the predicted value of the orbital Thirring-Lense effect remained.
- At the moment (September 2006), another mission is still on the way: Gravity Probe B, a co-operation of the NASA with Stanford University. Its goal is to directly confirm the dragging effect on four gyroscopes with quartz rotors – the roundest objects ever made – placed in a satellite on a polar orbit, that was launched in April 2004. Using all sorts of high precision technology, this experiment is expected to confirm the dragging effect at an accuracy of 1% .

Hence, we should accustom to the idea that what “non-rotating” means depends on whether there is something else rotating in our vicinity!

## A rotating universe: Kurt Gödel (1949)

On the occasion of Einstein's 70<sup>th</sup> birthday, Kurt Gödel published a remarkable solution of the field equations of general relativity, the so-called "Gödel universe", described by the simple expression <sup>6</sup>

$$ds^2 = \frac{1}{2\omega^2} \left( -(dt + e^x dz)^2 + dx^2 + dy^2 + \frac{1}{2} e^{2x} dz^2 \right), \quad (4)$$

where  $\omega$  is a constant (of the dimension of a frequency). Let us have a look at some of its properties:

- It describes a universe which is filled with a perfect fluid with very high pressure, or equivalently, with pressureless dust together with a negative cosmological constant. This is somewhat unusual, but not entirely forbidden by the fundamental laws of nature.
- It is completely regular. (In technical terms, it is singularity-free and geodesically complete).
- It is homogeneous (i.e. it looks the same when viewed from different positions) and stationary (i.e. it looks the same when viewed at different times) but not static (i.e. matter is moving, just as a uniformly rotating sphere is stationary but not static).

So far, this is good news. Let's continue:

- Two nearby observers, both at rest with respect to matter, rotate with respect to each other. This means that any observer who is at rest with respect to matter and always looks towards a particular nearby observer feels extra forces, i.e. gets dizzy! Conversely, any observer who is at rest with respect to matter and orients himself along a fixed direction of his local inertial frame (such that he will *not* get dizzy) sees all nearby observers rotating around him with angular velocity  $\omega$ . His impression must be that the whole universe is rotating around him! However, since space-time is homogeneous, there is *no distinguished axis of rotation* of the universe! Hence, in this sense, local inertial frames rotate with respect to each other.
- By appropriately moving around, one may travel into his own past! (In technical terms: Gödel's universe contains closed timelike curves).

Although the second of these amazing properties is more startling, it is the first one that is more relevant to our topic: Since the development of general relativity, the question arose whether it incorporated Mach's principle in a stronger form than just predicting Machian effects. Is there a sense in which one may state that inertia emerges from the distribution and motion of matter in the universe? Gödel's universe stimulated the discussion about this issue: does it confirm or contradict Mach's principle? Surprisingly, the answer not unique. In a sense, the (theoretical) existence of Gödel's universe may be viewed to *confirm* Mach's principle, because inertia and the properties of local inertial frames are tied to the global distribution and motion of matter. On the other hand, it may be viewed to *contradict* Mach's principle, because local inertial frames rotate with respect to each other, while the universe as a whole does not rotate around some particular axis. Since there has not been achieved a commonly agreed formulation of *what* Mach's principle precisely states, the question is rather open to speculation.

Apart from the discussion around Mach's principle, Gödel's discovery has had an important impact on the development of modern cosmology. It led to a deeper understanding of

cosmological solutions of Einstein's field equations and the development of more realistic models.

Let us finally ask whether there are rotational effects of the Gödelian type in *our* universe. Do observations indicate that the universe is rotating "around us"? The answer is that if there are such effects at all, they must be very small. The best bounds currently available have been obtained from observations on the cosmic microwave background radiation: All astrophysical and cosmological observations achieved so far are compatible with a Gödelian rotation of our universe provided that  $\omega < 10^{-9}$  arc seconds per century. This value which is much smaller than the angular frequencies we encountered in the above discussion of Machian effects near the earth! Recently, the use quantum gyroscopes has been suggested in order to get an even smaller bound.

Summarizing, the work of Mach, Thirring & Lense and Gödel have profoundly shaped the modern view of the nature of space, time, inertia and rotation.

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