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BELL'S INEQUALITIES AND DENSITY MATRICES. REVEALING "HIDDEN" NONLOCALITY.

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Abstract

As is well known, quantum mechanical behavior cannot, in general, be simulated by a local hidden variables model. Most -if not all- the proofs of this incompatibility refer to the correlations which arise when each of two (or more) systems separated in space is subjected

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to a single ideal measurement. This setting is good enough to show contradictions between local hidden variables models and quantum mechanics in the case of pure states. However, as shown here, it is not powerful enough in the case of mixtures. This is illustrated by an example. In this example, the correlations which arise when each of two systems separated in space is subjected to a single ideal measurement are classical; only when each system is subjected to a *sequence* of ideal measurements non-classical correlations are obtained. We also ask whether there are situations for which even this last procedure is not powerful enough and non-ideal measurements have to be considered as well.

As it is well known, when measurements are performed on two quantum systems separated in space their results are correlated in a manner which, in general, cannot be explained by a local hidden variables model. But 30 years after Bell's pioneering paper [1] we still lack a complete classification of quantum states into local and nonlocal ones. While the case of pure states is completely solved [2], for density matrices only partial results have been obtained so far [3]. In this letter I show that for solving this problem we have to change the usual way we think at Bell's inequalities and local hidden variable models.

Bell's original proof and most -if not all- the subsequent alternative proofs [4] (with or without inequalities) have a common aspect: They consider the case in which each of the two systems is subjected to a *single ideal* local measurement (chosen at random among many possible ideal measurements). For example in the case of spin 1/2 particles one usually considers that each particle is subjected to a single Stern-Gerlach measurement, that is, to a measurement of the spin along some arbitrary direction. More general, by ideal measurements I mean measurements as defined in the postulates of quantum mechanics [von Neumann, Dirac], that is, when a system is described by an n-dimensional Hilbert space, the observables to which measurements refer correspond to hermitian operators acting on this n-dimensional space, and the only possible results of a measurement are the eigenvalues of the measured operator. After such a measurement the state of the measured system becomes the projection of the initial state onto the subspace corresponding to the observed eigenvalue. Of course in principle one could consider that each of the two space separated systems is subjected to much more complicated experiments, that is, to a *sequence* of *non ideal* measurements instead of a single ideal one. Clearly, to prove that a given quantum state is nonlocal, it is enough to show that it violates some usual Bell inequality (obtained by considering a single ideal measurement performed on each side). On the other hand, to prove that a quantum state is local, one must show that the correlations between the results of *any* local experiments can be described by a local hidden variables model.

The above being said, the question still remains whether considering general local measurements gives us essential new information about the locality or nonlocality of quantum states. In this letter I will show that the answer is to this question is "yes" .

To put things in a better perspective, let us recall first the status of pure states. Although it might well be that considering general measurements will yield new and better inequalities, the basic question of which pure states are classical or not can be answered without going beyond the usual scheme. Indeed, as is well known [2] every entangled pure state violates some usual Bell inequality and it is therefore nonlocal. The only pure states which do not yield nonlocal correlations when a single ideal measurement is performed on each particle are the direct product states and they are obviously local. On the other hand, as I will show, the situation turns out to be different when one deals with mixed states. More precisely, I will give an example in which if each of two systems separated in space is subjected to a single ideal measurement the resulting correlations are classical but if each system is subjected to more complicated experiments the resulting correlations are nonlocal. The "more complicated experiments" I consider here are sequences of two ideal measurements.

The density matrices I analyze in this paper were discovered by Werner [5]. Consider two systems separated in space, each system living in a d dimensional Hilbert space, and let the (mixed) state of the two systems be

$$W = \frac{1}{d^2}\Big(\frac{1}{d}I^{(d\times d)} + 2\sum_{i<j;i,j=1}^{d} |S_{ij}\rangle\langle S_{ij}|\Big), \tag{1}$$

where $I^{(d\times d)}$ is the identity matrix in the $d \times d$ dimensional space of the two systems and $|S_{ij}\rangle$ is the "spin 1/2 singlet" state

$$|S_{ij}\rangle = \frac{1}{\sqrt{2}}\big(|i\rangle_1|j\rangle_2 - |j\rangle_1|i\rangle_2\big), \tag{2}$$

where $\{|i\rangle_1\}$ and $\{|j\rangle_2\}$ denote orthogonal bases in the hilbert spaces of particle 1 and 2 respectively. (The form (1) of the matrix W can be easily obtained from the original form given by Werner by noting that the "flip" operator V which appears in [5] can be expressed as $V = I - 2\sum_{i<j} |S_{ij}\rangle\langle S_{ij}|$.)

An important property of these density matrices is that they cannot be decomposed into mixtures of pure direct product states. As direct products are the onlyclassical pure states, and as Werner's matrices are not mixtures of such states, one feels intuitively that these density matrices are nonclassical. However Werner has proven that these density matrices do not violate any standard Bell inequality. That is, when each system is subjected to a single ideal measurement the resulting correlations are classical. (Werner has explicitly writtena local hidden variables model which simulates these correlations.) But is there a local hidden variables model which can simulate the results of *any* measurements performed on the two systems prepared in such a mixed state?

Very recently progress has been made in understanding the properties of these strange states. It has been shown [6] that when two spin 1/2 particles are in such a state (Werner's 2×2 dimensional matrix) they form a "quantum channel" which can be used for (imperfect) teleportation. Thus the 2×2 dimensional Werner matrix has indeed nonclassical aspects. However teleportation does not involve local measurements performed on each of the spins outside the light cone of each other, so from the fact that such a matrix can be used for teleportation we do not learn immediately anything about the nonlocal correlations it might generate. Nevertheless this suggests that we should consider more carefully the question of nonlocality of Werner type matrices and go beyond the usual scheme in which each system is subjected to a single ideal measurement.

In this letter I will consider Werner type matrices of dimension $\geq 5\times5$. Suppose now that each of the two particles is subjected to two consecutive ideal measurements. First each particle is subjected to a measurement of a 2-dimensional projection operator,

$$P = |1\rangle_{11}\langle 1| + |2\rangle_{11}\langle 2| \tag{3}$$

on particle 1 and

$$Q = |1\rangle_{22}\langle 1| + |2\rangle_{22}\langle 2| \tag{4}$$

on particle 2. *After* the first measurement is performed and its result is registered an observer situated near particle 1 chooses at random to measure one of the two operators A or A', whose exact form will be described below. It is important to note that the decision which operator to measure is taken only after the measurement of P is completed. Similarly, an observer situated near particle 2 chooses at random between a measurement of B or B'. This scheme is almost identical to the one usually used for deriving Bell's inequalities with the difference that A, A', B and B' are not measured directly on the particles in the original state but after the measurements of P and Q respectively.

The operators A, A', B and B' have each three different eigenvalues, 1, -1 and 0. The eigenvalues 1 and -1 are nondegenerate and the corresponding eigenstetes belong to the subspaces $\{|1\rangle_1, |2\rangle_1\}$ and $\{|1\rangle_2, |2\rangle_2\}$ respectively. The eigenvalue 0 is highly degenerate and corresponds to the rest of the hilbert spaces, that is to $\{|3\rangle_1, ..., |d\rangle_1\}$ and $\{|3\rangle_2, ..., |d\rangle_2\}$ respectively. The nondegenerate part of these operators is chosen such that they yield maximal violation of the Clauser, Horne, Shimony and Holt (CHSH) inequality [7] for the singlet state $|S_{12}\rangle$, that is

$$\langle S_{12}|AB + AB' + A'B - A'B'|S_{12}\rangle = 2\sqrt{2}. \tag{5}$$

Let us see now what happens if we start with an ensemble of pairs of particles in a Werner state $W^{(d\times d)}$ and subject each particle to the measurements described above. According to the results obtained in the measurements of P and Q the original ensemble splits into four subensembles (corresponding to the results $\{0,0\}$, $\{0,1\}$, $\{1,0\}$ and $\{1,1\}$). The most important point is that if the initial ensemble is classical, behaving according to a hidden variables model, than each of these four subensembles is classical. But then we get a contradiction with the quantum mechanical predictions. Indeed, the ensemble corresponding to P=1 and Q=1 is described by

$$W' = \frac{1}{N}PQWQP = \frac{2d}{2d+4}\left(\frac{1}{2d}I^{(2\times 2)} + |S_{12}\rangle\langle S_{12}|\right), \tag{6}$$

where N is a normalization factor and and $I^{(2\times 2)}$ is a 2×2 identity matrix acting in the $\{|1\rangle_1, |2\rangle_1\} \otimes \{|1\rangle_2, |2\rangle_2\}$subspace and zero in rest. In this state the CHSH inequality is violated. Indeed,

$$TrW'(AB + AB' + A'B - A'B') = \frac{2d}{2d+4} 2\sqrt{2} \geq 2 \quad for \quad d \geq 5. \tag{7}$$

In conclusion, although a local hidden variables model can simulate all the correlations which arise when only a single, ideal measurement is performed on each of the two particles, such a model cannot account for the correlations which arise when two consecutive measurements are performed on each particle.

To better understand the above result, let us compare the case in which the measurements of A or A' and B or B' take place directly on the original state W and the case when P and Q were measured first. According to quantum mechanics the correlations between the outcomes of A, A', B and B' are the same in both cases, as all these operators commute with P and Q. Still, a local hidden variables model can simulate these correlations in the first case but not in the second one. This happens because for a hidden variables model the two situations are dramatically different. Suppose first that one of the operators A or A' is measured on particle 1 directly in the initial state. Then the particle has from the beginning all the information about the detailed question which is asked about the $\{|1\rangle_1, |2\rangle_1\}$ subspace and it can use this information to avoid "unpleasant" questions. Indeed, quantum mechanics imposes that a measurement of A yields the outcome 0 with the same probability as a measurement of A' (as the corresponding eigenspaces are identical). In a local hidden variables model this means that

$$\int d\lambda \rho(\lambda) P_1(A = 0, \lambda) = \int d\lambda \rho(\lambda) P_1(A' = 0, \lambda), \tag{8}$$

where λ is the hidden variable, $\rho(\lambda)$ is the probability distribution of λ over the initial ensemble and $P_1(A = 0, \lambda)$ and $P_1(A' = 0, \lambda)$ are the probabilities that particle 1 gives the answers A=0 and A'=0 respectively. But from eq. (8) it *does not* follow that

$$P_1(A=0,\lambda)=P_1(A'=0,\lambda). \tag{9}$$

The same is true for the sum of probabilities to obtain the results 1 and -1, that is,

$$\int d\lambda\rho(\lambda)\big[P_1(A=1,\lambda)+P_1(A=-1,\lambda)\big]=\int d\lambda\rho(\lambda)\big[P_1(A'=1,\lambda)+P_1(A'=-1,\lambda)\big], \tag{10}$$

but in general

$$P_1(A=1,\lambda)+P_1(A=-1,\lambda)\neq P_1(A'=1,\lambda)+P_1(A'=-1,\lambda). \tag{11}$$

The meaning of these relations is that particle 1, in some pairs of the original ensemble, might have a local hidden variable according to which if it is subjected to a measurement of A, say, it will yield one of the answers 1 or -1, but if it is subjected instead to a measurement of A', it will yield 0. Analogously, similar behaviour could characterize also particle 2 in some of the pairs. This freedom to choose between ±1 and 0, depending on the measurement to which the particles are subjected, is vital for the success of Werner's local hidden variables model, as he himself remarked. (Actually in Werner's model only particle 1 in some of the pairs uses this freedom, while particle 2 is less sophisticated.) On the other hand, if we first subject particle 1 to a measurement of P, the particle commits itself to yield, in a subsequent measurement of either A or A', one of the outcomes ±1 or 0 *before* knowing which of these measurements will actually be performed. Indeed, $P=1$ forces the particle to yield 1 or -1 subsequently no matter if A or A' is measured; the outcome 0 is no longer a valid option. Similarly, $P=0$ forces the outcome 0 in a subsequent measurement of either A or A'. At this point no local hidden variables model could simulate the quantum mechanical behavior.

Another way of understanding the role of the measurements of P and Q is the following: they are used to select a subensemble of pairs from the original ensemble in a way *independent* of the measurements which are used to test it. Because of this independence one could apply the CHSH inequality directly to this subensemble and conclude that the larger than 2 value of the CHSH correlation predicted by quantum mechanics for this subensemble (eq. 7) is incompatible with a local hidden variables model. On the other hand, one could think of measuring A or A' and B or B' directly on the original ensemble and then selecting the subensemble of pairs for which both particle 1 and particle 2 have yielded ± 1. But in a local hidden variables model such as Werner's, the subensemble selected this way *depends* on the measurements used to test it. Indeed, suppose that particle 1 in a particular pair has a local hidden variable according to which, if it is subjected to a measurement of A, it will yield, say, 1, while to a measurement of A' it will yield 0. Then if it happens that particle 1 is subjected to a measurement of A, the pair might be included in the selected subensemble (depending on a ± 1 outcome yielded by particle 2), while if it happens that particle 1 is subjected to a measurement of A' the pair will not be selected. For such test-dependent subensembles Bell's inequalities cannot be applied directly and, as Werner shows, values larger than 2 for the CHSH correlations can be consistent with local hidden variables models. Other, much more famous, examples of test-dependent subensembles and apparent Bell's inequality violations are the "detector efficiency" problem, and the "space distribution" problem raised by E. Santos [8]. Another example of this type, probably the simplest possible, has been given by D. Mermin [9].

There are, of course, many open questions left. In the example presented in this letter the nonlocality of this special state was revealed by considering a sequence of ideal measurements. But it might be that a single but *non-ideal* measurement performed on each particle suffices. For example, we could bring along each of the original two particles some supplementary quantum system (called sometimes "ancilla") and then perform on each side an ideal measurement on the original particle and the corresponding ancilla together. Such a measurement can, in principle yield more outcomes that d, the dimension of the hilbert space of the original particle. This scheme is known in the literature as the implementation of a positive valued operator measure (POVM) and it is known that in certain situations it can yield more information about a quantum system than an ideal measurement performed only on the system itself. Using non-ideal measurements is obviously necessary in the case of Werner's 2×2 matrix. Indeed, in this case the strategy of subjecting each system to a sequence of ideal measurements does not work, as after the first measurement the systems are left in a direct product state and therefore any subsequent measurements yield classical correlations.

Finally, an open question is what is the class of classical states. Is it the case that only the mixtures which are decomposable as a sum of direct product states are classical and all the others are not, although superficially they seem to be? As a first test case one should probably attack the remaining Werner type matrices ($d \leq 4$).

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References

1) J. S. Bell, Physics 1, 195, (1964).

2) N. Gisin, Phys. Lett. A 145, 201, (1991); S. Popescu and D. Rohrlich, Phys. Lett. A 166, 293, (1992).

3) S. Braunstein, A. Mann and M. Revzen, Phys. Rev. Lett. 68, 3259, (1992); A. Mann, K. Nakamura and M. Revzen, J. of Phys. A L851, (1992).

4) For an extensive reference list see L. Khalfin and B. Tsirelson, Found. of Phys. 22, 879, (1992).

5) R. F. Werner, Phys. Rev. A 40, 4277, (1989).

6) S. Popescu, Phys. Rev. Lett. 72, 797, (1994).

7) J. Clauser, M. Horne, A. Shimony and R. Holt, Phys. Rev. Lett. 23, 880, (1969).

8) E. Santos, Phys. Rev. Lett. 66, 1388, (1991); Phys. Rev. Lett. 68, 2702, (1992); Phys. Rev. A 46, 3646, (1992).

9) D. Mermin, "Boojums All the Way Through", Cambridge University Press, NY (1990) pps. 167-169.