Polynomiality phenomena for FPLs and AS-trapezoids

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SFB meeting, JKU Linz
12.5.2017

Outline

Fully packed loops

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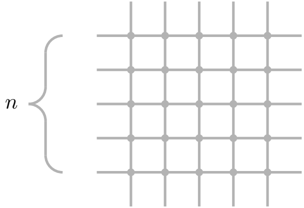
Definition

A fully packed loop (FPL) F of size n is a subgraph of the $n \times n$ grid with n external edges on every side s.t.:

- All vertices of the grid have degree 2 in F.
- F contains every other external edge, beginning with the topmost at the left side.



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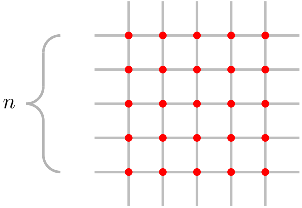

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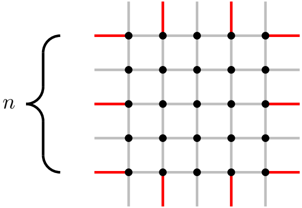

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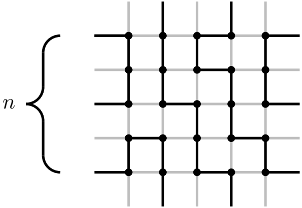
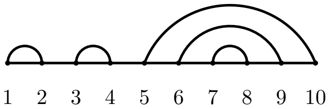


Noncrossing matchings

Definition

A noncrossing matching of size $2n$ is a matching of the numbers $1, \dots, 2n$ by arches such that there exists no pair of crossing arches.



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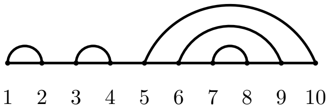
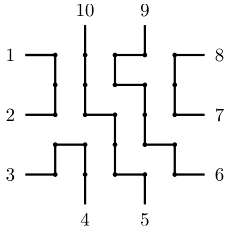


- We write $()_m$ for the noncrossing matching consisting of m nested arches. The above noncrossing matching is $()()()_3$.
- The set of noncrossing matchings of size $2n$ is denoted by NC_{2n}.

Link pattern

Definition

We number the external edges of an FPL F of size n counter clockwise with 1 up to $2n$. The link pattern $\pi(F)$ is the noncrossing matching such that i and j are matched in $\pi(F)$ iff i and j are connected by a path in F.



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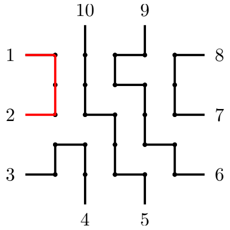

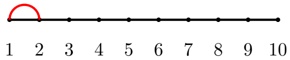

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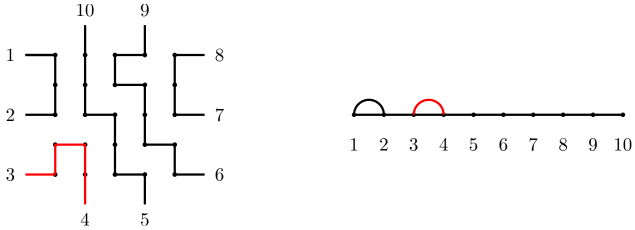

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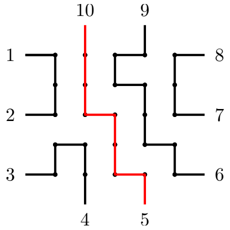

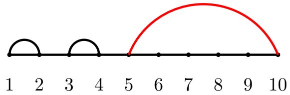

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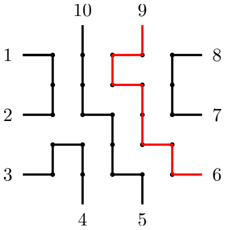

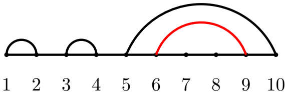

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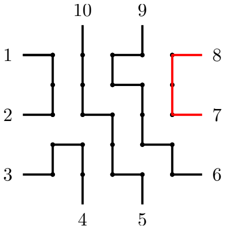

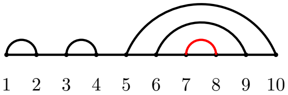

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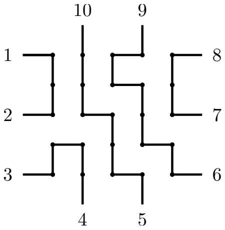

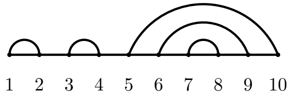

We denote by A_π the number of FPLs with link pattern π.

From noncrossing matchings to Young diagrams

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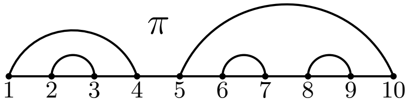
  - Draw a north-step if an arc is open.
  - Draw a east-step if an arc is closed.
  - The Young diagram $\lambda(\pi)$ is the area between the above path and the path consisting of n consecutive north-steps followed by n consecutive east-steps.

Example



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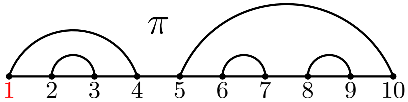

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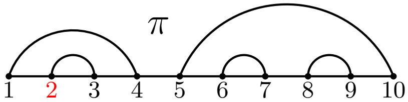

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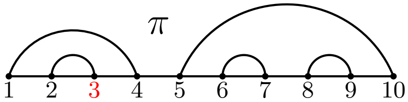

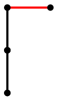

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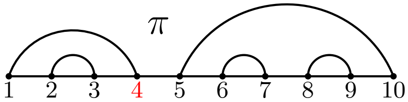

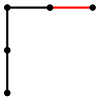

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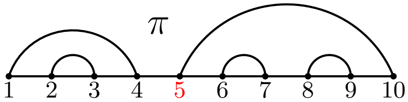

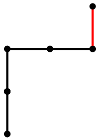

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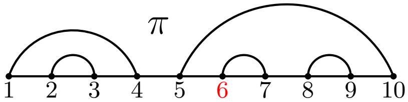

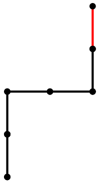

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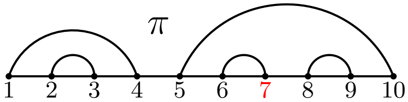

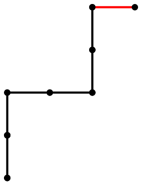

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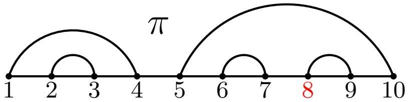

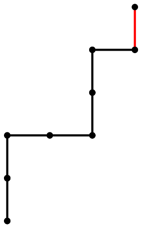

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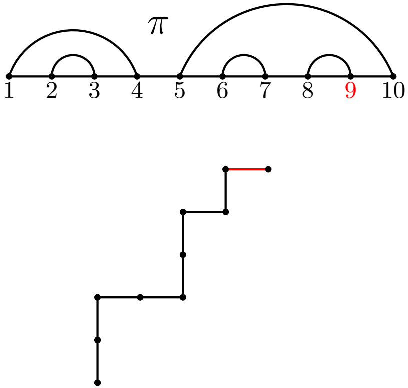

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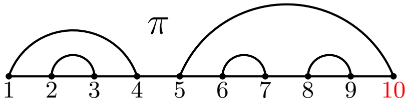

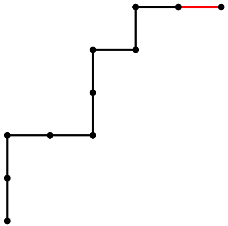

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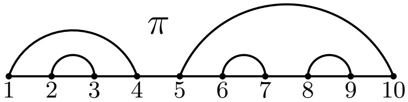

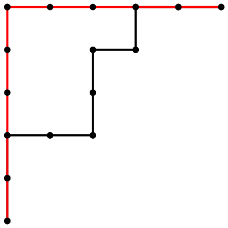

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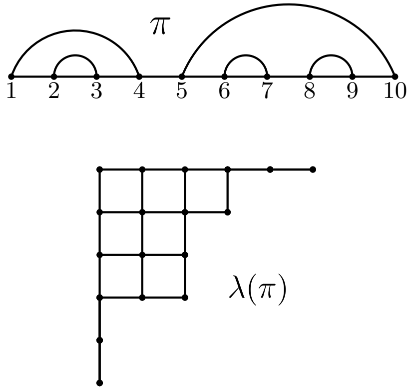
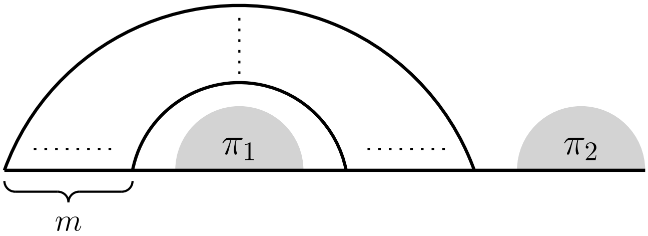


A polynomiality phenomenon

Theorem (Caselli et al. 2004, A. 2016)

Let π_1, π_2 be two noncrossing matchings, then $A_{(\pi_1)_m \pi_2}$ is a polynomial in m of degree $|\lambda(\pi_1)| + |\lambda(\pi_2)|$ and leading coefficient

$$\frac{\#(\text{SYT of shape } \lambda(\pi_1))}{|\lambda(\pi_1)|!} \frac{\#(\text{SYT of shape } \lambda(\pi_2))}{|\lambda(\pi_2)|!}.$$



Alternating sign triangles

Definition

An alternating sign triangle (AST) of order n is a configuration $(a_{i,j})_{1 \leq i \leq n, i \leq j \leq 2n-i}$ with entries $-1, 0, 1$ such that

$$
\begin{array}{ccccccccc}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & \cdots & \cdots & a_{1,2n-2} & a_{1,2n-1} \\
 & a_{2,2} & a_{2,3} & \cdots & \cdots & \cdots & a_{2,2n-2} & \\
 & & & \vdots & \vdots & \vdots & & \\
 & & & a_{n-1,n-1} & a_{n-1,n} & a_{n-1,n+1} & & \\
 & & & & a_{n,n} & & &
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- In all rows and columns the non-zero entries alternate,

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The following is an example of an AST of order 4.

$$\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & 0 & 1 & 0 & -1 & 1 & \\
 & & 0 & 0 & 1 & & \\
 & & & 1 & & &
\end{array}$$

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Theorem (Ayyer - Behrend - Fischer, 2016)

ASTs of order n and FPLs of size n are equinumerous.

Centred Catalan sets

Definition

A centred Catalan set S of size n is an n-subset of $\{-(n-1), -(n-2), \ldots, n-1\}$ such that $|S \cap \{-i, \ldots, i\}| \geq i+1$ for all $0 \leq i \leq n-1$.

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Theorem (A.,2016)

Let A be an AST of order n. We label the columns of A form left to right with $-(n-1), \ldots, n-1$ and define $S(A)$ to be the set of labels of columns with positive column-sum. Then $S(A)$ is a centred Catalan set of size n.

An example

$$A = \begin{array}{ccccccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
 & & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & & & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & & & \\
 & & & & 0 & 0 & 0 & 1 & 0 & -1 & 1 & & & & \\
 & & & & & 1 & 0 & 0 & 0 & 0 & & & & & \\
 & & & & & & 1 & -1 & 1 & & & & & & \\
 & & & & & & & 1 & & & & & & &
\end{array}$$

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 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
 & & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & & & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & & & \\
 & & & & 0 & 0 & 0 & 1 & 0 & -1 & 1 & & & & \\
 & & & & & 1 & 0 & 0 & 0 & 0 & & & & & \\
 & & & & & & 1 & -1 & 1 & & & & & & \\
 & & & & & & & 1 & & & & & & & \\
 & & & & & & & & & & & & & & \\
\overline{7} & \overline{6} & \overline{5} & \overline{4} & \overline{3} & \overline{2} & \overline{1} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}$$

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A = \begin{array}{ccccccccccccccc}
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 & & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & & & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & & & \\
 & & & & 0 & 0 & 0 & 1 & 0 & -1 & 1 & & & & \\
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 & & & & & & & & & & & & & & \\
\bar{7} & \bar{6} & \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
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$$S(A) = \{-6, -4, -2, -1, 0, 1, 3, 6\}$$

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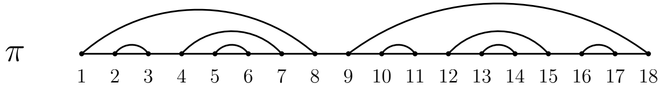

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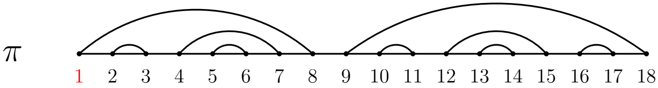

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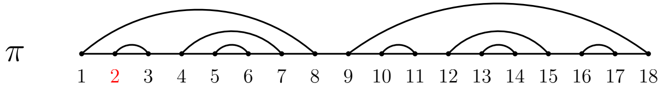

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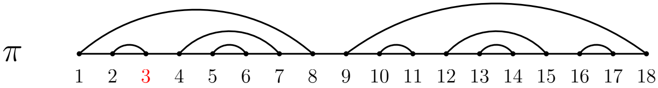

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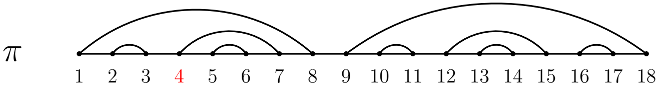

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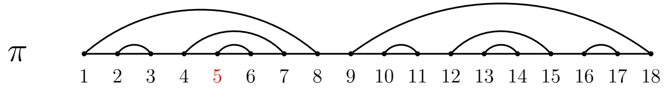

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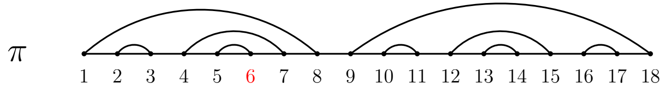

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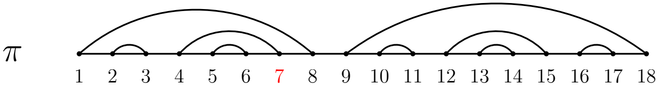

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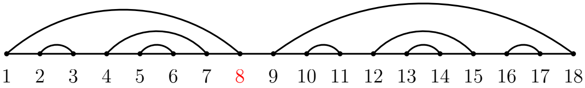

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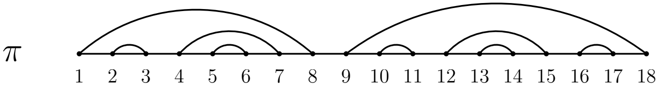

$$\Leftrightarrow \qquad S(\pi) = \{0, -1, -2, 2, 4, -5, -6, 6, -8\}$$

Concatenating centred Catalan sets

Let $l \in \mathbb{N}$. The dilation operator $\mathfrak{s}_l : \mathbb{Z} \to \mathbb{Z}$ is defined by

$$\mathfrak{s}_l(x) = \begin{cases} x + l & x > 0, \\ 0 & x = 0, \\ x - l & x < 0. \end{cases}$$

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For two centred Catalan sets S_1, S_2 of size n_1 or n_2 respectively, we define the concatenation (S_1, S_2) as

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Example

$(\{-4, -2, -1, 0, 1, 3\}, \{-1, 0, 1\}) =$
$\{-4, -2, -1, 0, 1, 3\} \cup \mathfrak{s}_5(\{-1, 0, 1\}) = \{-6, -4, -2, -1, 0, 1, 3, 6\}$

An example

$$A = \begin{array}{ccccccccccccccccccccccccccccc}
0 && 0 && 0 && 1 && 0 && 0 && 0 && 0 && 0 && 0 && 0 && 0 && 0 && 0 && 0 \\
& 1 && 0 && -1 && 0 && 0 && 0 && 0 && 0 && 0 && 0 && 0 && 0 && 1 & \\
&& 0 && 0 && 0 && 1 && 0 && 0 && 0 && 0 && 0 && 0 && 0 && \\
&&& 1 && 0 && -1 && 0 && 0 && 0 && 1 && 0 && 0 &&& \\
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AS-trapezoids

Definition

An (n,l)-AS-trapezoid is a configuration $(a_{i,j})_{1\leq i\leq n, i\leq j\leq 2(n+l)-i}$ with entries $-1,0,1$ such that

- In all rows and columns the non-zero entries alternate,
- all row-sums are 1,
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We associate to an (n,l)-AS-trapezoid A a centred Catalan set $S(A)$ of size $n+1$ such that $\mathfrak{s}_{l-1}(S(A)) \setminus \{0\}$ is the set of column labels of A with positive column-sum.

A splitting Theorem

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Theorem (A., 2016)

Let S_1 and S_2 be centred Catalan sets of size n_1 or n_2 respectively. Then holds

$$w((S_1, S_2)) = w(S_1 \cup \mathfrak{s}_{n_1-1}(S_2)) = w(S_1) w_{n_1}(S_2).$$

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$\{-i, i\} \subseteq S$	E	N
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- The skew shaped Young diagram $\sigma(S)/\mu(S)$ is the area between the paths σ and μ.

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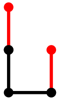

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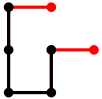

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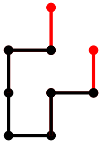

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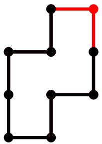

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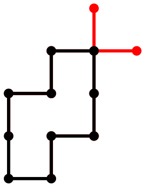

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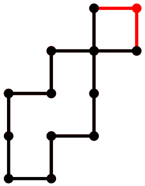

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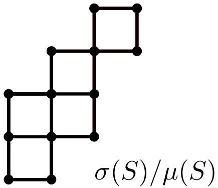

Another polynomiality phenomenon

Theorem (A., 2016)

Let S be a centred Catalan set. The weight function $w_l(S)$ is a polynomial in l of degree $|\sigma(S)/\mu(S)|$ with leading coefficient

$$\frac{2^{|\sigma(S)/\mu(S)|}\#\left(SYT\ of\ shape\ \sigma(S)/\mu(S)\right)}{|\sigma(S)/\mu(S)|!}.$$

About the roots

Conjecture

Let S be an irreducible centred Catalan set and define $(x)_j := x(x+1)(x+2)\ldots(x+j-1)$. Then holds the following

$$\{-k,\ldots,k\} \subseteq S \quad \Leftrightarrow \quad \prod_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} (2l+1+3i)_{(k-2i)} \Bigg| \, w_l(S).$$

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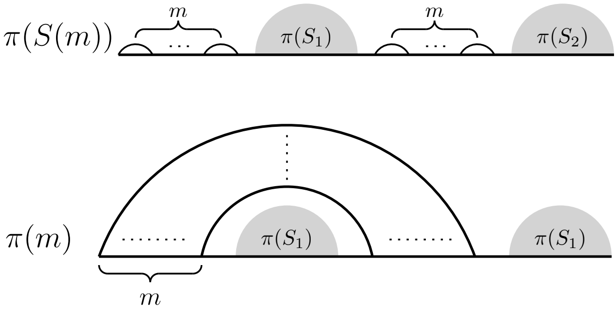
Conjecture

Let S be an irreducible centred Catalan set of size $n \geq 10$. *Then the rational roots of* $w_l(S)$ *are in the set* $\{-\frac{1}{2}, -1, \dots, \frac{-2n+5}{2}, -n+2, -\frac{n^2-5n+7}{2(n-3)}\}$.

Centred Catalan sets of a certain form

Let S_1, S_2 be centred Catalan sets and define

$$S(m) := (\{0,1,\dots,m\}, S_1, \{0,1,\dots,m\}, S_2),$$
$$\pi(m) := (\pi(S_1))_m \pi(S_2).$$



A comparison between two theorems

The number $A_{\pi(m)}$ of FPL with link pattern $\pi(m)$ is a polynomial in m of degree

$$\sum_{i=1}^{2} |\lambda(\pi(S_i))|,$$

and leading coefficient

$$\prod_{i=1}^{2} \frac{\#(SYT \text{ of shape } \lambda(\pi(S_i)))}{|\lambda(\pi(S_i))|!}$$

The number $w(S(m))$ of AS--trapezoids with centred Catalan set $S(m)$ is a polynomial in m of degree

$$\sum_{i=1}^{2} |\sigma(S_i)/\mu(S_i)|,$$

and leading coefficient

$$\prod_{i=1}^{2} \frac{2^{|\sigma(S_i)/\mu(S_i)|} \#\,(SYT \text{ of } \sigma(S_i)/\mu(S_i))}{|\sigma(S_i)/\mu(S_i)|!}$$