Fully packed loop configurations: polynomiality and nested arches

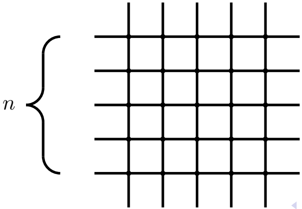
Florian Aigner
University of Vienna

77$^{\text{th}}$ Séminaire Lotharingien de Combinatoire
Strobl, 13.9.2016

Fully packed loop configurations

Definition

A fully packed loop configuration (FPL) F of size n is a subgraph of the $n \times n$ grid with n external edges on every side s.t.:

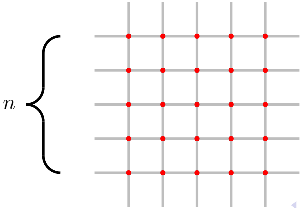


Fully packed loop configurations

Definition

A fully packed loop configuration (FPL) F of size n is a subgraph of the $n \times n$ grid with n external edges on every side s.t.:

- F contains all degree 4 vertices of the $n \times n$ grid and every degree 4 vertices of the grid has degree 2 in F.

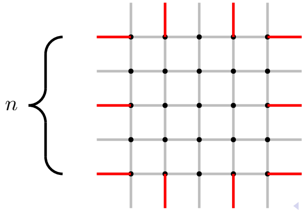


Fully packed loop configurations

Definition

A fully packed loop configuration (FPL) F of size n is a subgraph of the $n \times n$ grid with n external edges on every side s.t.:

- F contains all degree 4 vertices of the $n \times n$ grid and every degree 4 vertices of the grid has degree 2 in F.
- F contains every other external edge, beginning with the topmost at the left side.

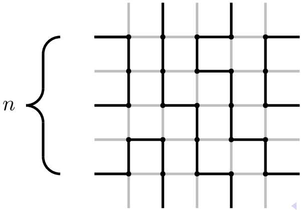


Fully packed loop configurations

Definition

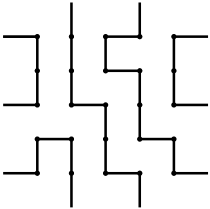
A fully packed loop configuration (FPL) F of size n is a subgraph of the $n \times n$ grid with n external edges on every side s.t.:

- F contains all degree 4 vertices of the $n \times n$ grid and every degree 4 vertices of the grid has degree 2 in F.
- F contains every other external edge, beginning with the topmost at the left side.



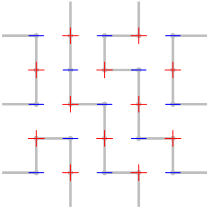
Why are FPLs interesting?

1. FPLs are in bijection to ASMs



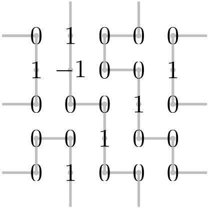
Why are FPLs interesting?

1. FPLs are in bijection to ASMs



Why are FPLs interesting?

1. FPLs are in bijection to ASMs



Why are FPLs interesting?

1. FPLs are in bijection to ASMs

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{matrix}$$

Why are FPLs interesting?

1. FPLs are in bijection to ASMs

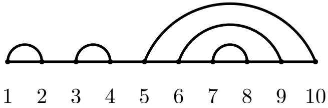
$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{matrix}$$

2. Connection to statistical physics

Noncrossing matchings

Definition

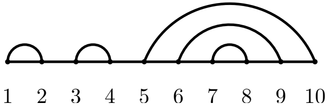
A noncrossing (nc) matching π of size $2n$ is a matching of the numbers $1, \ldots, 2n$ by arches such that no two arches cross. We denote by NC_{2n} the set of nc matchings of size $2n$.



Noncrossing matchings

Definition

A noncrossing (nc) matching π of size $2n$ is a matching of the numbers $1, \ldots, 2n$ by arches such that no two arches cross. We denote by NC_{2n} the set of nc matchings of size $2n$.

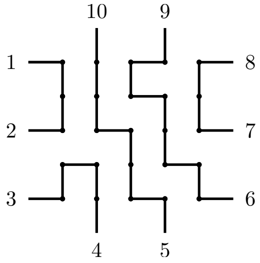


We write $()_m$ for the nc matching consisting out of m nested arches. The above nc matching is $()()()_3$.

Link pattern

Definition

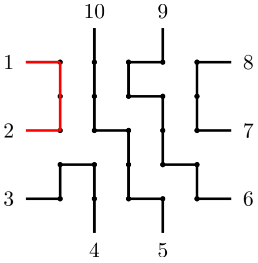
We number the external edges of a FPL F counter-clockwise with 1 up to $2n$. The link pattern $\pi(F)$ is the noncrossing matching such that i and j are matched in $\pi(F)$ iff i and j are connected by a path in F.

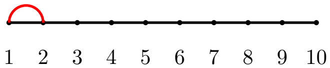


Link pattern

Definition

We number the external edges of a FPL F counter-clockwise with 1 up to $2n$. The link pattern $\pi(F)$ is the noncrossing matching such that i and j are matched in $\pi(F)$ iff i and j are connected by a path in F.

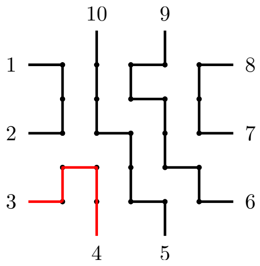


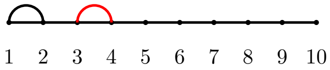


Link pattern

Definition

We number the external edges of a FPL F counter-clockwise with 1 up to $2n$. The link pattern $\pi(F)$ is the noncrossing matching such that i and j are matched in $\pi(F)$ iff i and j are connected by a path in F.

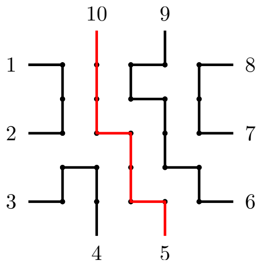


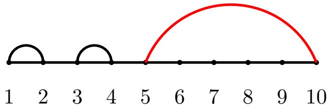


Link pattern

Definition

We number the external edges of a FPL F counter-clockwise with 1 up to $2n$. The link pattern $\pi(F)$ is the noncrossing matching such that i and j are matched in $\pi(F)$ iff i and j are connected by a path in F.

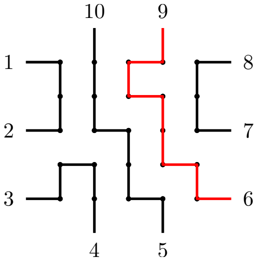


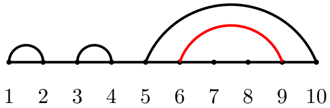


Link pattern

Definition

We number the external edges of a FPL F counter-clockwise with 1 up to $2n$. The link pattern $\pi(F)$ is the noncrossing matching such that i and j are matched in $\pi(F)$ iff i and j are connected by a path in F.

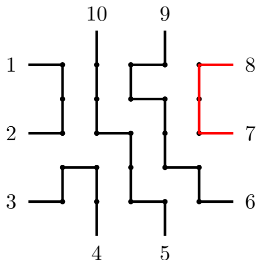


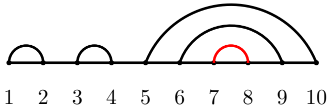


Link pattern

Definition

We number the external edges of a FPL F counter-clockwise with 1 up to $2n$. The link pattern $\pi(F)$ is the noncrossing matching such that i and j are matched in $\pi(F)$ iff i and j are connected by a path in F.

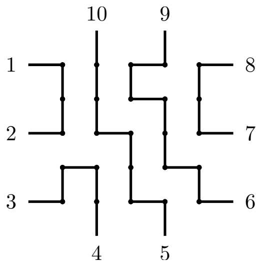


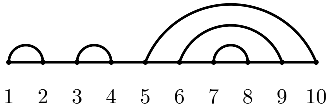


Link pattern

Definition

We number the external edges of a FPL F counter-clockwise with 1 up to $2n$. The link pattern $\pi(F)$ is the noncrossing matching such that i and j are matched in $\pi(F)$ iff i and j are connected by a path in F.





From noncrossing matchings to Young diagrams

Let $\pi \in \mathrm{NC}_{2n}$. Denote by $\lambda(\pi)$ the Young diagram obtained by the following algorithm:

From noncrossing matchings to Young diagrams

Let $\pi \in \mathrm{NC}_{2n}$. Denote by $\lambda(\pi)$ the Young diagram obtained by the following algorithm:

- Draw a north-step if an arc is open.

From noncrossing matchings to Young diagrams

Let $\pi \in \mathrm{NC}_{2n}$. Denote by $\lambda(\pi)$ the Young diagram obtained by the following algorithm:

- Draw a north-step if an arc is open.
- Draw a east-step if an arc is closed.

From noncrossing matchings to Young diagrams

Let $\pi \in \mathrm{NC}_{2n}$. Denote by $\lambda(\pi)$ the Young diagram obtained by the following algorithm:

- Draw a north-step if an arc is open.
- Draw a east-step if an arc is closed.
- The Young diagram $\lambda(\pi)$ is the area between the above path and the path which consists out of n consecutive north-steps followed by n consecutive east-steps.

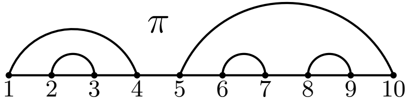
From noncrossing matchings to Young diagrams

Let $\pi \in \mathrm{NC}_{2n}$. Denote by $\lambda(\pi)$ the Young diagram obtained by the following algorithm:

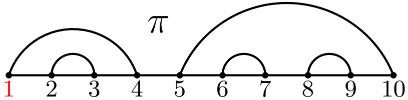
- Draw a north-step if an arc is open.
- Draw a east-step if an arc is closed.
- The Young diagram $\lambda(\pi)$ is the area between the above path and the path which consists out of n consecutive north-steps followed by n consecutive east-steps.

This yields an bijection between NC_{2n} and the Young diagrams with at most $n-i$ boxes in the i-th row from top.

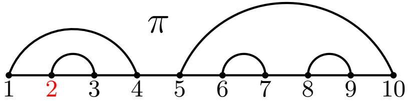
An example



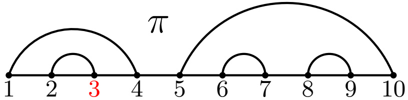
An example

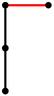


An example

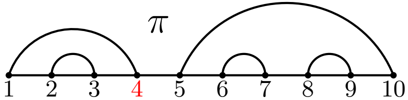


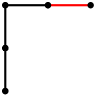
An example



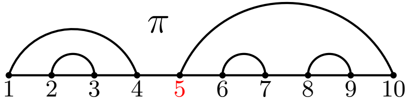


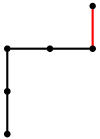
An example



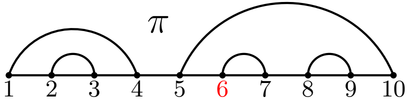


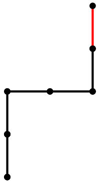
An example



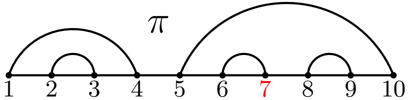


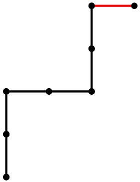
An example



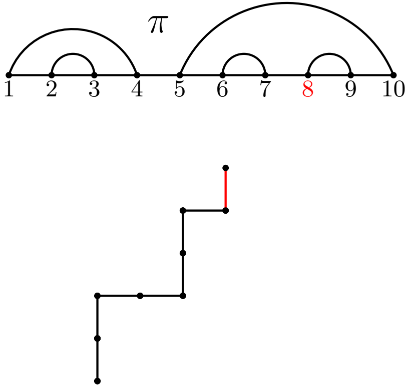


An example

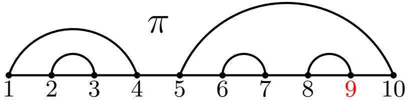


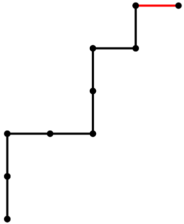


An example

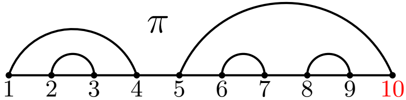


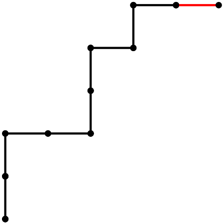
An example



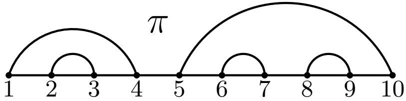


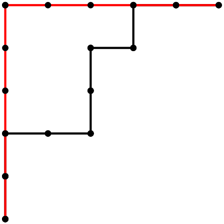
An example



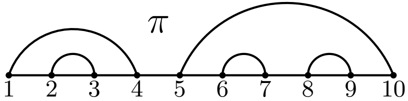


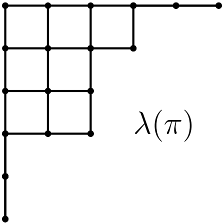
An example





An example



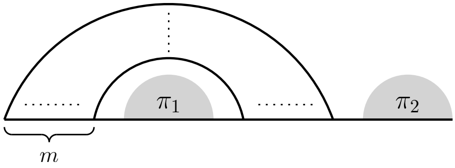


The goal

- Denote by A_π the number of FPLs with link pattern π. Calculating A_π for general π is too difficult, hence we concentrate on certain families of π.

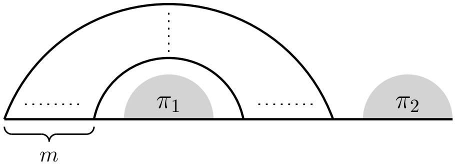
The goal

- Denote by A_π the number of FPLs with link pattern π. Calculating A_π for general π is too difficult, hence we concentrate on certain families of π.
- Zuber conjectured that $A_{(\pi_1)_m\pi_2}$ is a polynomial in m of degree $|\lambda(\pi_1)| + |\lambda(\pi_2)|$.



The goal

- Denote by A_π the number of FPLs with link pattern π. Calculating A_π for general π is too difficult, hence we concentrate on certain families of π.
- Zuber conjectured that $A_{(\pi_1)_m\pi_2}$ is a polynomial in m of degree $|\lambda(\pi_1)| + |\lambda(\pi_2)|$.

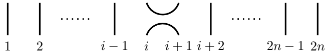


- Our goal is to prove this conjecture.

Temperley-Lieb operators

We define the i-th Temperley-Lieb operator e_i as

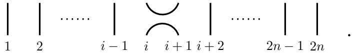
e_i :



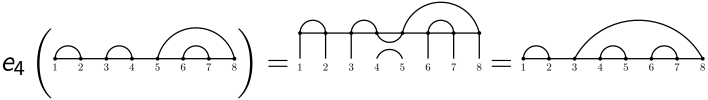
.

Temperley-Lieb operators

We define the i-th Temperley-Lieb operator e_i as

e_i : 

The Temperley-Lieb operators map nc matchings on nc matchings.



The RS-CS-Theorem

Theorem (Razumov-Stroganov-Cantini-Sportiello)

The vector $(A_\pi)_{\pi \in \mathrm{NC}_{2n}}$ *is up to normalization the unique solution of*

$$\sum_{i=1}^{2n} (e_i - \mathrm{Id})(A_\pi)_{\pi \in \mathrm{NC}_{2n}} = 0,$$

where $e_i((A_\pi)_{\pi \in \mathrm{NC}_{2n}}) = (\sum_{\pi' : e_i(\pi') = \pi} A_{\pi'})_{\pi \in \mathrm{NC}_{2n}}$.

Wheel polynomials I

Definition

Let n be an integer. A polynomial $p \in \mathbb{Q}(q)[z_1, \ldots, z_{2n}]$ is called wheel polynomial of order n if p is homogeneous of degree $n(n-1)$ and satisfies the wheel condition:

$$p(z_1, \ldots, z_{2n})_{|q^4 z_i = q^2 z_j = z_k} = 0,$$

for all $1 \leq i < j < k \leq 2n$.

Wheel polynomials I

Definition

Let n be an integer. A polynomial $p \in \mathbb{Q}(q)[z_1, \ldots, z_{2n}]$ is called wheel polynomial of order n if p is homogeneous of degree $n(n-1)$ and satisfies the wheel condition:

$$p(z_1, \ldots, z_{2n})_{|q^4 z_i = q^2 z_j = z_k} = 0,$$

for all $1 \leq i < j < k \leq 2n$.

Example

$$\prod_{1 \leq i < j \leq n} (q z_i - q^{-1} z_j) \prod_{n+1 \leq i < j \leq 2n} (q z_i - q^{-1} z_j)$$

is a wheel polynomial of order n.

A family of operator

Definition

For $1 \leq k \leq 2n$ define the linear maps
$S_k, D_k : \mathbb{Q}(q)[z_1, \ldots, z_{2n}] \longrightarrow \mathbb{Q}(q)[z_1, \ldots, z_{2n}]$ via

$$S_k(f)(z_1, \ldots, z_{2n}) := f(z_1, \ldots, z_{k-1}, z_{k+1}, z_k, z_{k+2}, \ldots, z_{2n}),$$

$$D_k(f)(z_1, \ldots, z_{2n}) := \frac{q z_k - q^{-1} z_{k+1}}{z_{k+1} - z_k}(S_k(f) - f).$$

Wheel Polynomials II

Denote by $W_n[z]$ the $Q(q)$-vector space of wheel polynomials.

Theorem (Zinn-Justin, Di Francesco)

There exists a $\mathbb{Q}(q)$*-basis* $\{\Psi_\pi \,|\pi \in \mathrm{NC}_{2n}\}$ *of* $W_n[z]$ *s.t.:*

Wheel Polynomials II

Denote by $W_n[z]$ the $Q(q)$-vector space of wheel polynomials.

Theorem (Zinn-Justin, Di Francesco)

There exists a $\mathbb{Q}(q)$-basis $\{\Psi_\pi \mid \pi \in \mathrm{NC}_{2n}\}$ of $W_n[z]$ s.t.:

- $\Psi_{()_n} = \prod_{1 \leq i < j \leq n} \frac{q z_i - q^{-1} z_j}{q - q^{-1}} \frac{q z_{n+i} - q^{-1} z_{n+j}}{q - q^{-1}}$.

Wheel Polynomials II

Denote by $W_n[z]$ the $Q(q)$-vector space of wheel polynomials.

Theorem (Zinn-Justin, Di Francesco)

There exists a $\mathbb{Q}(q)$-basis $\{\Psi_\pi \, | \pi \in \mathrm{NC}_{2n}\}$ of $W_n[z]$ s.t.:

- $\Psi_{()_n} = \prod_{1 \le i < j \le n} \frac{qz_i - q^{-1} z_j}{q - q^{-1}} \frac{qz_{n+i} - q^{-1} z_{n+j}}{q - q^{-1}}$.
- $\Psi_\pi(z) = D_j(\Psi_\sigma) - \sum_{\tau \in e_j^{-1}(\sigma) \setminus \{\sigma, \pi\}} \Psi_\tau$, *if* $\sigma \nearrow_j \pi$.

Wheel Polynomials II

Denote by $W_n[z]$ the $Q(q)$-vector space of wheel polynomials.

Theorem (Zinn-Justin, Di Francesco)

There exists a $\mathbb{Q}(q)$-basis $\{\Psi_\pi \mid \pi \in \mathrm{NC}_{2n}\}$ of $W_n[z]$ s.t.:

- $\Psi_{()_n} = \prod_{1 \leq i < j \leq n} \frac{qz_i - q^{-1}z_j}{q - q^{-1}} \frac{qz_{n+i} - q^{-1}z_{n+j}}{q - q^{-1}}$.
- $\Psi_\pi(z) = D_j(\Psi_\sigma) - \sum_{\tau \in e_j^{-1}(\sigma) \setminus \{\sigma, \pi\}} \Psi_\tau$, *if* $\sigma \nearrow_j \pi$.
- *Set* $q = e^{\frac{2\pi i}{3}}$, *then* $\Psi_\pi(1, \ldots, 1) = A_\pi$ *holds for all* $\pi \in \mathrm{NC}_{2n}$.

A polynomiality theorem

Theorem (A.)

Set

$$P = \prod_{1 \leq i \neq j \leq 2n} \left(\frac{q z_i - q^{-1} z_j}{q - q^{-1}} \right)^{\alpha_{i,j}} \prod_{i=1}^{2n} \left(\frac{q - q^{-1} z_i}{q - q^{-1}} \right)^{\beta_i} \left(\frac{q z_i - q^{-1}}{q - q^{-1}} \right)^{\gamma_i}.$$

Let $1 \leq, i_1, \ldots, i_k \leq 2n$. *Then*

$$D_{i_1} \circ \cdots \circ D_{i_k}(P)_{|z_1 = \ldots = z_{2n} = 1}$$

is a polynomial in $\alpha_{i,j}, \beta_i, \gamma_i$ *of degree at most* k.

FPLs with nested arches

Theorem (Caselli-Krattenthaler-Lass-Nadeau, A.)

Let π_1, π_2 be two noncrossing matchings. The number $A_{(\pi_1)_m \pi_2}$ of FPLs with link pattern $(\pi_1)_m \pi_2$ is a polynomial of degree $|\lambda(\pi_1)| + |\lambda(\pi_2)|$ with leading coefficient $\frac{\dim(\lambda(\pi_1)) \dim(\lambda(\pi_2))}{|\lambda(\pi_1)!|\lambda(\pi_2)!}$.