A new determinant for the Q-enumeration of ASMs

Florian Aigner

Séminaire Combi
31.10.2018

VIENNA DOCTORAL SCHOOL MATHEMATICS

universität wien

Outline

Alternating sign matrices

Definition

An alternating sign matrix (or short ASM) of size n is an $n \times n$ matrix with entries $1, 0, -1$, such that

- all row- and column-sums are equal 1,
- in each row and column the non-zero entries alternate.

The ASMs of size 3 are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Counting ASMs

We denote by $\mathrm{A}_n(Q)$ the number of ASMs of size n, where each ASM has weight $Q^{\text{number of entries equal to } -1}$. This is called the Q-enumeration of ASMs.

n	1	2	3	4	5	6
$\mathrm{A}_n(0)$	1	2	6	24	120	720
$\mathrm{A}_n(1)$	1	2	7	42	429	7436
$\mathrm{A}_n(2)$	1	2	8	64	1024	32768
$\mathrm{A}_n(3)$	1	2	9	90	2025	102060

Counting ASMs

We denote by $\mathrm{A}_n(Q)$ the number of ASMs of size n, where each ASM has weight $Q^{\text{number of entries equal to } -1}$. This is called the Q-enumeration of ASMs.

n	1	2	3	4	5	6	
$\mathrm{A}_n(0)$	1	2	6	24	120	720	# of permutations
$\mathrm{A}_n(1)$	1	2	7	42	429	7436	
$\mathrm{A}_n(2)$	1	2	8	64	1024	32768	
$\mathrm{A}_n(3)$	1	2	9	90	2025	102060	

Counting ASMs

We denote by $A_n(Q)$ the number of ASMs of size n, where each ASM has weight $Q^{\text{number of entries equal to } -1}$. This is called the Q-enumeration of ASMs.

n	1	2	3	4	5	6	
$A_n(0)$	1	2	6	24	120	720	# of permutations
$A_n(1)$	1	2	7	42	429	7436	# ASMs
$A_n(2)$	1	2	8	64	1024	32768	
$A_n(3)$	1	2	9	90	2025	102060	

Counting ASMs

We denote by $\mathrm{A}_n(Q)$ the number of ASMs of size n, where each ASM has weight $Q^{\text{number of entries equal to } -1}$. This is called the Q-enumeration of ASMs.

n	1	2	3	4	5	6	
$\mathrm{A}_n(0)$	1	2	6	24	120	720	# of permutations
$\mathrm{A}_n(1)$	1	2	7	42	429	7436	# ASMs
$\mathrm{A}_n(2)$	1	2	8	64	1024	32768	# perfect matchings of
$\mathrm{A}_n(3)$	1	2	9	90	2025	102060	Atzec diamonds

The ASM Conjecture

Conjecture (Mills-Robbins-Rumsey, 1982)

The number of ASMs of size n is given by

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

The ASM Conjecture

Conjecture (Mills-Robbins-Rumsey, 1982)

The number of ASMs of size n is given by

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Descending Plane Partitions

$$\begin{array}{ccccc} 6 & 6 & 5 & 3 & 1 \\ & 5 & 3 & & \\ & & 2 & & \end{array}$$

The ASM Conjecture

Conjecture (Mills-Robbins-Rumsey, 1982)

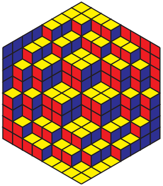
The number of ASMs of size n is given by

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Descending Plane Partitions

6	6	5	3	1
	5	3		
		2		

Totally Symmetric Self Complementary Plane Partitions



Proofs of the ASM Conjecture

- 1996: D. Zeilberger. "*Proof of the Alternating Sign Matrix Conjecture*".
- 1996: G. Kuperberg. "*Another proof of the alternating sign matrix conjecture*".
- 2007: I. Fischer. "*A new proof of the refined alternating sign matrix theorem*".
- 2016: I. Fischer. "*Short proof of the ASM theorem avoiding the six-vertex model*".

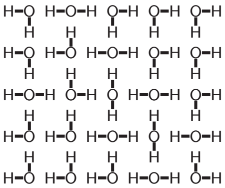
Proofs of the ASM Conjecture

- 1996: D. Zeilberger. "*Proof of the Alternating Sign Matrix Conjecture*".
- 1996: G. Kuperberg. "*Another proof of the alternating sign matrix conjecture*".
- 2007: I. Fischer. "*A new proof of the refined alternating sign matrix theorem*".
- 2016: I. Fischer. "*Short proof of the ASM theorem avoiding the six-vertex model*".

"However, the greatest, still unsolved, mystery concerns the question what plane partitions have to do with alternating sign matrices." – Krattenthaler.

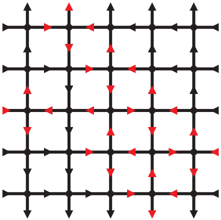
The many faces of ASMs

square ice



monotone triangle

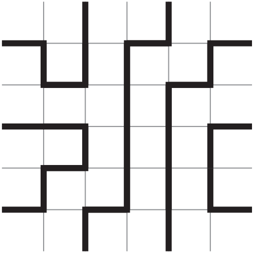
2
2 3
1 2 4
1 2 3 5
1 2 3 4 5



six-vertex configuration

0	1	0	0	0
0	0	1	0	0
1	0	−1	1	0
0	0	1	−1	1
0	0	0	1	0

ASM

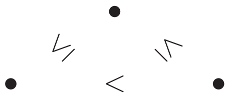


FPL

Monotone triangles

A *monotone triangle* with n rows is a triangular array $(a_{i,j})_{1 \leq j \leq i \leq n}$ of integers of the following form

$$\begin{array}{ccccccc} & & & a_{n,1} & & & \\ & & a_{n-1,1} & & a_{n-1,2} & & \\ & \iddots & & & & \ddots & \\ a_{1,1} & & a_{1,2} & & \cdots & & a_{1,n} \end{array}$$



whose entries are weakly increasing along north-east $a_{i+1,j} \leq a_{i,j}$ and south-east diagonals $a_{i,j} \leq a_{i+1,j+1}$ and strictly increasing along rows $a_{i,j} < a_{i,j+1}$.

A bijection between ASMs and monotone triangles

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}$$

A bijection between ASMs and monotone triangles

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} \qquad 0$$

- We replace every entry by the partial column-sum of its column from top to bottom.

A bijection between ASMs and monotone triangles

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 \\ 0 \end{matrix}$$

- We replace every entry by the partial column-sum of its column from top to bottom.

A bijection between ASMs and monotone triangles

$$\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \qquad \begin{array}{c} 0 \\ 0 \\ 1 \\ \\ \\ \end{array}$$

- We replace every entry by the partial column-sum of its column from top to bottom.

A bijection between ASMs and monotone triangles

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 \\ 0 \\ 1 \\ 1 \\ \end{matrix}$$

- We replace every entry by the partial column-sum of its column from top to bottom.

A bijection between ASMs and monotone triangles

0	1	0	0	0	0
0	0	1	0	0	0
1	0	−1	1	0	1
0	0	1	−1	1	1
0	0	0	1	0	1

- We replace every entry by the partial column-sum of its column from top to bottom.

A bijection between ASMs and monotone triangles

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{matrix}$$

- We replace every entry by the partial column-sum of its column from top to bottom.

A bijection between ASMs and monotone triangles

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{matrix} \qquad \begin{matrix} & & & & 2 & & & & \\ & & & 2 & & 3 & & & \\ & & 1 & & 2 & & 4 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{matrix}$$

- We replace every entry by the partial column-sum of its column from top to bottom.
- We form an triangular array by writing in the i-th row the labels of the columns with entry 1 in the i-th row.

An operator formula

We define the forward difference $\overline{\Delta}_x$ as the operator

$$\overline{\Delta}_x f(x) = f(x+1) - f(x).$$

Theorem (Fischer, 2010)

The Q-enumeration of ASMs of size n is given by

$$\prod_{1 \leq i < j \leq n} \left(Q \, \mathrm{Id} + (Q-1)\overline{\Delta}_{x_i} + \overline{\Delta}_{x_j} + \overline{\Delta}_{x_i}\overline{\Delta}_{x_j} \right) \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{j - i} \Bigg|_{x_i = i}.$$

The operator formula for $n = 2$.

Remember: $\overline{\Delta}_x f(x) = f(x+1) - f(x)$.

$$\left(Q\mathrm{Id} + (Q-1)\overline{\Delta}_{x_1} + \overline{\Delta}_{x_2} + \overline{\Delta}_{x_1}\overline{\Delta}_{x_2}\right)(x_2 - x_1)\Big|_{x_1=1,x_2=2}$$

The operator formula for $n = 2$.

Remember: $\overline{\Delta}_x f(x) = f(x+1) - f(x)$.

$$\left(Q\mathrm{Id} + (Q-1)\overline{\Delta}_{x_1} + \overline{\Delta}_{x_2} + \overline{\Delta}_{x_1}\overline{\Delta}_{x_2}\right)(x_2 - x_1)\Big|_{x_1=1,x_2=2}$$

$$= \left(Q(x_2 - x_1) + (Q-1)(-1) + 1 + 0\right)\Big|_{x_1=1,x_2=2} = 2.$$

A new determinant for the Q-enumeration

Theorem (A. 2018)

The $(q^{-1}+2+q)$-enumeration of ASMs is given by

$$\mathrm{A}_n(q^{-1}+2+q) = \det_{1\le i,j\le n}\left(\binom{i+j-2}{j-1}\frac{1-(-q)^{1+j-i}}{1+q}\right).$$

Sketch of the proof

Set $Q = q^{-1} + 2 + q$.

1. Start with the operator formula.

$$\prod_{1 \le i < j \le n} \left(Q \operatorname{Id} + (Q-1)\overline{\Delta}_{x_i} + \overline{\Delta}_{x_j} + \overline{\Delta}_{x_i}\overline{\Delta}_{x_j} \right) \prod_{1 \le i < j \le n} \frac{x_j - x_i}{j - i} \Bigg|_{x_i = i}$$

Sketch of the proof

Set $Q = q^{-1} + 2 + q$.

1. Start with the operator formula.
2. Rewrite it to a constant term formula.

$$\mathrm{CT}_{x_1,\ldots,x_n} \frac{\mathcal{AS}_{x_1,\ldots,x_n}\left(\prod_{i=1}^{n}(1+x_i)^i \prod_{1\le i<j\le n}(Q+(Q-1)x_i+x_j+x_ix_j)\right)}{\prod_{1\le i<j\le n}(x_j-x_i)}$$

Sketch of the proof

Set $Q = q^{-1} + 2 + q$.

1. Start with the operator formula.
2. Rewrite it to a constant term formula.
3. Use a general Lemma (Fonseca, Zinn-Justin, '08; Fischer, '18) which transforms the antisymmetriser into a determinant.

$$\mathrm{CT}_{x_1,\ldots,x_n}\left((-1)^{\frac{n(n+1)}{2}} q^n (q-q^{-1})^{\frac{n(n+3)}{2}} \prod_{i=1}^{n} (1+x_i)^{n+1}(x_i+1+q)^{-2}\right.$$

$$\times \lim_{y_1,\ldots,y_n \to 1} \det_{1\le i,j\le n}\left(\frac{1}{\left(y_j - \frac{x_i+1+q^{-1}}{x_i+1+q}\right)\left(y_j - q^2\frac{x_i+1+q^{-1}}{x_i+1+q}\right)}\right)$$

$$\left.\times \prod_{1\le i<j\le n} (x_j-x_i)^{-1}(y_j-y_i)^{-1}\right)$$

Sketch of the proof

Set $Q = q^{-1} + 2 + q$.

1. Start with the operator formula.
2. Rewrite it to a constant term formula.
3. Use a general Lemma (Fonseca, Zinn-Justin, '08; Fischer, '18) which transforms the antisymmetriser into a determinant.
4. Use algebraic manipulations and a trick (Behrend, Di Franceso,Zinn-Justin, 2012) to obtain the wanted formula.

$$\det_{1 \leq i,j \leq n} \left(\binom{i+j-2}{j-1} \frac{1-(-q)^{1+j-i}}{1+q} \right)$$

A more general determinant

Instead of the previous determinant we consider

$$d_{n,k}(x,q) := \det_{1 \le i,j \le n} \left(\binom{x+i+j-2}{j-1} \frac{1-(-q)^{k+j-i}}{1+q} \right),$$

with $k \in \mathbb{Z}$ and x is a variable.

The weighted enumeration of ASMs is $d_{n,1}(0,q)$.

Desnanot-Jacobi / Condensation method

Theorem (Desnanot-Jacobi, Condensation method)

Let n be a positive integer and A an $n \times n$ matrix, then holds

$$\det A \det A_{1,n}^{1,n} = \det A_1^1 \det A_n^n - \det A_1^n \det A_n^1,$$

where A is an $n \times n$ matrix and $A_{j_1,\cdots,j_k}^{i_1,\cdots,i_k}$ denotes the submatrix of A in which the $i_1,\cdots,i_k$-th rows and $j_1,\cdots,j_k$-th columns are omitted.

Desnanot-Jacobi applied

$$\det_{1\le i,j\le n}\left(\binom{x+i+j-2}{j-1}\frac{1-(-q)^{1+j-i}}{1+q}\right)_1^1$$

$$= \begin{vmatrix} \binom{x}{0} & \frac{1-q^2}{1+q}\binom{x+1}{1} & \frac{1+q^3}{1+q}\binom{x+2}{2} & \frac{1-q^4}{1+q}\binom{x+3}{3} & \frac{1+q^5}{1+q}\binom{x+4}{4} & \cdots \\ 0 & \binom{x+2}{1} & \frac{1-q^2}{1+q}\binom{x+3}{2} & \frac{1+q^3}{1+q}\binom{x+4}{3} & \frac{1-q^4}{1+q}\binom{x+5}{4} & \cdots \\ \frac{1}{q}\binom{x+2}{1} & 0 & \binom{x+4}{2} & \frac{1-q^2}{1+q}\binom{x+5}{3} & \frac{1+q^3}{1+q}\binom{x+6}{4} & \cdots \\ \frac{1-q^{-2}}{1+q}\binom{x+3}{0} & \frac{1}{q}\binom{x+4}{1} & 0 & \binom{x+6}{3} & \frac{1-q^2}{1+q}\binom{x+7}{4} & \cdots \\ \frac{1+q^{-3}}{1+q}\binom{x+4}{0} & \frac{1-q^{-2}}{1+q}\binom{x+5}{1} & \frac{1}{q}\binom{x+6}{2} & 0 & \binom{x+8}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

Desnanot-Jacobi applied

$$\det_{1\le i,j\le n}\left(\binom{x+i+j-2}{j-1}\frac{1-(-q)^{1+j-i}}{1+q}\right)_1^1$$

$$= \begin{vmatrix} \binom{x+2}{1} & \frac{1-q^2}{1+q}\binom{x+3}{2} & \frac{1+q^3}{1+q}\binom{x+4}{3} & \frac{1-q^4}{1+q}\binom{x+5}{4} & \cdots \\ 0 & \binom{x+4}{2} & \frac{1-q^2}{1+q}\binom{x+5}{3} & \frac{1+q^3}{1+q}\binom{x+6}{4} & \cdots \\ \frac{1}{q}\binom{x+4}{1} & 0 & \binom{x+6}{3} & \frac{1-q^2}{1+q}\binom{x+7}{4} & \cdots \\ \frac{1-q^{-2}}{1+q}\binom{x+5}{1} & \frac{1}{q}\binom{x+6}{2} & 0 & \binom{x+8}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

Desnanot-Jacobi applied

$$\det_{1\le i,j\le n}\left(\binom{x+i+j-2}{j-1}\frac{1-(-q)^{1+j-i}}{1+q}\right)_1^1$$

$$=\frac{(x+2)(x+3)(x+4)(x+5)\dots(x+n)}{2\cdot 3\cdot 4\cdot \dots \cdot (n-1)}$$

$$\times\begin{vmatrix} \binom{x+2}{0} & \frac{1-q^2}{1+q}\binom{x+3}{1} & \frac{1+q^3}{1+q}\binom{x+4}{2} & \frac{1-q^4}{1+q}\binom{x+5}{3} & \dots \\ 0 & \binom{x+4}{1} & \frac{1-q^2}{1+q}\binom{x+5}{2} & \frac{1+q^3}{1+q}\binom{x+6}{3} & \dots \\ \frac{1}{q}\binom{x+4}{0} & 0 & \binom{x+5}{2} & \frac{1-q^2}{1+q}\binom{x+6}{3} & \dots \\ \frac{1-q^{-2}}{1+q}\binom{x+5}{0} & \frac{1}{q}\binom{x+6}{1} & 0 & \binom{x+8}{3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

Desnanot-Jacobi applied

$$\det_{1\le i,j\le n}\left(\binom{x+i+j-2}{j-1}\frac{1-(-q)^{1+j-i}}{1+q}\right)_1^1$$
$$=\binom{x+n}{n-1}\det_{1\le i,j\le n-1}\left(\binom{x+i+j}{j-1}\frac{1-(-q)^{1+j-i}}{1+q}\right).$$

What do we obtain for general q?

The determinant $d_{n,k}(x,q)$ has the form

$$d_{n,k}(x,q) = q^{c_q(n,k)} p_{n,k}(x) f_{n,k}(x,q),$$

with

$$p_{n,k}(x) = \prod_{i=1}^{n-1} \frac{\lfloor \frac{i}{2} \rfloor !}{i!} \prod_{i=0}^{\lfloor \frac{n-|k|-1}{2} \rfloor} (x + |k| + 2i + 1),$$

$$c_q(n,k) = \begin{cases} 0 & k > 0, n \leq k, \\ n\,k & k < 0, n \leq -k, \\ -\sum_{i=1}^{n-k} \lfloor \frac{i}{2} \rfloor & \text{otherwise,} \end{cases}$$

and $f_{n,k}(x,q)$ being a polynomial in x and q which is given recursively.

A factorisation theorem for general q

Theorem (Kuperberg 1996, A. 2018)

Denote by $\mathrm{A}_n(Q)$ the Q-enumeration of ASMs of size n. Then there exists polynomials $r_n(Q)$ such that

$$\mathrm{A}_{2n}(Q) = 2r_{2n}(Q)r_{2n+1}(Q),$$
$$\mathrm{A}_{2n+1}(Q) = r_{2n+1}(Q)r_{2n+2}(Q).$$

This was conjectured by Mills-Robbins-Rumsey for $\mathrm{A}_n(Q)$ and by Fischer for the evaluation of the determinant.

Various specialisations for q

0-enumeration:	$q=-1$	(primitive second root of unity),
1-enumeration:	$q=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$	(primitive third root of unity),
2-enumeration:	$q=\pm i$	(primitive fourth root of unity),
3-enumeration:	$q=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$	(primitive sixth root of unity),
4-enumeration:	$q=1$	(primitive first root of unity).

Various specialisations I

Theorem (A.)

The 0-enumeration case:

$$d_{n,1}(x,-1)=\left(2\left\lfloor\frac{n+1}{2}\right\rfloor-1\right)!!\prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}(x+2i).$$

Various specialisations I

Theorem (A.)

The 0-enumeration case:

$$d_{n,1}(x,-1) = \left(2\left\lfloor\frac{n+1}{2}\right\rfloor - 1\right)!! \prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}(x+2i).$$

Corollary

There are $n!$ permutations of size n.

Various specialisations II

Theorem (A.)

The 1-enumeration case: let q be a primitive third root of unity.

$$d_{n,6k+1}(x,q) = 2^{\lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor} \prod_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(i-1)!}{(n-i)!}$$

$$\times \prod_{i \geq 0} \left(\frac{x}{2} + 3i + 1 \right)_{\lfloor \frac{n-4i}{2} \rfloor} \left(\frac{x}{2} + 3i + 3 \right)_{\lfloor \frac{n-4i-3}{2} \rfloor}$$

$$\times \prod_{i \geq 0} \left(\frac{x}{2} + n - i + \frac{1}{2} \right)_{\lfloor \frac{n-4i-1}{2} \rfloor} \left(\frac{x}{2} + n - i - \frac{1}{2} \right)_{\lfloor \frac{n-4i-2}{2} \rfloor},$$

where $(a)_i := a(a+1)\cdots(a+i-1)$.

Corollary

This implies the enumeration formula of ASMs.

Various specialisations III

Theorem (A.)

The 2-enumeration case: let q be a primitive fourth root of unity.

$$d_{n,4k+1}(x,q) = 2^{\left\lfloor \frac{n}{2} \right\rfloor} \prod_{i=1}^{n-1} \frac{4^{\left\lfloor \frac{i}{2} \right\rfloor} \left\lfloor \frac{i}{2} \right\rfloor !}{i!} \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left(\frac{x}{2} + i \right)_{n-2i+1}.$$

Corollary (A.)

This implies the 2-enumeration formula of ASMs.

Various specialisations IV

Theorem (A.)

The 3-enumeration case: let q be a primitive sixth root of unity.

$$d_{n,3k+1}(x,q) = c(n) \prod_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} (x+2+3i)_{n-1-2i},$$

with

$$c(n) = \begin{cases} 3^{\frac{(n-2)n}{4}} \prod_{i=0}^{n-1} \frac{\lfloor \frac{i}{2} \rfloor !}{i!} & n \text{ is even,} \\ 3^{\frac{(n-1)^2}{4}} \prod_{i=0}^{n-1} \frac{\lfloor \frac{i}{2} \rfloor !}{i!} & \text{otherwise.} \end{cases}$$

Corollary

This implies the 3-enumeration formula of ASMs.

Various specialisations V

Theorem (A.)

The 4-enumeration case: $q = 1$.

$$d_{n,2k+1}(x,1) = \prod_{i=2}^{\lfloor \frac{n}{2} \rfloor} (2i-1)^{-(n+1-2i)} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (x+2i) p_n(x) p_{n-1}(x),$$

with

$$p_1(x) = 1, \quad p_3(x) = 2x+5, \quad p_{2n}(x) = p_{2n-1}(x+2),$$

$$p_{2n+1}(x) = \Big((x+2n+1)(x+2n+2) p_{2n-1}(x) p_{2n-1}(x+4) - \\ -(x+1)(x+2) p_{2n-1}(x+2)^2 \Big) \left(2n p_{2n-3}(x+4)\right)^{-1}.$$

Connection to another determinant

Let q be a sixth root of unity, then holds

$$d_{n,3-k}(x, q^2) = q^{-n} \det_{1 \leq i,j \leq n} \left(\binom{x+i+j-2}{j-1} + q^k \delta_{i,j} \right).$$

Theorem (Ciucu-Eisenkölbl-Krattenthaler-Zare, 2001)

The above determinant counts weighted cyclically symmetric lozenge tilings of a hexagon with a triangular hole of size x.

Outlook I - What is x?

general x :

$$d_{n,k}(x,q)$$

specialising x :

Outlook I - What is x?

general x :

$d_{n,k}(x,q)$

$\downarrow$

specialising x : ASMs

ASMs Alternating Sign Matrices

Outlook I - What is x?

general x :

$$d_{n,k}(x,q) \quad \leftrightarrow \quad \det_{1 \le i,j \le n}\left(\binom{x+i+j-2}{j-1} + q^k \delta_{i,j}\right)$$

$\downarrow$

specialising x : ASMs

ASMs Alternating Sign Matrices

Outlook I - What is x?

general x :

$$d_{n,k}(x,q) \quad \leftrightarrow \quad \det_{1 \leq i,j \leq n}\left(\binom{x+i+j-2}{j-1} + q^k \delta_{i,j}\right)$$

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

specialising x : ASMs $\qquad\qquad$ DPPs

ASMs — Alternating Sign Matrices
DPPs — Descending Plane Partitions

Outlook I - What is x?

general x :

$$d_{n,k}(x,q) \quad \leftrightarrow \quad \det_{1\leq i,j\leq n}\left(\binom{x+i+j-2}{j-1}+q^k\delta_{i,j}\right)$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad \searrow$$

specialising x : ASMs DPPs ASTs

ASMs	Alternating Sign Matrices
DPPs	Descending Plane Partitions
ASTs	Alternating Sign Triangles

Outlook I - What is x?

general x : d-DPPs

$$\uparrow$$

$$d_{n,k}(x,q) \quad \leftrightarrow \quad \det_{1 \le i,j \le n}\left(\binom{x+i+j-2}{j-1} + q^k \delta_{i,j} \right)$$

$$\downarrow \qquad \downarrow \qquad \searrow$$

specialising x : ASMs DPPs ASTs

ASMs	Alternating Sign Matrices
DPPs	Descending Plane Partitions
ASTs	Alternating Sign Triangles
d-DPPs	d-Descending Plane Partitions

Outlook I - What is x?

$$\begin{array}{lcccc} \text{general } x: & & & d\text{-DPPs} & \text{ASts}\\ & & & \uparrow & \nearrow \\ & d_{n,k}(x,q) & \leftrightarrow & \det_{1\le i,j\le n}\left(\binom{x+i+j-2}{j-1}+q^k\delta_{i,j}\right) & \uparrow \\ & \downarrow & & \downarrow & \searrow \\ \text{specialising } x: & \text{ASMs} & & \text{DPPs} & \text{ASTs}\end{array}$$

ASMs	Alternating Sign Matrices
DPPs	Descending Plane Partitions
ASTs	Alternating Sign Triangles
d-DPPs	d-Descending Plane Partitions
ASts	Alternating Sign trapezoids

Outlook I - What is x?

general x :	?		d-DPPs	ASts
	$\uparrow$		$\uparrow$	$\nearrow$
	$d_{n,k}(x,q)$	$\leftrightarrow$	$\det_{1 \leq i,j \leq n} \left(\binom{x+i+j-2}{j-1} + q^k \delta_{i,j} \right)$	$\uparrow$
	$\downarrow$		$\downarrow$	$\searrow$
specialising x :	ASMs		DPPs	ASTs

ASMs	Alternating Sign Matrices
DPPs	Descending Plane Partitions
ASTs	Alternating Sign Triangles
d-DPPs	d-Descending Plane Partitions
ASts	Alternating Sign trapezoids

Outlook II - Different starting points

We started with an operator formula for ASMs weighted by their number of -1's. However there are also further operator formulas, e.g., for

- ASMs with respect to their inv and inv$'$ statistic (this is a refinement of the above),
- vertically symmetric ASMs.

Outlook III - symmetric functions

We had in the proof sketch:

$$
\mathrm{A}_n(Q) = \mathrm{CT}_{x_1,\ldots,x_n} \frac{\mathcal{AS}_{x_1,\ldots,x_n}\left(\prod_{i=1}^n (1+x_i)^i \prod_{1\le i<j\le n}(Q+(Q-1)x_i+x_j+x_ix_j)\right)}{\prod_{1\le i<j\le n}(x_j-x_i)}.
$$

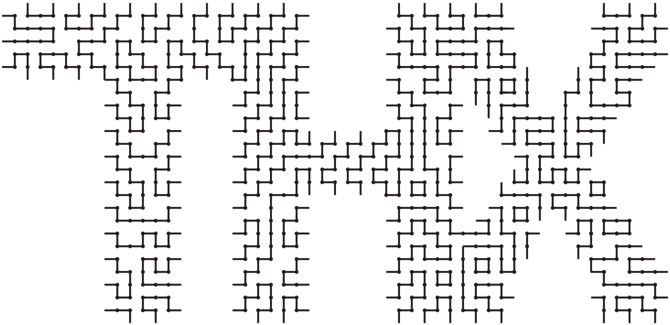
Outlook III - symmetric functions

We had in the proof sketch:

$$\mathrm{A}_n(Q) = \mathrm{CT}_{x_1,\ldots,x_n} \frac{\mathcal{AS}_{x_1,\ldots,x_n}\left(\prod_{i=1}^n (1+x_i)^i \prod_{1\le i<j\le n}(Q+(Q-1)x_i+x_j+x_ix_j)\right)}{\prod_{1\le i<j\le n}(x_j-x_i)}.$$

By substituting $x_i \mapsto x_i - 1$ and dividing by $x_1 \ldots x_n$ we obtain

$$\mathrm{A}_n(Q) = \left.\frac{\mathcal{AS}_{x_1,\ldots,x_n}\left(\prod_{i=1}^n (x_i)^{i-1} \prod_{1\le i<j\le n}(1+(Q-2)x_i+x_ix_j)\right)}{\prod_{1\le i<j\le n}(x_j-x_i)}\right|_{x_1=\ldots=x_n=1}.$$


THX