# An introduction to alternating sign matrices
## A combinatorial story of missing bijections

Florian Aigner

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VIENNA DOCTORAL SCHOOL MATHEMATICS

universität wien

# Outline

# A naive perspection on enumerative Combinatorics

A combinatorial structure (Species) $F$ is an association from finite sets to finite sets, associating to every set $X$ the set of combinatorial objects on $X$

$$X \mapsto F(X) = \{\text{combinatorial objects on } X\}.$$

This association should be "independent of the nature" of the elements in $X$, i.e., for every bijection $\varphi : X \to Y$ there exists and bijection $F(\varphi)$

$$F(\varphi) : F(X) \to F(Y).$$

# Examples

Denote by $\mathcal{S}$ the combinatorial structure of Permutations, i.e.,

$$\mathcal{S}(X) = \{\pi | \pi \text{ is a bijection from } X \text{ to } X\}.$$

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$$\mathcal{S}(\{1,2\}) = \{\begin{pmatrix} 1 \mapsto 1 \\ 2 \mapsto 2 \end{pmatrix}, \begin{pmatrix} 1 \mapsto 2 \\ 2 \mapsto 1 \end{pmatrix}\}.$$

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Denote by $L$ the combinatorial structure of liner orders, i.e.,

$$L(X) = \{ \text{ linear orders on } X \}.$$

**Example**

$$L(\{1, 2\}) = \{12, 21\}, \qquad L(\{\bigstar, \bullet\}) = \{\bigstar\bullet, \bullet\bigstar\}.$$

# How many are there?

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The number of permutations on $X$ is $|\mathcal{S}(X)| = |X|!$.

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Given a combinatorial structure $F$, one of the most fundamental question is what is the size of $F(X)$?

**Example (I)**

The number of permutations on $X$ is $|\mathcal{S}(X)| = |X|!$.

**Example (II)**

The number of linear order on $X$ is $|L(X)| = |X|!$.

# Combinatorial structures with the same number

Let $F, G$ be two combinatorial structures such that

$$|F(X)| = |G(X)|.$$

for all finite sets $X$. Hence there exists for all finite sets $X$ a bijection

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Combinatorialists are interested in "nice" bijections.

*What are "nice" bijections*?

# Natural isomorphism

A natural Isomorphism between two combinatorial structures $F, G$ is a family of bijections $\eta_X : F(X) \to G(X)$ for all finite sets $X$ such that the following diagram commutes for all finite sets $X, Y$ of same cardinality and all bijections $\varphi : X \to Y$,

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(\varphi)}\downarrow & & \downarrow{\scriptstyle G(\varphi)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

i.e., $\eta_Y \circ F(\varphi) = G(\varphi) \circ \eta_X$.

# An example with natural isomorphism

- Denote by $S$ the species of subsets. For $\varphi : X \to Y$ the associated map $S(\varphi)$ is given by

$$S(\varphi)(A) = \varphi(A) := \{\varphi(a) | a \in A\}.$$

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Then the following is natural isomorphism between $S$ and $P$

$$\begin{aligned} \eta_X : S(X) &\to P(X) \\ A &\mapsto (A, X \setminus A). \end{aligned}$$

# An example without natural isomorphism

The species $\mathcal{S}$ of permutations and $L$ of linear order are not natural isomorphic.

- Let $X = \{1, 2\}$ and $\varphi : X \to X$ with $\varphi(1) = 2$ and $\varphi(2) = 1$.

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- $$\mathcal{S}(\varphi)\left(\begin{pmatrix} 1 \mapsto 1 \\ 2 \mapsto 2 \end{pmatrix}\right) = \begin{pmatrix} 2 \mapsto 2 \\ 1 \mapsto 1 \end{pmatrix}, \quad \mathcal{S}(\varphi)\left(\begin{pmatrix} 1 \mapsto 2 \\ 2 \mapsto 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \mapsto 1 \\ 1 \mapsto 2 \end{pmatrix}.$$

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- $$L(\varphi)(12) = 21, \qquad L(\varphi)(21) = 12.$$

**Theorem (Andrews '79,'94, Zeilberger '96, Ayyer-Behrend-Fischer '16)**

*The following combinatorial objects are enumerated by*

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

**ASMs '82**

| | | | |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 |
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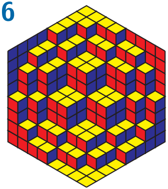

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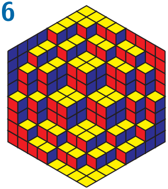

### ASTs '16

$$\begin{array}{ccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & 1 & 0 & -1 & 1 & 0 & \\ & & 1 & 0 & 0 & & \\ & & & 1 & & & \end{array}$$

# Descending plane partitions

**Definition (Andrews '79)**

A descending plane partition (DPP) of size $n$ is an array of successively indented rows filled with positive integers less than or equal $n$ such that

- the entries are weakly decreasing along rows and strictly decreasing along columns,
- the first entry in each row is larger than the length of its row and does not exceed the number of entries in the preceding row.

The DPPs of size 3 are

$$\emptyset \qquad 2 \qquad 3 \qquad \begin{matrix} 3 & 1 \end{matrix} \qquad \begin{matrix} 3 & 2 \end{matrix} \qquad \begin{matrix} 3 & 3 \end{matrix} \qquad \begin{matrix} 3 & 3 \\ & 2 \end{matrix}$$

# Alternating sign matrices

**Definition (Mills-Robbins-Rumsey '82)**

An alternating sign matrix (ASM) of size $n$ is an $n \times n$ matrix with entries $1, 0, -1$, such that

- all row- and column-sums are equal 1,
- in each row and column the non-zero entries alternate.

The ASMs of size 3 are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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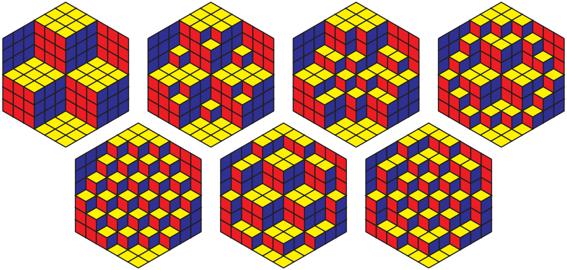
# TSSCPPs

**Definition (Mills-Robbins-Rumsey '86)**

A Totally symmetric self complementary plane partitions (TSSCPP) of size $n$ is a filling of an $2n \times 2n \times 2n$ box with unit cubes which is invariant under change of axis and coincides with its "empty filling".

The TSSCPPs of size 3 are



# Alternating sign triangles

**Definition (Ayyer-Behrend-Fischer ’16)**

An alternating sign triangle (AST) of size $n$ is a configuration of $n$ centred rows where the $i$-th row, counted from the bottom, has $2i-1$ elements, with entries $-1, 0$ or $1$ such that

- all row-sums are equal 1,
- in each row and column the non-zero entries alternate,
- the first non-zero entry from top is positive.

The ASTs of size 3 are

$$\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} 0 & 0 & 0 & 1 & 0 \\ & 1 & 0 & 0 & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & \\ & & 1 & & \end{array}$$

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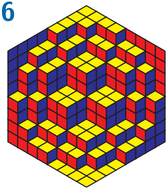

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- Maybe there are no "nice" bijections.
- Generalisations, refinements and symmetry classes can give more insight.
- A change of perspective could also help, e.g., we can interpret ASMs and TSSCPPs as order ideals in Posets.

# Generalisations

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  - Gog- and Magog-trapezoids (one extra parameter): equinumerousity proven by Zeilberger '96.

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- DPPs and ASTs have been generalised to
  - $d$-DPPs (one extra parameter) have been introduced by Andrews '79.
  - AS-trapezoids (one extra parameter) have been announced by Ayyer-Behrend-Fischer '16 and introduced by Aigner '17; equinumerousity was proven by Fischer '18.

## A refinement of ASMs

Let $A$ be an ASM of size $n$. We denote by

- $\nu(A) = \sum_{\substack{1 \leq i < i' \leq n \\ 1 \leq j' < j \leq n}} A_{ij} A_{i'j'}$ the inversion number of $A$,
- $\mu(A)$ the number of $-1$'s in $A$,
- $\rho_1(A)$ the number of 0's to the left of the topmost 1,
- $\rho_2(A)$ the number of 0's to the right of the bottommost 1.

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} \nu = 9 \\ \mu = 2 \\ \rho_1 = 1 \\ \rho_2 = 3 \end{matrix}$$

# A refinement of DPPs

Let $D$ be a DPP of size $n$. We denote by

- $\nu(D)$ the number of parts $D_{ij}$ with $D_{ij} > j - i$,
- $\mu(D)$ the number of parts $D_{ij}$ with $D_{ij} \leq j - i$,
- $\rho_1(D)$ the number of $n$'s in $D$,
- $\rho_2(D)$ the number of $(n-1)$'s in $D$ plus the number of rows of $D$ with length $n-1$.

| | | | | | |
|---|---|---|---|---|---|
| 6 | 6 | 6 | 5 | 1 | $\nu = 7$ |
| | 5 | 4 | 2 | | $\mu = 3$ |
| | | 3 | 1 | | $\rho_1 = 3$ |
| | | | | | $\rho_2 = 3$ |

# A refinement of ASMs and DPPs

- Write $\mathrm{ASM}_n(a, b, c, d)$ for the number of ASMs $A$ of size $n$ with $\nu(A) = a, \mu(A) = b, \rho_1(A) = c, \rho_2(A) = d$.
- Write $\mathrm{DPP}_n(a, b, c, d)$ for the number of DPPs $D$ of size $n$ with $\nu(D) = a, \mu(D) = b, \rho_1(D) = c, \rho_2(D) = d$.

**Theorem (Behrend-Di Francesco- Zinn-Justin, '13)**

$$\mathrm{ASM}_n(a, b, c, d) = \mathrm{DPP}_n(a, b, c, d).$$