Fully Complementary Higher Dimensional Partitions

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Overview

- Partitions in various dimensions
- Fully complementary higher dimensional partitions
- Restricting to the two dimensional case

Part 1
Partitions in various dimensions

Partitions I

A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a weakly decreasing sequence of non-negative integers with all but finitely many entries equal to 0. We define the size $|\lambda| = \lambda_1 + \lambda_2 + \cdots$.

The partitions of size 5 are

$$(5)$$
$$(4,1)$$
$$(3,2)$$
$$(3,1,1)$$
$$(2,2,1)$$
$$(2,1,1,1)$$
$$(1,1,1,1,1)$$

Partitions II

Theorem

The generating function for partitions is

$$\sum_{\lambda} q^{|\lambda|} = \prod_{i \geq 1} \frac{1}{1 - q^i}.$$

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$$\sum_{\lambda} q^{|\lambda|}$$

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Define $m_i(\lambda)$ as the number of parts in λ equal to i. Then

$$\sum_{\lambda} q^{|\lambda|} = \sum_{\lambda} \prod_{i\geq 1} q^{m_i(\lambda)i} = \prod_{i\geq 1}(1 + q^i + q^{2i} + \cdots)$$

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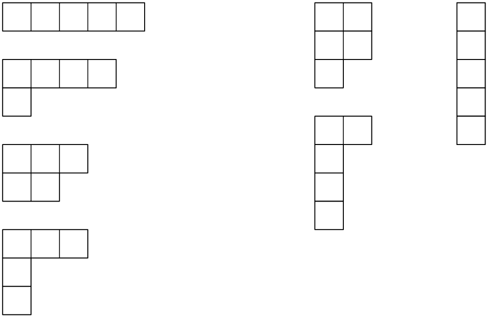
$$= \prod_{i\geq 1} \frac{1}{1-q^i}. \qquad \square$$

Partitions III

A Young diagram λ is a finite subset of $\mathbb{N}^2_{>0}$ such that $(x_1, x_2) \in \lambda$ implies $(y_1, y_2) \in \lambda$ for $1 \leq y_i \leq x_i$ for $1 \leq i \leq 2$. We represent Young diagrams as collection of boxes:



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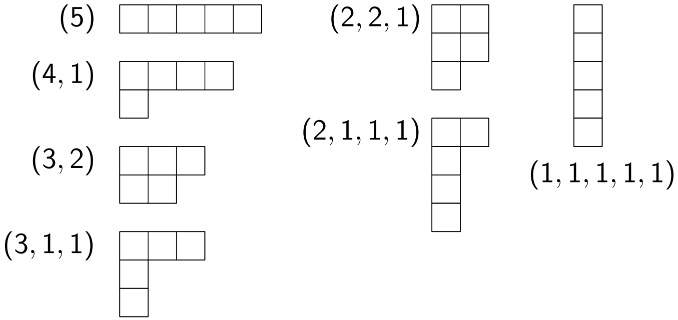

In the following we identify partitions and Young diagrams.

Partitions IV

We say that a Young diagram λ is contained in an (a, b)-box if $\lambda \subseteq [a] \times [b]$.

Theorem

The generating function of Young diagrams inside an (a, b)-box is

$$\sum_{\lambda} q^{|\lambda|} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{1-q^{i+j}}{1-q^{i+j-1}}.$$

Plane partitions I

A plane partition π is an array $(\pi_{i,j})$ of non-negative integers and finite support, which is weakly decreasing along rows and columns.

$$\begin{matrix} 3 & 2 & 2 & 1 \\ 2 & 2 & 1 & \\ 1 & & & \end{matrix}$$

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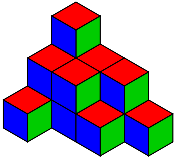

A 2-dimensional Young diagram λ is a finite subset of $\mathbb{N}^3_{>0}$ such that $(x_1, x_2, x_3) \in \lambda$ implies $(y_1, y_2, y_3) \in \lambda$ for $1 \leq y_i \leq x_i$ for $1 \leq i \leq 3$.

Plane partitions II

Theorem (MacMahon)

The generating function for plane partitions is

$$\sum_{\pi} q^{|\pi|} = \prod_{i \geq 1} \frac{1}{(1-q^i)^i}.$$

One can prove this theorem by using the Cauchy identity for Schur polynomials together with the RSK algorithm.

Plane partitions III

A 2-dimensional Young diagram λ is contained in an (a, b, c)-box if $\lambda \subseteq [a] \times [b] \times [c]$.

Theorem (MacMahon)

The generating function for 2-dimensional Young diagrams inside an (a, b, c)-box is

$$\sum_{\lambda} q^{|\lambda|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}.$$

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- The above can be proven by using Lattice paths together with the Lindström–Gessel–Viennot lemma.
- The left hand side can be interpreted as the principal specialisation of a Schur polynomial of rectangular shape which can be evaluated using Stanley's hook-content formula.

d-dimensional partitions I

A d-dimensional partition π is an array $(\pi_{i_1,\dots,i_d})$ of non-negative integers and finite support, such that

$$\pi_{i_1,\dots,i_d} \geq \pi_{i_1,\dots,i_k+1,\dots,i_d},$$

for all $i_1, \dots, i_d \in \mathbb{N}_{>0}$ and $1 \leq k \leq d$.

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$$\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \qquad \Leftrightarrow \qquad \{(1,1,1,1),(1,1,2,1)\}$$

A d-dimensional Young diagram λ is a finite subset of of $\mathbb{N}_{>0}^{d+1}$ such that $\mathbf{x} \in \lambda$ implies $\mathbf{y} \in \lambda$ for $1 \leq y_i \leq x_i$ for $1 \leq i \leq d+1$.

d-dimensional partitions II

Conjecture (MacMahon)

The generating function of d-dimensional partitions π is

$$\sum_{\pi} q^{|\pi|} = \prod_{i \geq 1} \frac{1}{(1-q^i)^{\binom{d+i-2}{d-1}}}.$$

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Theorem (Amanov–Yeliussizov, 2023)

The generating function of d-dimensional partitions π with respect to two statistics cor and $|\cdot|_{ch}$ is given by

$$\sum_{\pi} t^{cor(\pi)} q^{|\pi|_{ch}} = \prod_{i\geq 1} (1-tq^i)^{-\binom{i+d-2}{d-1}}.$$

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- Up to dimension 2, the enumeration formula is a product formula, i.e., of the form

$$\prod_{i_1=1}^{n_1} \cdots \prod_{i_{d+1}=1}^{n_{d+1}} \frac{f(i_1, \ldots, i_{d+1})}{g(i_1, \ldots, i_{d+1})},$$

  where f, g are linear polynomials in $i_1, \ldots, i_{d+1}$.

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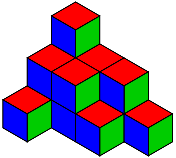
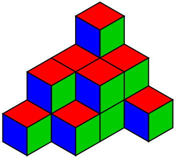
  where f, g are linear polynomials in $i_1, \ldots, i_{d+1}$.
- This implies that only "small" prime factors can appear.
- However the number of 3-dimensional partitions inside a $(2, 2, 2, 3)$-box is already 887, which is a prime.

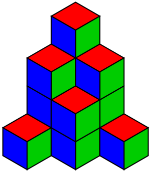
Part 2
Fully complementary higher dimensional partitions

Symmetries of boxed plane partitions

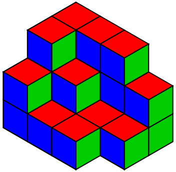




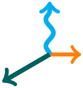
reflection

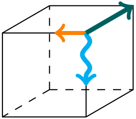


rotation



complementation



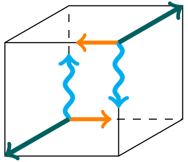


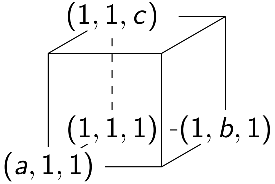
Self-complementary VS fully complementary

A $2d$-Young diagram λ inside an (a, b, c)-box is called self-complementary if it is equal to its complementation.

Self-complementary VS fully complementary

A $2d$-Young diagram λ inside an (a, b, c)-box is called self-complementary if λ, and λ "placed" at the corner (a, b, c) fills the (a, b, c)-box without overlap.





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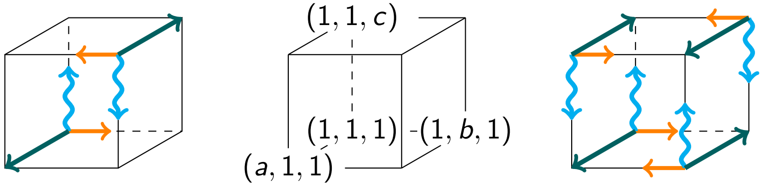

A $2d$-Young diagram λ inside an (a, b, c)-box is called fully complementary if λ, and λ "placed" at the corners $(a, b, 1)$, $(a, 1, c)$ and $(1, b, c)$ fill the (a, b, c)-box without overlap.

Fully Complementary in higher dimensions (intuitive)

Let $\mathbf{n} = (n_1, \ldots, n_{d+1})$ be a sequence of positive integers and λ a d-dimensional Young diagram.

- We take 2^d copies of λ and "place" them at the corners $(c_1, \ldots, c_{d+1})$ where an even number of the c_i are of the form $2n_i$, all others are 1.

We call λ fully complementary inside a $(2n_1, \ldots, 2n_{d+1})$-box if

- no two copies of λ overlap,
- the union of all copies is the full $(2n_1, \ldots, 2n_{d+1})$-box.

Fully Complementary in higher dimensions (precise)

Let $\mathbf{n} = (n_1, \ldots, n_{d+1})$ be a sequence of positive integers and $I \subseteq [d+1] = \{1, 2, \ldots, d+1\}$. We define for $\mathbf{x} = (x_1, \ldots, x_{d+1})$

$$\rho_{I,\mathbf{n}}(\mathbf{x}) := \left(\begin{cases} x_i & i \notin I, \\ 2n_i + 1 - x_i & i \in I, \end{cases} \right)_{1 \le i \le d+1}.$$

A d-dimensional Young diagram λ is called fully complementary inside a $(2n_1, \ldots, 2n_{d+1})$-box if

- for all even sized $I \neq J \subseteq [d+1]$ holds $\rho_{I,\mathbf{n}}(\lambda) \cap \rho_{J,\mathbf{n}}(\lambda) = \emptyset$,
- and $\displaystyle\bigcup_{\substack{I \subseteq [d+1] \\ |I| \text{ even}}} \rho_{I,\mathbf{n}}(\lambda) = [2n_1] \times \cdots \times [2n_{d+1}]$.

An example

For $\mathbf{n} = (1,1,1,1)$ the 3-dimensional Young diagram $\lambda = \{(1,1,1,1),(2,1,1,1)\}$ is fully complementary inside the $(2,2,2,2)$-box.

$$\rho_{\{1,2\}}(\lambda) = \{(2,2,1,1),(1,2,1,1)\}, \qquad \rho_{\{2,3\}}(\lambda) = \{(1,2,2,1),(2,2,2,1)\},$$
$$\rho_{\{1,3\}}(\lambda) = \{(2,1,2,1),(1,1,2,1)\}, \qquad \rho_{\{2,4\}}(\lambda) = \{(1,2,1,2),(2,2,1,2)\},$$
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$$\rho_{\{1,2,3,4\}}(\lambda) = \{(2,2,2,2),(1,2,2,2)\}.$$

There are three further Young diagrams which are fully complementary inside $(2,2,2,2)$:

$$\{(1,1,1,1),(1,2,1,1)\}$$
$$\{(1,1,1,1),(1,1,2,1)\}$$
$$\{(1,1,1,1),(1,1,1,2)\}$$

A main theorem

We call a d-dimensional partition π fully complementary inside a $(2n_1, \ldots, 2n_{d+1})$-box if its Young diagram is fully complementary in this box.

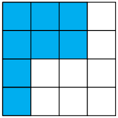
Theorem (SA, 2023)

Let $\mathbf{x} = (x_1, \ldots, x_{d+1})$, $\mathbf{n} = (n_1, \ldots, n_{d+1}) \in \mathbb{N}_{>0}^{d+1}$ *and denote by* $\mathrm{FCP}(\mathbf{n})$ *the set of fully complementary partitions inside a* $(2n_1, \ldots, 2n_{d+1})$*-box. Then*

$$\sum_{\mathbf{n} \in \mathbb{N}_{>0}^{d+1}} |\,\mathrm{FCP}(\mathbf{n})|\mathbf{x}^{\mathbf{n}} = \frac{\left(d+1-\sum_{i=1}^{d+1} x_i\right)\prod_{i=1}^{d+1} x_i}{\left(1-\sum_{i=1}^{d+1} x_i\right)\prod_{i=1}^{d+1}(1-x_i)}.$$

Stretching maps (intuitive)

We want to map d-dimensional partitions from $\mathrm{FCP}(\mathbf{n})$ injectively to $\mathrm{FCP}(\mathbf{n} + e_k)$ by a stretching map φ_k. For $d = 1$, this map can be defined by as shown next:



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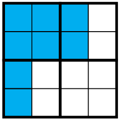

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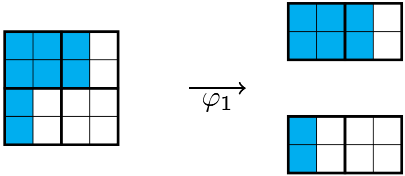

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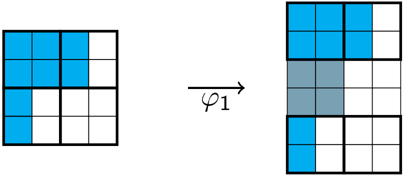

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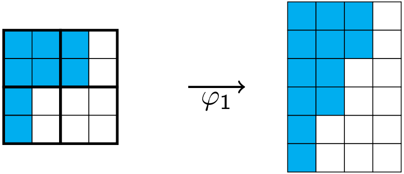

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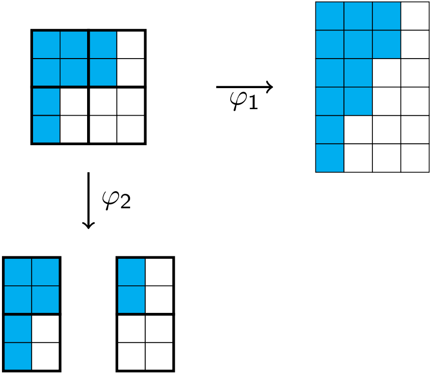

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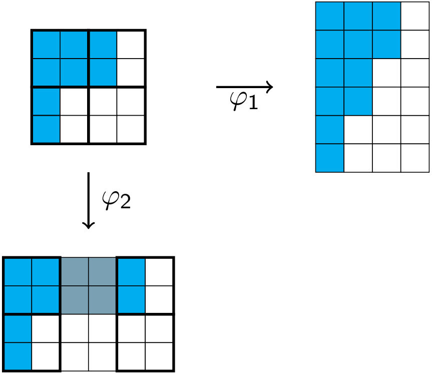

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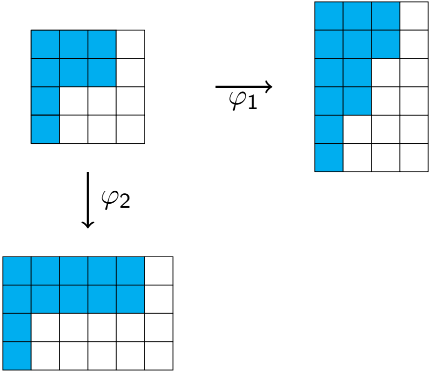

Stretching maps (precise)

For $1 \leq k \leq d$ define the map $\varphi_k : \mathrm{FCP}(\mathbf{n}) \to \mathrm{FCP}(\mathbf{n} + e_k)$

$$\varphi_k(\pi)_{i_1,\dots,i_d} = \begin{cases} \pi_{i_1,\dots,i_d} & i_k \leq n_k, \\ n_{d+1} & i_k \in \{n_k+1, n_k+2\} \\ & \text{ and } i_j \leq n_j \text{ for all } 1 \leq j \neq k \leq d, \\ \pi_{i_1,\dots,i_k-2,\dots,i_d} & i_k > n_k + 2, \\ 0 & \text{otherwise,} \end{cases}$$

and the map $\varphi_{d+1} : \mathrm{FCP}(\mathbf{n}) \to \mathrm{FCP}(\mathbf{n} + e_{d+1})$ as

$$\varphi_{d+1}(\pi)_{i_1,\dots,i_d} = \begin{cases} \pi_{i_1,\dots,i_d} + 2 & i_j \leq n_j \text{ for all } 1 \leq j \leq d, \\ \pi_{i_1,\dots,i_d} & \text{ otherwise.} \end{cases}$$

A recursive structure

Proposition

Let $\mathbf{n} = (n_1, \ldots, n_{d+1})$ *be a sequence of positive integers. Then* $\mathrm{FCP}(\mathbf{n})$ *is equal to the disjoint union*

$$\mathrm{FCP}(\mathbf{n}) = \dot{\bigcup}_{1 \le k \le d+1} \varphi_k\big(\mathrm{FCP}(\mathbf{n} - e_k)\big).$$

Note, that if exactly one $n_i = 0$, we have to define $\mathrm{FCP}(\mathbf{n})$ to consist of "the empty array" and extend the definitions of the φ_k appropriately.

Part 3
Restricting to the 2-dimensional case

Symmetry classes of FCPs I

There are many interesting symmetry classes of plane partitions. Is the same true for fully complementary plane partitions?

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symmetric	$\Rightarrow$	quasi symmetric
cyclically symmetric	$\Rightarrow$	?
self-complementary	$\Rightarrow$	self-complementary
transpose-complementary	$\Rightarrow$	quasi transpose-complementary

Symmetry classes of FCPs II

A plane partition π inside an (a, a, c)-box is called

- quasi-symmetric if

$$\pi_{i,j} = \pi_{j,i},$$

  for all $1 \leq i, j \leq a$ with $i + j \neq a + 1$,

- quasi transpose-complementary if

$$\pi_{i,j} + \pi_{a+1-j,a+1-i} = c,$$

  holds for all $1 \leq i, j \leq n$ with $i + j \neq a + 1$.

Symmetry classes of FCPs III

Proposition

1. *Denote by* $\mathrm{QS}(a,c)$ *the set of quasi symmetric fully complementary plane partitions (FCPP) inside an* (a,a,c)*-box, then holds*

$$\sum_{a,c\geq 0} |\,\mathrm{QS}(a,c)|x^a y^c = \frac{x+y-2x^2-xy}{(1-x)(1-2x-y)}.$$

2. *The number of self-complementary FCPPs inside an* $(2a,2b,2c)$*-box is* $\binom{a+b}{a}$.

3. *The number of quasi transpose-complementary FCPPs inside an* $(2a,2a,2c)$*-box is* 2^a.

Quasi transpose-complementary PPs

Theorem

The number of quasi transpose-symmetric plane partitions inside an (a, a, c)-box is equal to the number of symmetric plane partitions inside an (a, a, c)-box.

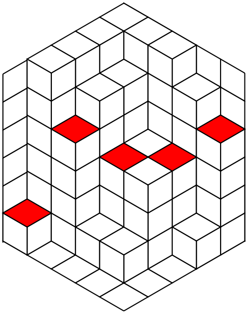
As a side product of the proof of the above theorem, we stumble upon the relation

$$2^{n-1}\,\mathrm{TCPP}(n, n, 2c) = \mathrm{SPP}(n-1, n-1, 2c+1),$$

where TCPP denotes the number of transpose-complementary plane partitions and SPP the number of symmetric plane partitions.

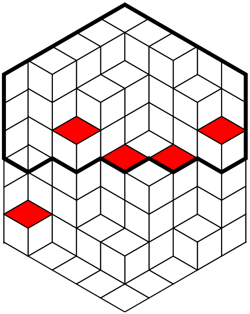
Proof idea

6	6	6	5	4
6	5	3	3	1
6	5	3	3	0
6	4	1	1	0
1	0	0	0	0



Proof idea

6	6	6	5	4
6	5	3	3	1
6	5	3	3	0
6	4	1	1	0
1	0	0	0	0



Proof idea

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6	5	3	3	1
6	5	3	3	0
6	4	1	1	0
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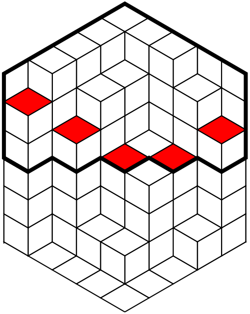

- We can restrict ourselves to the "fundamental domain" of the region, by reflecting diagonal entries if necessary and count them with weight 2 if reflected.

Proof idea

6	6	6	5	4
6	5	3	3	1
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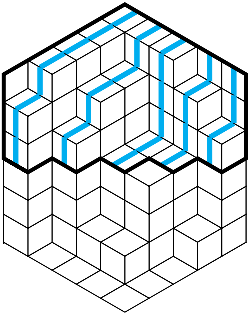

- We can restrict ourselves to the “fundamental domain” of the region, by reflecting diagonal entries if necessary and count them with weight 2 if reflected.
- Use the typical lattice path approach; however the first east step might have weight 2.

Some conjectures I

Conjecture (1)

Denote by $\mathrm{qspp}(a,c)$ *the number of quasi symmetric plane partitions inside an* (a,a,c)*-box. Then,*

$$\mathrm{qspp}(a,c-a)=\begin{cases} c\binom{c+a-1}{2a-1}p_a(c) & a \text{ is even},\\ \binom{c+a-1}{2a-1}p_a(c) & a \text{ is odd},\end{cases}$$

where $p_a(c)$ *is an irreducible polynomial in* $\mathbb{Q}[c]$ *that is even, i.e.,* $p_a(c)=p_a(-c)$.

We call a plane partition π inside an (a, a, c)-box quasi transpose complementary of second kind (QTC2), if

$$\pi_{i,j} + \pi_{a+1-j,a+1-i} = c,$$

for all $1 \leq i, j \leq a$ with $i \neq j$.

Conjecture (2)

Then for $a \geq 2$, the number $\mathrm{qtcpp}_2(a, c)$ of QTC2 plane partitions inside an (a, a, c)-box is given by

$$\mathrm{qtcpp}_2(a, c - \frac{a}{2}) = \begin{cases} c\binom{c+\frac{a}{2}-1}{a-1} p_a(c) & a \text{ is even}, \\ \binom{c+\frac{a}{2}-1}{a-1} p_a(c) & a \text{ is odd}, \end{cases}$$

where $p_a(c)$ is an irreducible polynomial in $\mathbb{Q}[c]$ that is even.

Conjecture (3)

Denote by $\mathrm{qtcspp}_2(a,c)$ *the number of symmetric QTC2 plane partitions. Then for* $a \geq 2$*,*

$$\mathrm{qtcspp}_2(a, c - \frac{a}{2}) = \begin{cases} c\binom{c+\frac{a}{2}-1}{a-1} p_a(c) & a \text{ is even,} \\ \binom{c+\frac{a}{2}-1}{a-1} p_a(c) & a \text{ is odd,} \end{cases}$$

where $p_a(c)$ *is an irreducible polynomial in* $\mathbb{Q}[c]$ *that is even.*