Alternating sign matrices and totally symmetric plane partitions

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joint work with:
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Applied Algebra Seminar - York University

Outline

- Alternating sign matrices
- Plane partitions
- Refined enumerations of alternating sign matrices
- The interplay of ASMs, TSPPs and Schur polynomials

Alternating sign matrices

Definition

An alternating sign matrix (or short ASM) of size n is an $n \times n$ matrix with entries $1, 0, -1$, such that

- all row- and column-sums are equal to 1,
- in each row and column, the non-zero entries alternate.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Motivation

- ASMs appear in different areas of Mathematics: λ-determinant, MacNeille completion of the Bruhat order on S_n, connections to representation theory, symmetric functions.

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- ASMs appear in different areas of Mathematics: λ-determinant, MacNeille completion of the Bruhat order on S_n, connections to representation theory, symmetric functions.
- They are connected to statistical physics: six-vertex model (square ice), Razumov-Stroganov-Cantini-Sportiello Theorem, $O(\tau)$ loop model.
- ASMs are mysteriously connected to certain families of plane partitions.

Enumeration of ASMs

The enumeration formula for ASMs was conjectured by Robbins and Rumsey in the early 1980s.

Theorem (Zeilberger, 1996)

The number of ASMs of size n is given by

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

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Further proofs were found by

- Kuperberg in 1996 using the six-vertex model approach,
- Fischer in 2007 using her *operator formula* (a short version of this paper appeared in 2016),
- A. in 2018 which is also based on the operator formula.

Plane partitions

Definition

A *plane partition* $\pi = (\pi_{i,j})$ is an (a, b, c)-box is an array of non-negative integers

$$\begin{matrix} \pi_{1,1} & \pi_{1,2} & \cdots & \pi_{1,b} \\ \pi_{2,1} & \pi_{2,2} & \cdots & \pi_{2,b} \\ \vdots & \vdots & & \vdots \\ \pi_{a,1} & \pi_{a,2} & \cdots & \pi_{a,b} \end{matrix}$$

such that $\pi_{i,j} \leq c$ and all rows and columns are weakly decreasing.

Plane partitions were first introduced by MacMahon in the end of the 19th century.

Five times plane partitions

4	3	3	1
4	2	1	0
2	0	0	0

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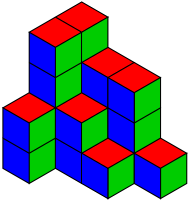

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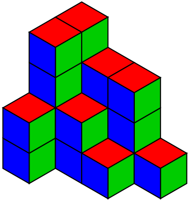

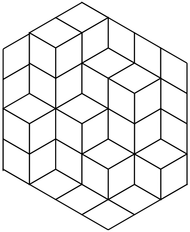

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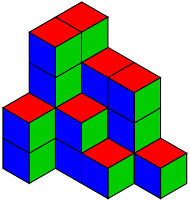
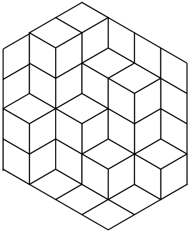
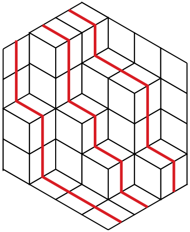
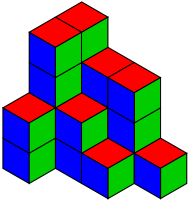


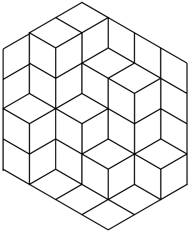


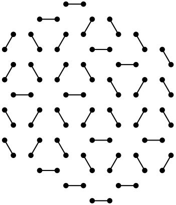


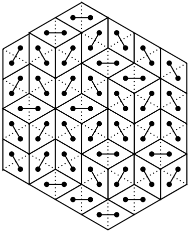
Five times plane partitions

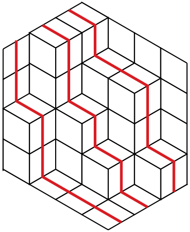
$$\begin{matrix} 4 & 3 & 3 & 1 \\ 4 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \end{matrix}$$



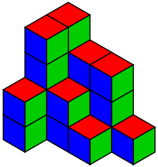




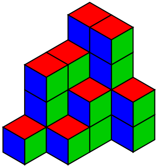




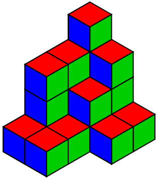
Symmetry operations on plane partitions



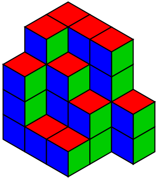
π



reflection / transposition



rotation



completion π^c

$$\pi^c_{i,j} := c - \pi_{a+1-i,b+1-j}$$

Three important symmetry classes

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Theorem (Andrews 1979, 1994, Zeilberger 1996)

ASMs of size n, 0-DPPs where the entries are at most n and TSSCPPs inside an $(2n, 2n, 2n)$-box are equinumerous.

Refined enumerations of ASMs

Let A be an ASM of size n.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We denote by $\rho(A)$ the column of the unique 1 in the top row of A and define the inversion number $\operatorname{inv}(A)$ and complementary inversion number $\operatorname{inv}'(A)$ by

$$\operatorname{inv}(A) := \sum_{\substack{1 \le i' < i \le n \\ 1 \le j' \le j \le n}} a_{i',j} a_{i,j'} \qquad \operatorname{inv}'(A) := \sum_{\substack{1 \le i' < i \le n \\ 1 \le j \le j' \le n}} a_{i',j} a_{i,j'},$$

In the above example we have $(\rho(A), \operatorname{inv}(A), \operatorname{inv}'(A)) = (2, 3, 2)$.

An antisymmetrizer formula

For a positive integer n, define

$$\mathcal{A}_n(u,v;\mathbf{x}) := \frac{\mathbf{ASym}_{x_1,\ldots,x_n}\left[\prod_{i=1}^{n} x_i^{i-1} \prod_{1\le i<j\le n}(v+(1-u-v)x_i+ux_ix_j)\right]}{\prod_{1\le i<j\le n}(x_j-x_i)},$$

where $\mathbf{ASym}_{x_1,\ldots,x_n} f(x_1,\ldots,x_n) = \sum_{\sigma\mathfrak{S}_n} \operatorname{sgn}(\sigma) f(x_{\sigma(1)},\ldots,x_{\sigma(n)})$.

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Theorem (AFKNT, 2020)

The number of ASMs A of size n with $(\rho(A),\operatorname{inv}(A),\operatorname{inv}'(A)) = (a,b,c)$ is the coefficient of $z^{a-1}u^bv^c$ in $\mathcal{A}_n(u,v;z,1,\ldots,1)$.

This is a generalisation of a result by Fischer-Riegler 2015.

Some examples

$$\begin{aligned}\mathcal{A}_1(u,v;\mathbf{x}) &= 1,\\ \mathcal{A}_2(u,v;\mathbf{x}) &= v + u x_1 x_2\\ \mathcal{A}_3(u,v;\mathbf{x}) &= v^3 + uv^2(x_1x_2 + x_1x_3 + x_2x_3) + uv(1-u-v)x_1x_2x_3\\ &\quad + u^2v(x_2x_3x_1^2 + x_2x_3^2x_1 + x_2^2x_3x_1) + u^3x_1^2x_2^2x_3^2,\end{aligned}$$

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Some examples

$$\begin{aligned}\mathcal{A}_4(u,v;\mathbf{x}) &= v^6 + uv^5 s_{(1,1)}(\mathbf{x}) + (1-u-v)uv^4 s_{(1,1,1)}(\mathbf{x}) \\ &\quad + (1-u-v)^2 uv^3 s_{(1,1,1,1)}(\mathbf{x}) + u^2v^4 s_{(2,1,1)}(\mathbf{x}) \\ &\quad + 2(1-u-v)u^2v^3 s_{(2,1,1,1)}(\mathbf{x}) + u^3v^3 s_{(2,2,2)}(\mathbf{x}) \\ &\quad + 2(1-u-v)u^3v^2 s_{(2,2,2,1)}(\mathbf{x}) + u^3v^3 s_{(3,1,1,1)}(\mathbf{x}) \\ &\quad + (1-u-v)^2 u^3 v s_{(2,2,2,2)}(\mathbf{x}) + u^4v^2 s_{(3,2,2,1)}(\mathbf{x}) \\ &\quad + (1-u-v)u^4 v s_{(3,2,2,2)}(\mathbf{x}) + u^5 v s_{(3,3,2,2)}(\mathbf{x}) \\ &\quad + u^6 s_{(3,3,3,3)}(\mathbf{x}).\end{aligned}$$

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- Is $\mathcal{A}_n$ 'Schur-positive'?
- Can we describe the coefficients?
- Which Schur polynomials appear in $\mathcal{A}_n$?

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n	#(different Schur polynomials)
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2	2
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Definition

A partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is called *modified balanced* of size n if $\lambda_1 \leq n-1$ and $\lambda_i < \lambda'_i$ whenever $\lambda_i \geq i$, where $\lambda' = (\lambda'_1, \ldots, \lambda'_n)$ denotes the conjugate partition.

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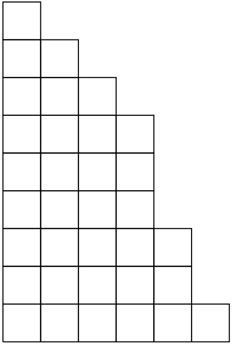
Denote by l the length of the Durfee square of λ. The Frobenius notation of λ is $(\lambda_1 - 1, \ldots, \lambda_l - l | \lambda_1' - 1, \ldots, \lambda_l' - l)$. A partition $\lambda = (a_1, \ldots, a_l | b_1, \ldots, b_l)$ is modified balanced iff $a_i < b_i$ for $1 \leq i \leq l$.

An example



$$\lambda = (6, 5, 5, 4, 4, 4, 3, 2, 1)$$

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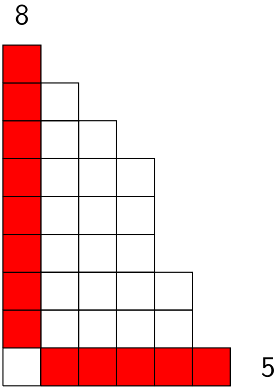

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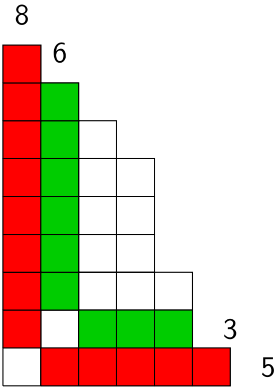

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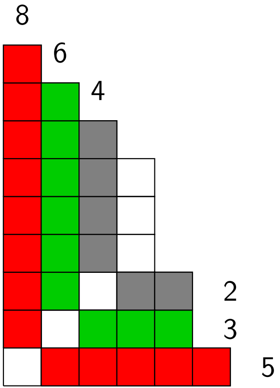

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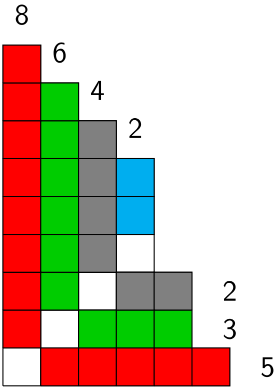

$$\lambda = (6,5,5,4,4,4,3,2,1)$$
$$= (5,3,2,0|8,6,4,2)$$

A bijection to Dyck paths

Given a modified balanced partition $\lambda = (a_1, \ldots, a_l | b_1, \ldots, b_l)$ of size n we can construct a Dyck path of length $2n$ bijectively via

$$\lambda \mapsto N^{b_l} E^{a_l+1} N^{b_{l-1}-b_l} E^{a_{l-1}-a_l} \cdots N^{b_2-b_1} E^{a_2-a_1} N^{n-b_1} E^{n-a_1-1}.$$

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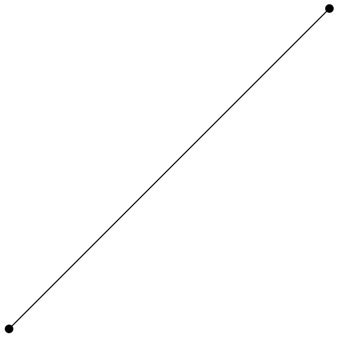
For $\lambda = (5, 3, 2, 0 | 8, 6, 4, 2)$ as before and $n = 9$, we obtain



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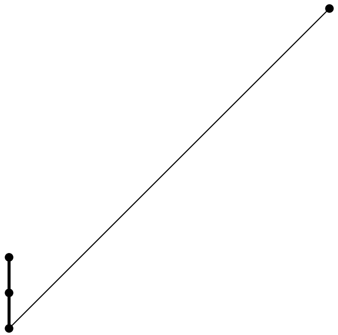

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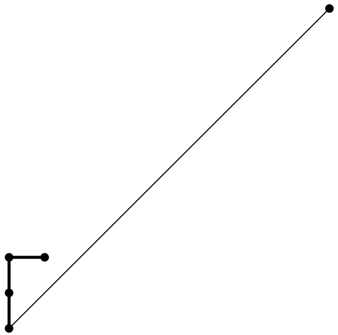

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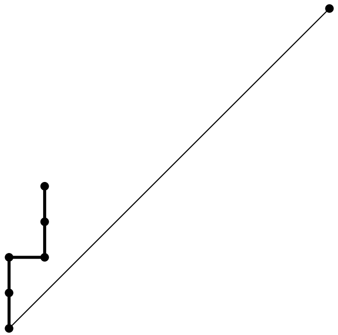

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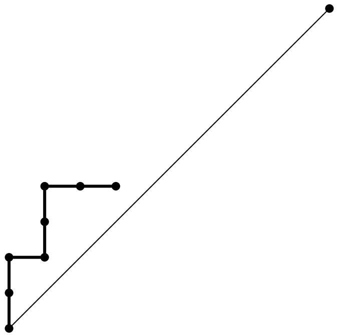

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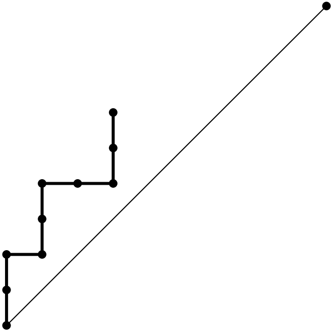

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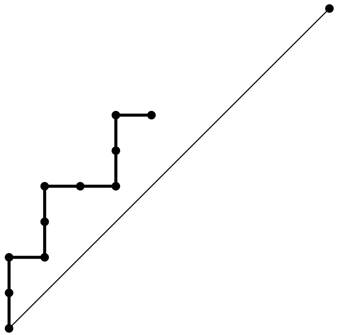

A bijection to Dyck paths

Given a modified balanced partition $\lambda = (a_1, \ldots, a_l | b_1, \ldots, b_l)$ of size n we can construct a Dyck path of length $2n$ bijectively via

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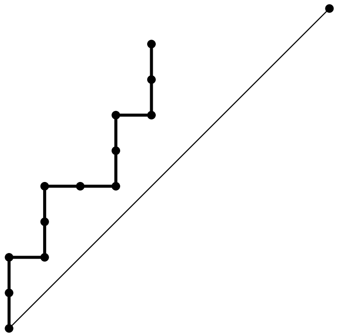

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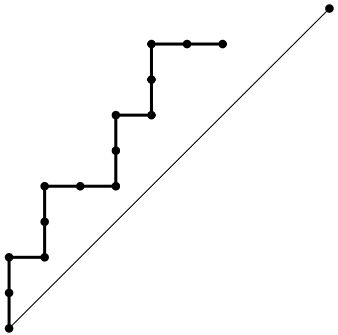

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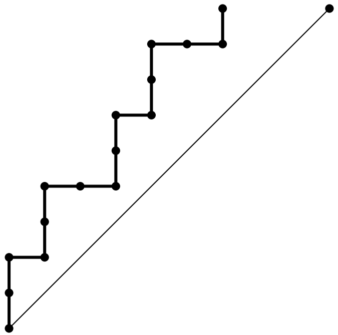

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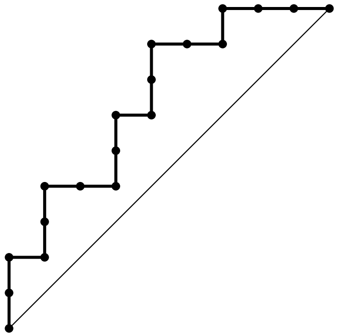

What are the coefficients in $\mathcal{A}_n$

$$\begin{aligned}\mathcal{A}_4(u,v;\mathbf{x}) = v^6 + uv^5 s_{(1,1)}(\mathbf{x}) + (1-u-v)uv^4 s_{(1,1,1)}(\mathbf{x}) \\ + (1-u-v)^2 uv^3 s_{(1,1,1,1)}(\mathbf{x}) + u^2v^4 s_{(2,1,1)}(\mathbf{x}) \\ + 2(1-u-v)u^2v^3 s_{(2,1,1,1)}(\mathbf{x}) + u^3v^3 s_{(2,2,2)}(\mathbf{x}) \\ + 2(1-u-v)u^3v^2 s_{(2,2,2,1)}(\mathbf{x}) + (1-u-v)^2 u^3 v s_{(2,2,2,2)}(\mathbf{x}) \\ + u^3v^3 s_{(3,1,1,1)} + u^4v^2 s_{(3,2,2,1)} + (1-u-v)u^4 v s_{(3,2,2,2)}(\mathbf{x}) \\ + u^5 v s_{(3,3,2,2)} + u^6 s_{(3,3,3,3)}\end{aligned}$$

What are the coefficients in $\mathcal{A}_n$

$$\begin{aligned}\mathcal{A}_4(u,v;\mathbf{x}) = {} & 1v^6 + 1uv^5 s_{(1,1)}(\mathbf{x}) + 1(1-u-v)uv^4 s_{(1,1,1)}(\mathbf{x}) \\ & + 1(1-u-v)^2uv^3 s_{(1,1,1,1)}(\mathbf{x}) + 1u^2v^4 s_{(2,1,1)}(\mathbf{x}) \\ & + 2(1-u-v)u^2v^3 s_{(2,1,1,1)}(\mathbf{x}) + 1u^3v^3 s_{(2,2,2)}(\mathbf{x}) \\ & + 2(1-u-v)u^3v^2 s_{(2,2,2,1)}(\mathbf{x}) + 1(1-u-v)^2u^3v s_{(2,2,2,2)}(\mathbf{x}) \\ & + 1u^3v^3 s_{(3,1,1,1)} + 1u^4v^2 s_{(3,2,2,1)} + 1(1-u-v)u^4v s_{(3,2,2,2)}(\mathbf{x}) \\ & + 1u^5v s_{(3,3,2,2)} + 1u^6 s_{(3,3,3,3)}\end{aligned}$$

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$$
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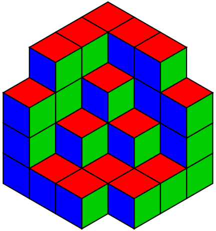
& + 1u^5v s_{(3,3,2,2)} + 1u^6 s_{(3,3,3,3)}
\end{aligned}
$$

n	sum of coefficients
1	1
2	2
3	5
4	16
5	66
6	352

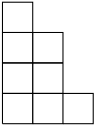
TSPPs with given diagonal I

Denote by TSPP_n the set of totally symmetric plane partitions inside an (n, n, n)-box. For $T \in \mathrm{TSPP}_n$ define

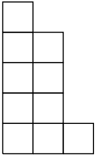
$$\operatorname{diag}(T) := (T_{i,i})' = (a_1, \ldots, a_l | b_1, \ldots, b_l)$$
$$\pi(T) := (a_1, \ldots, a_l | b_1 + 1, \ldots, b_l + 1).$$



T



$\operatorname{diag}(T)$



$\pi(T)$

TSPPs with given diagonal II

Proposition (AFKNT, 2020)

1. *The map* $T \mapsto \pi(T)$ *is a surjection from totally symmetric plane partitions inside an* (n, n, n)*-box to modified balanced partitions of size* $n + 1$*.*
2. *Let* $\lambda = (a_1, \ldots, a_l | b_1, \ldots, b_l)$ *be a modified balanced partition. The number of TSPPs* T *with* $\pi(T) = \lambda$ *is equal to*

$$\det_{1 \leq i,j \leq l} \left(\binom{b_j - 1}{a_i} \right).$$

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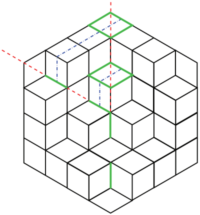


The main Theorem

For a modified balanced partition $\lambda = (a_1, \ldots, a_l | b_1, \ldots, b_l)$ define

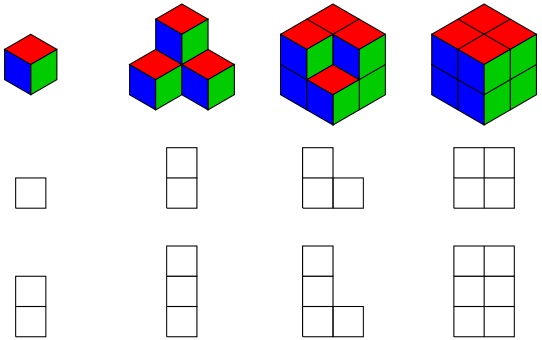
$$\omega_\lambda(u,v) := u^{\sum_i (a_i+1)} (1-u-v)^{\sum_i (b_i - a_i)} v^{\binom{n}{2} - \sum_i b_i}.$$

Theorem (AFKNT, 2020)

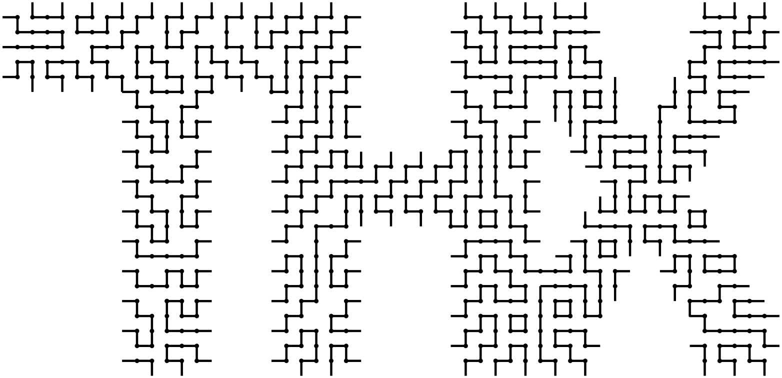
Let n be a positive integer. Then

$$\mathcal{A}_n(u,v;\mathbf{x}) = \sum_{T \in \mathrm{TSPP}_{n-1}} \omega_{\pi(T)}(u,v) s_{\pi(T)}(\mathbf{x}).$$

An explicit example for $n = 3$



T:	$\emptyset$				
$\mathrm{diag}(T)$:	$\emptyset$				
$\pi(T)$:	$\emptyset$				
	$(\vert)$	$(0\vert 1)$	$(0\vert 2)$	$(1\vert 2)$	$(1,0\vert 2,1)$
$\omega_{\pi(T)}(u,v))$:	v^3	uv^2	$u(1-u-v)v$	u^2v	u^3


THX