(-1)-Enumerations of arrowed Gelfand–Tsetlin patterns

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joint work with I. Fischer

Overview

- Arrowed Gelfand–Tsetlin patterns
- A (-1)-enumeration of arrowed GT patterns
- Proof Sketch

Classical GT pattern

A Gelfand-Tsetlin pattern (GT) is a triangular array of integers of the form

$$
\begin{array}{ccccccc}
 & & & T_{1,1} & & & \\
 & & T_{2,1} & & T_{2,2} & & \\
 & \iddots & & & & \ddots & \\
T_{n,1} & & \cdots & \cdots & \cdots & & T_{n,n}
\end{array}
\qquad
\begin{array}{ccc}
 & T_{i,i} & \\
\rotatebox{45}{$\le$} & & \rotatebox{-45}{$\le$} \\
T_{i+1,j} & \le & T_{i+1,j+1}
\end{array}
$$

The weight of a GT pattern T is $\mathbf{x}^T := \prod_{i=1}^{n} x_i^{\sum_{j=1}^{i}(T_{i,j}) - \sum_{j=1}^{i-1}(T_{i-1,j})}$.

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For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, the Schur polynomial s_λ is

$$s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T,$$

where the sum is over all GTs T with bottom row $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

An Example

For $\lambda = (2,2,1)$ we have

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 2 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 2 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 2 & & 2 & \\ 1 & & 2 & & 2 \end{array}$$

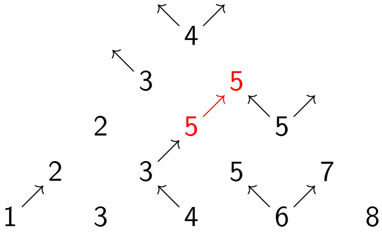
$$x_1 x_2^2 x_3^2 \qquad x_1^2 x_2 x_3^2 \qquad x_1^2 x_2^2 x_3$$

$$s_{(2,2,1)}(x_1, x_2, x_3) = x_1 x_2^2 x_3^2 + x_1^2 x_2 x_3^2 + x_1^2 x_2^2 x_3.$$

Arrowed Gelfand Tsetlin pattern

An arrowed Gelfand-Tsetlin pattern is a GT pattern $(T_{i,j})$ together with a decoration of the entries by the symbols $\emptyset, \nwarrow, \nearrow, \nwarrow\!\!\!\nearrow$ such that

$$T_{i+1,j} = T_{i,j} \text{ and } T_{i+1,j} \text{ is decorated by } \nearrow \text{ or } \nwarrow\!\!\!\nearrow,$$
$$\Leftrightarrow$$
$$T_{i+1,j+1} = T_{i,j} \text{ and } T_{i+1,j+1} \text{ is decorated by } \nwarrow \text{ or } \nwarrow\!\!\!\nearrow.$$



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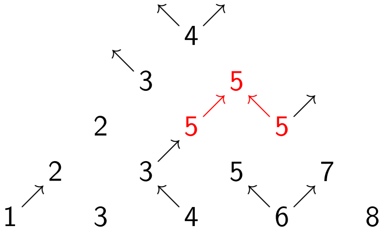

The weight of an AGT

We call the following local configurations special little triangles

$$
\begin{array}{ccc} & x & \\ x \nearrow & & \nwarrow x \end{array}
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$$

The sign of an AGT T is

$$\operatorname{sgn}(T) = (-1)^{\#\text{ of special little triangles in } T}.$$

We define the weight $W(A)$ of A as

$$\operatorname{sgn}(T) \cdot t^{\#\emptyset} u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\!\!\!\nearrow} \cdot \mathbf{x}^T \prod_{i=1}^{n} x_i^{\#\nearrow \text{ in row i} - \#\nwarrow \text{ in row i}}.$$

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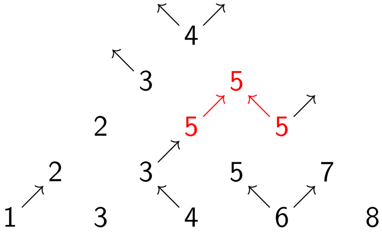
The arrowed Gelfand–Tsetlin pattern



has weight $-t^7u^3v^2w^3x_1^4x_2^3x_3^5x_4^6x_5^5$.

A multivariate generating function for AGTs

Denote by E_x the *shift operator* $E_x f(x) = f(x+1)$.

Theorem (Fischer – S.A., 2023)

The weighted enumeration $\mathcal{A}_\lambda(t,u,v,w;\mathbf{x})$ of all AGTs with bottom row $(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$ is given by

$$\mathcal{A}_\lambda(t,u,v,w;\mathbf{x}) = \prod_{i=1}^{n} \left(ux_i + vx_i^{-1} + w + t\right) \times \prod_{1 \le i < j \le n} \left(t\,\mathrm{id} + uE_{\lambda_j} + vE_{\lambda_i}^{-1} + wE_{\lambda_j}E_{\lambda_i}^{-1}\right) s_\lambda(\mathbf{x}).$$

Known specialisations

$$t^{\#\emptyset} u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\!\nearrow}$$

- $\mathcal{A}_\lambda(1,0,0,0;\mathbf{x}) = s_\lambda(\mathbf{x})$,
- $\mathcal{A}_\lambda(0,0,1,0;\mathbf{x}) = s_{(\lambda_1-n,\lambda_2-n+1,\ldots,\lambda_n-1)}(\mathbf{x})$,

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- $\mathcal{A}_{(n,n-1,\ldots,1)}(0,u,v,w;\mathbf{x})$ yields a weighted enumeration of alternating sign matrices,
- $\mathcal{A}_{(2n,2n-2,\ldots,2)}(0,u,v,w;\mathbf{x})$ yields a weighted enumeration of vertically symmetric alternating sign matrices.

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For this talk we are interested in

- $\mathcal{A}_\lambda(1,1,1,-1;\mathbf{x})\big|_{x_i=1}$ and $\mathcal{A}_\lambda(1,1,1,0;\mathbf{x})\big|_{x_i=1}$.

The main result

Theorem (Fischer – S.A.)

For positive integers n, m we have

$$\sum_{0 \le \lambda_n < \cdots < \lambda_1 \le m} \mathcal{A}_\lambda(1,1,1,-1;\mathbf{1}) = 2^n \prod_{i=1}^{n} \frac{(m-n+3i+1)_{i-1}(m-n+i+1)_i}{\left(\frac{m-n+i+2}{2}\right)_{i-1}(i)_i},$$

$$\sum_{0 \le \lambda_n < \cdots < \lambda_1 \le m} \mathcal{A}_\lambda(1,1,1,0;\mathbf{1}) = 3^{\binom{n+1}{2}} \prod_{i=1}^{n} \frac{(2n+m+2-3i)_i}{(i)_i},$$

where $(a)_i = (a)(a+1)\cdots(a+i-1)$ is the Pochhammer symbol.

The case $m = n - 1$

Setting $m = n - 1$ implies $\lambda = (n-1, n-2, \ldots, 1, 0)$ and hence

$$2^{-n}\mathcal{A}_{(n-1,n-2,\ldots,1,0)}(1,1,1,-1;\mathbf{1}) = 2^{n(n-1)/2}\prod_{i=0}^{n-1}\frac{(4i+2)!}{(n+2j+1)!}$$
$$= 1, 4, 60, 3328, 678912, \ldots\ .$$

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- configurations of the 20 vertex model in a certain domain, and
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This was proved by Koutschan and extended in a recent preprint by Corteel, Huang and Krattenthaler.

Overview of the proof of the main results

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2. Guess a (partial) LU decomposition.
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3. Proof the LU decomposition
   - It "suffices" to prove a hypergeometric triple sum.
   - For this we use Mathematica implementations of Sister Celine's algorithm and creative telescoping.

Ad 1) Littlewood identities

We have the operator formula for evaluating $\mathcal{A}_\lambda$

$$\mathcal{A}_\lambda(t,u,v,w;\mathbf{x}) = \prod_{i=1}^{n} \left(ux_i + vx_i^{-1} + w + t\right)$$
$$\times \prod_{1\le i<j\le n} \left(t\,\mathrm{id} + uE_{\lambda_j} + vE_{\lambda_i}^{-1} + wE_{\lambda_j}E_{\lambda_i}^{-1}\right) s_\lambda(\mathbf{x}).$$

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The classical Littlewood identity is

$$\sum_\lambda s_\lambda(\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1\le i<j\le n} \frac{1}{1-x_i x_j},$$

where $\mathbf{x} = (x_1,\dots,x_n)$.

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$$\sum_{\lambda \subseteq (m^n)} s_\lambda(\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1 \le i < j \le n} \frac{1}{1 - x_i x_j} \cdot \frac{1}{x_j - x_i}$$

$$\times \det_{1 \le i,j \le n} \left(x_i^{j-1} \qquad - x_i^{m+2n-j} \qquad \right),$$

where $\mathbf{x} = (x_1, \ldots, x_n)$.

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In the generalised setting we have

$$\sum_{0\le\lambda_n<\cdots<\lambda_1\le m} A_\lambda(1,1,1,w;\mathbf{x}) = \prod_{i=1}^{n}\frac{1}{1-x_i}\prod_{1\le i<j\le n}\frac{1}{1-x_ix_j}\cdot\frac{1}{x_j-x_i}$$

$$\times \det_{1\le i,j\le n}\left(x_i^{j-1}f_j(x_i) - x_i^{m+2n-j}f_j(x_i^{-1})\right)\prod_{i=1}^{n}(x_i^{-1}+1+w+x_i),$$

where $\mathbf{x}=(x_1,\ldots,x_n)$ and $f_j(x)=(1+x)^{j-1}(1+wx)^{n-j}$.

Ad 1) A simple determinant

By setting $m = 2l + 1$ and $x_1 = \cdots = x_n = 1$, we obtain

$$(3+w)^n 2^n \det_{1 \le i,j \le n} \left(\sum_{p,q} w^{n-j-q} (-1)^j \times \binom{j-1}{p} \binom{n-j}{q} \binom{p-q-\ell+i-2}{2i-1} \right).$$

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For $w = -1$ this can be simplified by using the Chu-Vandermonde identity

$$2^{2n} \det_{1\le i,j\le n}\left(\sum_{p}\binom{n-j}{p}\binom{\ell-p+i}{2i-j}\right).$$

Ad 2) Guessing the LU decomposition

Define $a_{i,j} = \sum_p \binom{n-j}{p}\binom{\ell-p+i}{2i-j}$ and

$$x_{i,j} = \begin{cases} (-1)^{i+1} \dfrac{(j)_j}{(2\ell-n+3j+2)_{j-1}(2\ell-n+i+2)_j} \\ \times \displaystyle\sum_t \Bigg(2^{2i-4t-n} (\ell - n/2 + j/2 + t + 3/2)_{i-2t-1} & i \le j, \\ \times \dfrac{(i-j-2t+1)_{2t}(i-2j+1)_{j-1-t}}{(1)_t (1)_{i-2t-1}} \Bigg) \\ 0 & \text{otherwise.} \end{cases}$$

Lemma

We have

$$\sum_{k=1}^{n} a_{i,k} x_{k,j} = \begin{cases} 1 & i = j, \\ 0 & i < j. \end{cases}$$

Ad 3) Next steps

- The expression in the sum can be simplified by using transformations for hypergeometric series (the package HYP by Krattenthaler was very useful for this!).

- However, the triple sum can not be evaluated by summation rules for hypergeometric series (as far as we are aware of).

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- The expression in the sum can be simplified by using transformations for hypergeometric series (the package HYP by Krattenthaler was very useful for this!).
- However, the triple sum can not be evaluated by summation rules for hypergeometric series (as far as we are aware of).
- We use two algorithms (Sister Celine's method, creative telescoping) which provide recursions for the triple sum and allow us to prove the Lemma.

Idea of Sister Celine's method

- Given a function $F(n) = \sum_k f(n,k)$ which we want to evaluate,
  - in our case: we want to show $F(n) = 0$ or $F(n) = 1$,
  - $f(n,k)$ consists of Pochhammer symbols.

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  - in our case: we want to show $F(n) = 0$ or $F(n) = 1$,
  - $f(n,k)$ consists of Pochhammer symbols.
- Assume we can find a recursion for f of the form

$$\sum_{r,s} a_{r,s}(n) f(n-r, k-s) = 0,$$

then we obtain

$$0 = \sum_k \left(\sum_{r,s} a_{r,s}(n) f(n-r, k-s) \right) = \sum_{r,s} a_{r,s} F(n-r).$$

Basic idea of creative telescoping

- Given a function $F(n) = \sum_{k=i}^{j} f(n,k)$ which we want to evaluate.
- Assume we can find a recursion

$$a(n)f(n,k) + b(n)f(n+1,k) = g(n,k+1) - g(n,k),$$

- then we obtain for $F(n)$

$$a(n)F(n) + b(n)F(n+1) = g(n,j+1) - g(n,i).$$

Careful checking is necessary!

Let $f(n) = (n+1)_n$ and remember

$$(x)_n = \begin{cases} (x)(x+1)\cdots(x+n-1) & n > 0, \\ 1 & n = 0, \\ \frac{1}{(x-1)(x-2)\cdots(x+n)} & n < 0. \end{cases}$$

The above algorithms will yield the recursion

$$f(n) = 2(2n-1)f(n-1),$$

which is however only true if $n \neq 0$.