Alternating sign matrices and totally symmetric plane partitions

Florian Aigner

joint work with I. Fischer

Combinatorial Coworkspace 2022

Schur polynomials

A Gelfand-Tsetlin pattern (GT) is an array of integers of the form

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\begin{array}{ccccccc}
 & & & T_{1,1} & & & \\
 & & T_{2,1} & & T_{2,2} & & \\
 & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\
T_{n,1} & & \cdots & \cdots & \cdots & & T_{n,n}
\end{array}
\qquad
\begin{array}{ccccc}
 & & T_{i,i} & & \\
 & \rotatebox{45}{$\le$} & & \rotatebox{-45}{$\le$} & \\
T_{i+1,j} & & \leq & & T_{i+1,j+1}
\end{array}
$$

The weight of a GT pattern T is $\mathbf{x}^T := \prod_{i=1}^{n} x_i^{\sum_{j=1}^{i}(T_{i,j}) - \sum_{j=1}^{i-1}(T_{i-1,j})}$.

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For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ we define the Schur polynomial s_λ as

$$s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T,$$

where the sum is over all GTs T with bottom row $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

An Example

For $\lambda = (2, 2, 1)$ we have

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 2 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 2 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 2 & & 2 & \\ 1 & & 2 & & 2 \end{array}$$

$$x_1 x_2^2 x_3^2 \qquad x_1^2 x_2 x_3^2 \qquad x_1^2 x_2^2 x_3$$

$$s_{2,2,1}(x_1, x_2, x_3) = x_1 x_2^2 x_3^2 + x_1^2 x_2 x_3^2 + x_1^2 x_2^2 x_3.$$

Monotone Triangles

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The number of ASMs of size n is given by

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

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$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = 3^{-\binom{n}{2}} s_{(n-1,n-1,n-2,n-2,\ldots,1,1)}(\mathbf{1}_{2n}).$$

Motivation

How to encounter ASMs:

- Leibniz formula for λ-determinant; (deformation of Weyl's denominator formula).
- 6-vertex model with DWBC
- MacNeille completion of the Bruhat order on S_n.

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ASMs are connected to:

- Plane partitions
- $O(\tau)$-loop model via the *Razumov-Stroganov-Cantini-Sportiello Theorem*

Arrowed monotone triangles

We define the signed interval

$$\underline{[a,b]} = \begin{cases} [a,b] & a \leq b, \\ \emptyset & a = b+1, \\ [b+1, a-1] & a > b+1, \end{cases} \qquad \operatorname{sgn}(\underline{[a,b]}) = \begin{cases} 1, \\ 1, \\ -1. \end{cases}$$

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A arrowed monotone triangle is a monotone triangle $T = (T_{i,j})$ where the entries have a decoration $d(T_{i,j}) \subseteq \{\nwarrow, \nearrow\}$, s.t.

$$T_{i,j} \in \underline{[T_{i+1,j} \qquad\qquad , T_{i,j+1} \qquad\qquad]}.$$

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Examples

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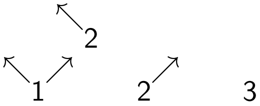

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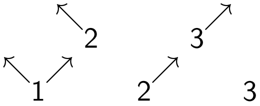

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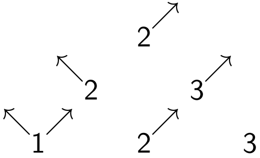

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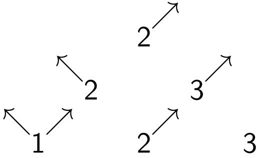

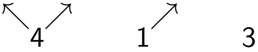

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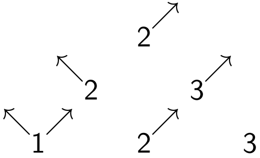

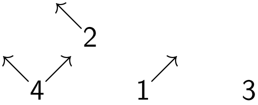

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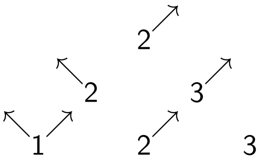

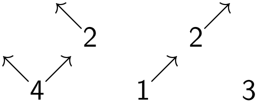

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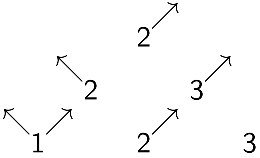

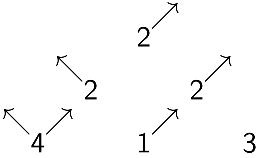

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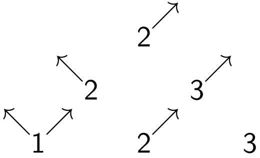

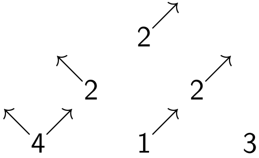

Question: What can one say about arrowed MTs from a geometrical point of view?

Weights for arrowed monotone triangles

We define the weight $M(T)$ of an arrowed MT T as

$$W(M) = \omega_{\emptyset}^{\#\emptyset} \omega_{\nearrow}^{\#\{\nearrow\}} \omega_{\nwarrow}^{\#\{\nwarrow\}} \omega_{\nwarrow\!\!\!\nearrow}^{\#\{\nwarrow,\nearrow\}}$$

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- $(\omega_{\emptyset}, \omega_{\nearrow}, \omega_{\nwarrow}, \omega_{\nwarrow\!\!\!\nearrow}) = (1, 0, 0, -t)$: Hall-Littlewood polynomials (up to rational function in t).

Denote by E_x denote the *shift operator* $E_x f(x) = f(x+1)$.

Theorem (A.-Fischer)

The multivariate generating function for arrowed monotone triangles with bottom row $(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$ *w.r.t. the weight* W *is*

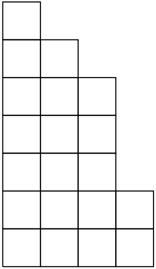
$$\sum_M W(M) = \prod_{i=1}^{n} (\omega_\emptyset + \omega_\nearrow x_i + \omega_\nwarrow x_i^{-1} + \omega_{\nwarrow\!\!\!\nearrow})$$

$$\times \prod_{1 \le i < j \le n} \left(\omega_\emptyset \,\mathrm{id} + \omega_\nearrow E_{k_i} + \omega_\nwarrow E_{k_j}^{-1} + \omega_{\nwarrow\!\!\!\nearrow} E_{k_i} E_{k_j}^{-1} \right) s_{(k_n, \ldots, k_1)}(\mathbf{x}) \Big|_{k_i = \lambda_i},$$

where the sum is over all monotone triangles with bottom row $(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$.

Frobenius notation for partitions

Let λ be a partition and l the length of the Durfee square. The Frobenius notation of λ is $(\lambda_1 - 1, \ldots, \lambda_l - l | \lambda'_1 - 1, \ldots, \lambda'_l - l)$.



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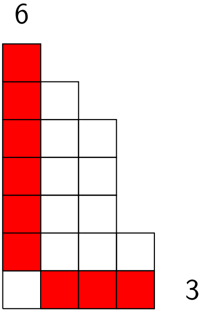

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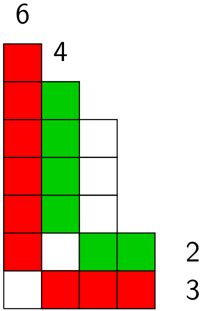

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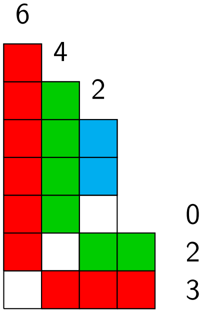

$$\lambda = (4,4,3,3,3,2,1)$$
$$= (3,2,0|6,4,2)$$

Plane partitions

Definition (MacMahon)

A *plane partition* $\pi = (\pi_{i,j})$ inside an (a, b, c)-box is an array of non-negative integers

$$\begin{matrix} \pi_{1,1} & \pi_{1,2} & \cdots & \pi_{1,b} \\ \pi_{2,1} & \pi_{2,2} & \cdots & \pi_{2,b} \\ \vdots & \vdots & & \vdots \\ \pi_{a,1} & \pi_{a,2} & \cdots & \pi_{a,b} \end{matrix}$$

such that $\pi_{i,j} \leq c$ and all rows and columns are weakly decreasing.

Five times plane partitions

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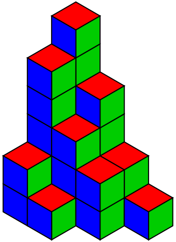

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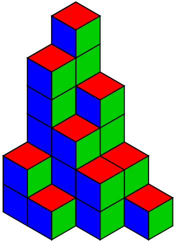

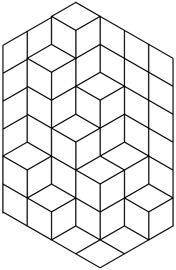

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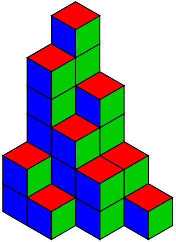
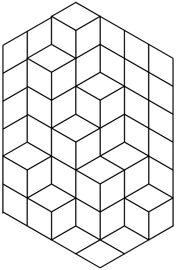
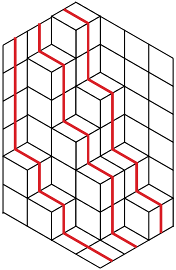
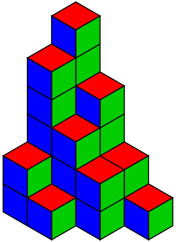


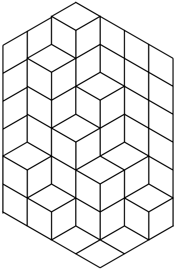


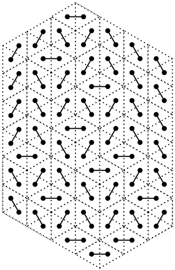


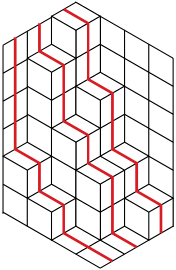
Five times plane partitions

$$\begin{matrix} 6 & 4 & 2 & 1 \\ 5 & 3 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{matrix}$$

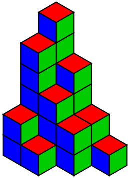




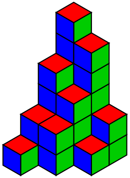




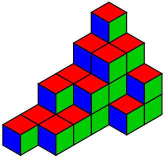
Symmetry operations on plane partitions



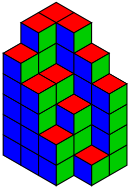
π



reflection / transposition



rotation

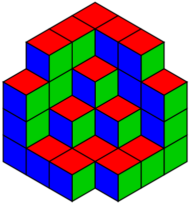


complement π^c

$$\pi^c_{i,j} := c - \pi_{a+1-i,b+1-j}$$

Denote by TSPP_n the set of totally symmetric plane partitions inside an (n, n, n)-box. For $T \in \mathsf{TSPP}_n$ define

$$\begin{matrix} 4 & 4 & 4 & 3 \\ 4 & 3 & 2 & 1 \\ 4 & 2 & 1 & 1 \\ 3 & 1 & 1 & 0 \end{matrix}$$



T

A new family of symmetric polynomials

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$$\mathrm{diag}(T) = (T_{i,i})' = (a_1, \ldots, a_l | b_1, \ldots, b_l),$$

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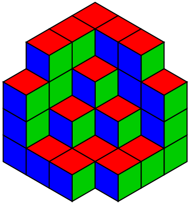
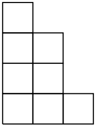


T



$\mathrm{diag}(T)$

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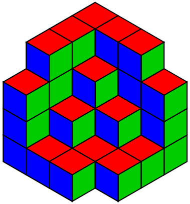
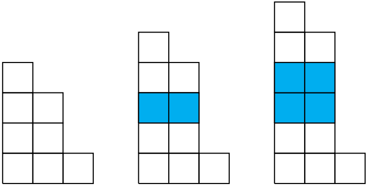
$$\pi_k(T) = (a_1, \ldots, a_l | b_1 + k, \ldots, b_l + k),$$

4	4	4	3
4	3	2	1
4	2	1	1
3	1	1	0



T



$\mathrm{diag}(T) = \pi_0(T)$ $\quad \pi_1(T)$ $\quad \pi_2(T)$

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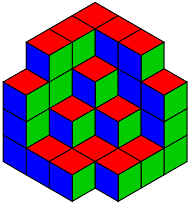
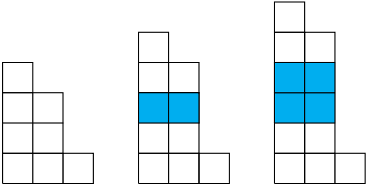
$$\omega_T(r, u, v, w) = r^l u^{\sum_{i=1}^{l}(a_i+1)} v^{\binom{n}{2} - \sum_{i=1}^{l}(b_i+1)} w^{\sum_{i=1}^{l}(b_i - a_i)}.$$

$$\begin{array}{cccc} 4 & 4 & 4 & 3 \\ 4 & 3 & 2 & 1 \\ 4 & 2 & 1 & 1 \\ 3 & 1 & 1 & 0 \end{array}$$



T



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We define the symmetric polynomial in $\mathbf{x} = (x_1,\ldots,x_{n+k-1})$

$$\mathcal{A}_{n,k}(r,u,v,w;\mathbf{x}) = \sum_{T \in \mathrm{TSPP}_{n-1}} \omega_T(r,u,v,w) s_{\pi_k(T)}(\mathbf{x}).$$

A multivariate generating function for ASMs

Theorem (A.-Fischer-Konvalinka-Nadeau-Tewari)

The multivariate generating function for ASMs w.r.t W is

$$\prod_{i=1}^{n} \frac{1}{(ux_i + vx_i^{-1} + w)^n} \sum_M W(M) = \mathcal{A}_{n,1}(1, u, v, w; \mathbf{x}),$$

where the sum is over all monotone triangles with bottom row $(1, 2, \ldots, n)$.

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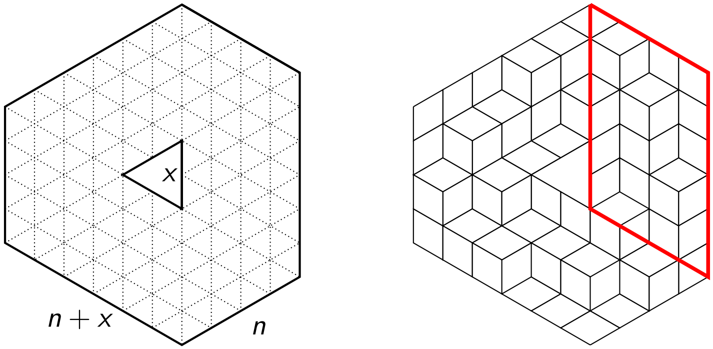
where the sum is over all monotone triangles with bottom row $(1,2,\ldots,n)$.

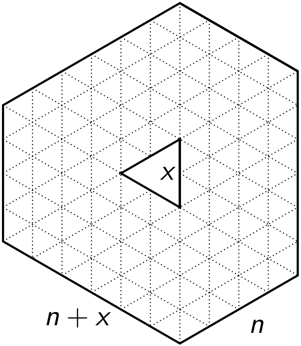
Question: Is there a representation theoretic interpretation?

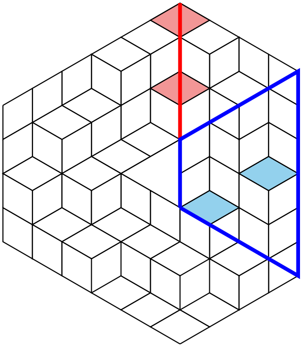
Cyclically symmetric lozenge tilings



Denote by $\mathrm{CS}_{n,x}(r,t)$ the generating function of cyclically symmetric lozenge tilings in a cored hexagon with side lengths $(n, n+x, n, n+x, n, n+x)$ with respect to the weight

Cyclically symmetric lozenge tilings





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$$r^{\#\lozenge \text{ on the red line}} \; t^{\#\lozenge \text{ in the blue region}}$$

Three enumeration formulas

Remember, the symmetric polynomials $\mathcal{A}_{n,k}(r,u,v,w;\mathbf{x})$ were defined as

$$\mathcal{A}_{n,k}(r,u,v,w;\mathbf{x}) = \sum_{T \in \mathrm{TSPP}_{n-1}} \omega_T(r,u,v,w) s_{\pi_k(T)}(\mathbf{x}).$$

Theorem (A.-Fischer)

Let n be a positive integer and let $\mathbf{1} = (1,\ldots,1)$. Then,

$$\mathcal{A}_{n,0}(r,1,t,1;\mathbf{1}) = \mathrm{CS}_{n-1,0}(r,t+2),$$
$$\mathcal{A}_{n,k}(r,1,-1,1;\mathbf{1}) = \mathrm{CS}_{n-1,2k}(r,1),$$
$$\mathcal{A}_{n,k}(r,1,0,1;\mathbf{1}) = \mathrm{CS}_{n-1,k}(r,2).$$