(-1)-Enumerations of arrowed Gelfand–Tsetlin patterns

Florian Schreier-Aigner

joint work with I. Fischer

Overview

- Arrowed Gelfand–Tsetlin patterns
- A (-1)-enumeration of arrowed GT patterns
  - The main results
  - Signless versions of our results
- Proof Sketch
  - A generalised bounded Littlewood identity
  - Determinant manipulations
  - Sister Celine's algorithm and creative telescoping

Classical GT pattern

A Gelfand-Tsetlin pattern (GT) is a triangular array of integers of the form

$$\begin{array}{ccccc} & & T_{1,1} & & \\ & T_{2,1} & & T_{2,2} & \\ \cdot^{\cdot^{\cdot}} & & & & \ddots \\ T_{n,1} & & \cdots\cdots & & T_{n,n} \end{array} \qquad \begin{array}{ccc} & T_{i,j} & \\ \rotatebox{45}{$\le$} & & \rotatebox{-45}{$\le$} \\ T_{i+1,j} & \le & T_{i+1,j+1} \end{array}$$

The weight of a GT pattern T is $\mathbf{x}^T := \prod_{i=1}^{n} x_i^{\sum_{j=1}^{i}(T_{i,j}) - \sum_{j=1}^{i-1}(T_{i-1,j})}$.

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For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, the Schur polynomial s_λ is

$$s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T,$$

where the sum is over all GTs T with bottom row $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

An Example

For $\lambda = (2,2,1)$ we have

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 2 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 2 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 2 & & 2 & \\ 1 & & 2 & & 2 \end{array}$$

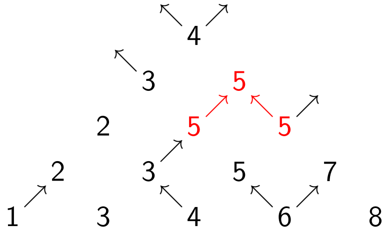
$$x_1x_2^2x_3^2 \qquad x_1^2x_2x_3^2 \qquad x_1^2x_2^2x_3$$

$$s_{(2,2,1)}(x_1, x_2, x_3) = x_1x_2^2x_3^2 + x_1^2x_2x_3^2 + x_1^2x_2^2x_3.$$

Arrowed Gelfand Tsetlin pattern

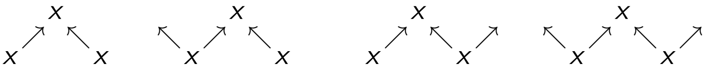
An arrowed Gelfand-Tsetlin pattern is a GT pattern $(T_{i,j})$ together with a decoration of the entries by the symbols $\emptyset, \nwarrow, \nearrow, \nwarrow\!\!\!\!\nearrow$ such that

$$T_{i+1,j} = T_{i,j} \text{ and } T_{i+1,j} \text{ is decorated by } \nearrow \text{ or } \nwarrow\!\!\!\!\nearrow,$$
$$\Leftrightarrow$$
$$T_{i+1,j+1} = T_{i,j} \text{ and } T_{i+1,j+1} \text{ is decorated by } \nwarrow \text{ or } \nwarrow\!\!\!\!\nearrow.$$



The weight of an AGT

We call the following local configurations special little triangles



The sign of an AGT T is

$$\operatorname{sgn}(T) = (-1)^{\#\text{ of special little triangles in } T}.$$

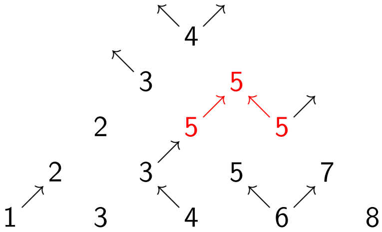
We define the weight $W(A)$ of A as

$$\operatorname{sgn}(T) \cdot t^{\#\emptyset} u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\!\nearrow} \cdot \mathbf{x}^T \prod_{i=1}^{n} x_i^{\#\nearrow \text{ in row i} - \#\nwarrow \text{ in row i}}.$$

An example

$$\operatorname{sgn}(T) \cdot t^{\#\emptyset} u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\!\nearrow} \cdot \mathbf{x}^{T} \prod_{i=1}^{n} x_i^{\#\nearrow \text{ in row i} - \#\nwarrow \text{ in row i}}.$$

The arrowed Gelfand–Tsetlin pattern



has weight $-t^7 u^3 v^2 w^3 x_1^4 x_2^3 x_3^5 x_4^6 x_5^5$.

A multivariate generating function for AGTs

Denote by E_x the *shift operator* $E_x f(x) = f(x+1)$.

Theorem (Fischer – S.A., 2023)

The weighted enumeration $\mathcal{A}_\lambda(t,u,v,w;\mathbf{x})$ of all AGTs with bottom row $(\lambda_n, \lambda_{n-1}, \dots, \lambda_1)$ is given by

$$\mathcal{A}_\lambda(t,u,v,w;\mathbf{x}) = \prod_{i=1}^{n} \left(ux_i + vx_i^{-1} + w + t\right) \times \prod_{1 \le i < j \le n} \left(t\,\mathrm{id} + uE_{\lambda_j} + vE_{\lambda_i}^{-1} + wE_{\lambda_j}E_{\lambda_i}^{-1}\right) s_\lambda(\mathbf{x}).$$

Known specialisations

$$t^{\#\emptyset} u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\!\nearrow}$$

- $\mathcal{A}_\lambda(1,0,0,0;\mathbf{x}) = s_\lambda(\mathbf{x})$,
- $\mathcal{A}_\lambda(0,0,1,0;\mathbf{x}) = s_{(\lambda_1-n,\lambda_2-n+1,\ldots,\lambda_n-1)}(\mathbf{x})$,

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- $\mathcal{A}_{(n,n-1,\ldots,1)}(0,u,v,w;\mathbf{x})$ yields a weighted enumeration of ASMs,
- $\mathcal{A}_{(2n,2n-2,\ldots,2)}(0,u,v,w;\mathbf{x})$ yields a weighted enumeration of vertically symmetric ASMs,

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For this talk we are interested in

- $\mathcal{A}_\lambda(1,1,1,-1;\mathbf{x})\big|_{x_i=1}$ and $\mathcal{A}_\lambda(1,1,1,0;\mathbf{x})\big|_{x_i=1}$.

The main results

Theorem (Fischer – S.A.)

For positive integers n, m *we have*

$$\sum_{0 \leq \lambda_n < \lambda_{n-1} < \ldots < \lambda_1 \leq m} \mathcal{A}_\lambda(1,1,1,-1;\mathbf{1}) = 2^n \prod_{i=1}^{n} \frac{(m-n+3i+1)_{i-1}(m-n+i+1)_i}{\left(\frac{m-n+i+2}{2}\right)_{i-1}(i)_i},$$

and

$$\sum_{0 \leq \lambda_n < \lambda_{n-1} < \ldots < \lambda_1 \leq m} \mathcal{A}_\lambda(1,1,1,0;\mathbf{1}) = 3^{\binom{n+1}{2}} \prod_{i=1}^{n} \frac{(2n+m+2-3i)_i}{(i)_i}.$$

The case $m = n - 1$

Setting $m = n - 1$ implies $\lambda = (n-1, n-2, \ldots, 1, 0)$ and hence

$$2^{-n}\mathcal{A}_{(n-1,n-2,\ldots,1,0)}(1,1,1,-1;\mathbf{1}) = 2^{n(n-1)/2}\prod_{i=0}^{n-1}\frac{(4i+2)!}{(n+2j+1)!}$$

$$= 1, 4, 60, 3328, 678912, \ldots \; .$$

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These numbers were conjectured by Di Francesco to enumerate

- configurations of the 20 vertex model in a certain domain, and
- domino tilings of Aztec-like triangles respectively.

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This was proved by Koutschan and extended in a recent preprint by Corteel, Huang and Krattenthaler.

A signless formulation

For a GT pattern A define $r(A)$ as the number of entries which are not equal to their north-east and north-west neighbours.

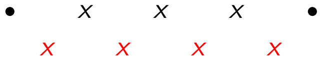
Theorem (Fischer – S.A.)

The (-1)-enumeration of AGT pattern with bottom row λ is equal to the weighted enumeration of GT pattern with bottom row λ where only the bottom row can contain three equal entries with respect to the weight $2^{r(A)}$.

We also have an analogous theorem in the $w = 0$ case.

Proof I

Assume there are four equal entries in one row. We regard the possible decorations of the top most such configuration:



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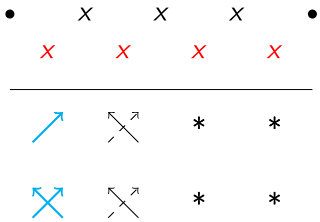

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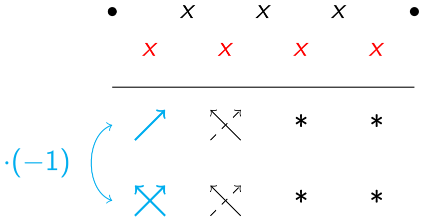

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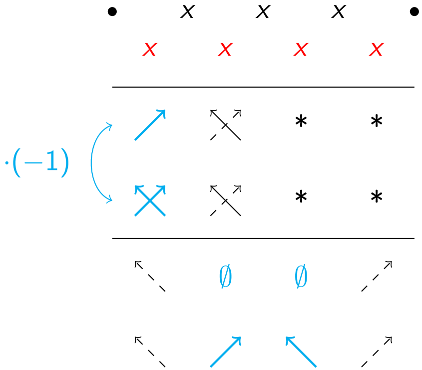

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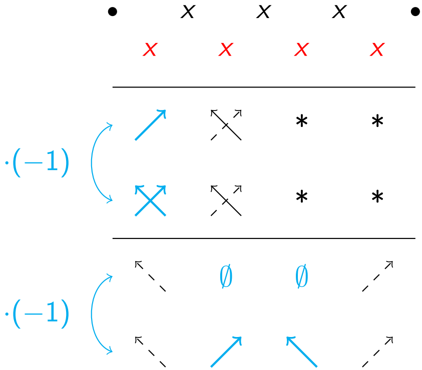

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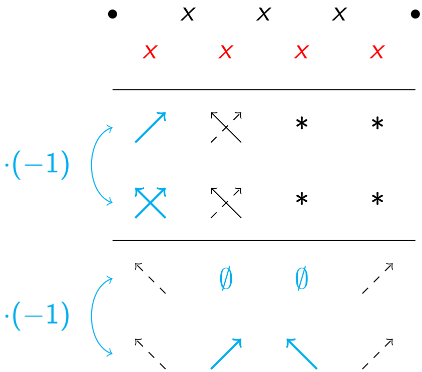

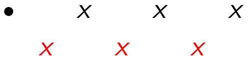

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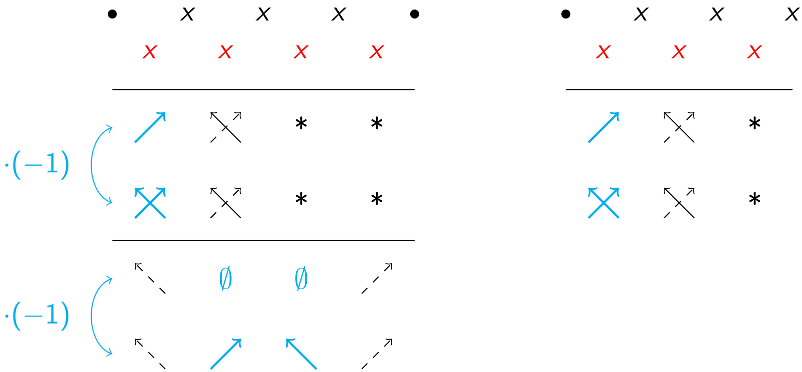

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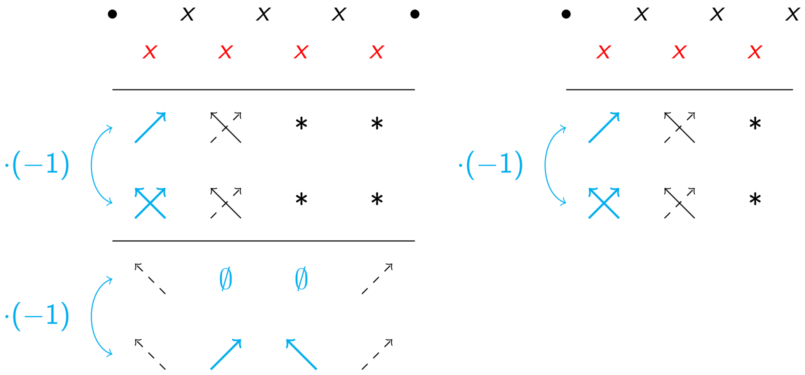

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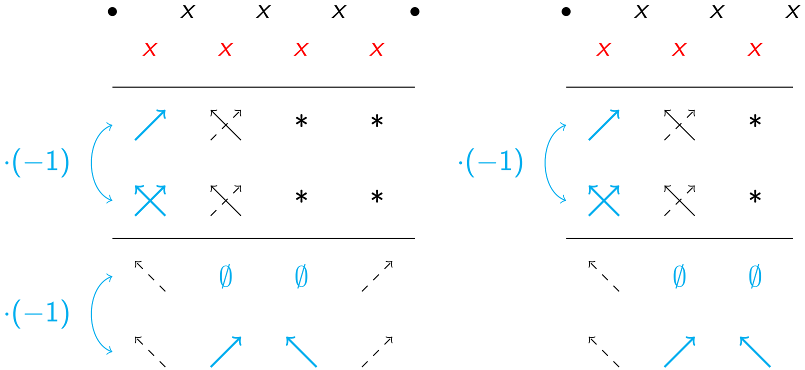

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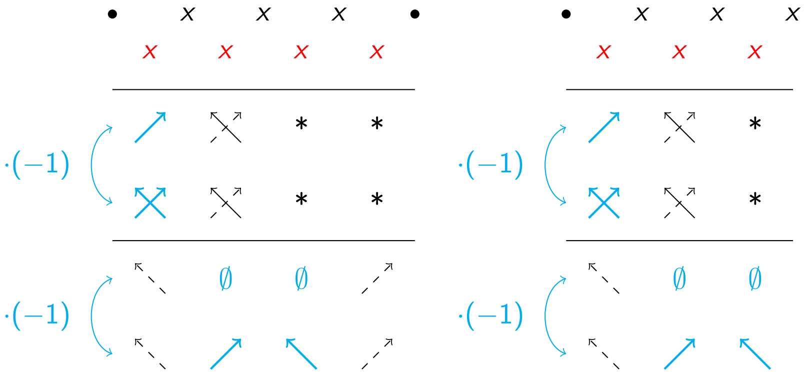

Proof II

three equal entries,
not in bottom row

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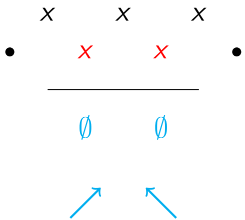

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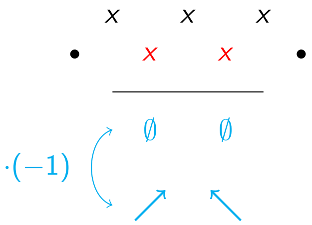

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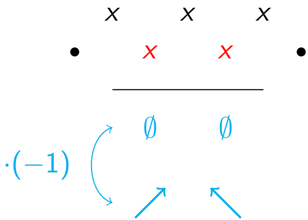

topmost
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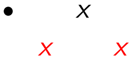

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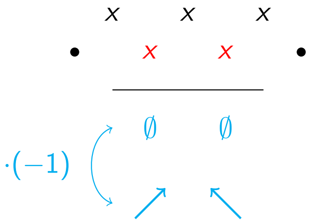

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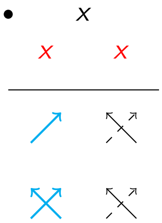

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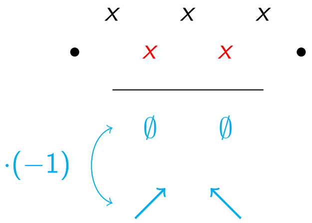

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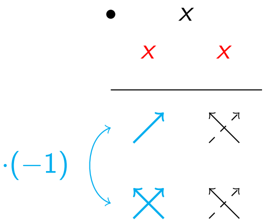

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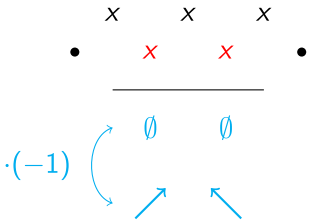

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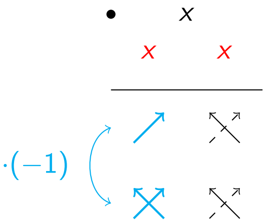
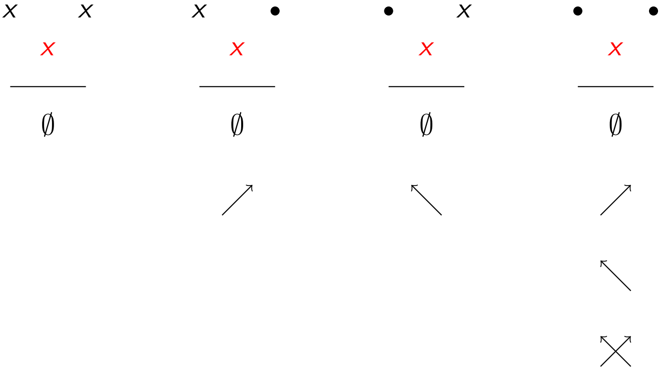


We can therefore assume:

- only the bottom row contains three equal entries,
- there are no special little triangles.

Proof III

Finally we regard the possible decorations of a single entry which do not yield a special little triangle.



Reminder of the main results

Theorem (Fischer – S.A.)

For positive integers n, m we have

$$\sum_{0 \le \lambda_n < \lambda_{n-1} < \ldots < \lambda_1 \le m} \mathcal{A}_\lambda(1,1,1,-1;\mathbf{1}) = 2^n \prod_{i=1}^{n} \frac{(m-n+3i+1)_{i-1}(m-n+i+1)_i}{\left(\frac{m-n+i+2}{2}\right)_{i-1}(i)_i},$$

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Overview of the proof of the main results

1. Obtain a determinant
2. Guess a (partial) LU decomposition.
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   - Avoid this by showing that it is a polynomial in m.
3. Proof the LU decomposition
   - It “suffices” to prove a hypergeometric triple sum.
   - For this we use Mathematica implementations of Sister Celine’s algorithm and creative telescoping

Reminder

We have the operator formula for evaluating $\mathcal{A}_\lambda$

$$\mathcal{A}_\lambda(t,u,v,w;\mathbf{x}) = \prod_{i=1}^{n} \left(ux_i + vx_i^{-1} + w + t\right)$$
$$\times \prod_{1\le i<j\le n} \left(t\,\mathrm{id} + uE_{\lambda_j} + vE_{\lambda_i}^{-1} + wE_{\lambda_j}E_{\lambda_i}^{-1}\right) s_\lambda(\mathbf{x}).$$

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The classical Littlewood identity is

$$\sum_\lambda s_\lambda(\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1\leq i<j\leq n} \frac{1}{1-x_i x_j},$$

where $\mathbf{x} = (x_1, \ldots, x_n)$.

A generalised bounded Littlewood identity

Theorem (Fischer, 2023+)

For positive integers n, m we have

$$\sum_{0 \leq \lambda_n < \lambda_{n-1} < \ldots < \lambda_1 \leq m} \mathcal{A}_\lambda(1,1,1,w;\mathbf{1}) =$$

$$\prod_{i=1}^{n} \left(x_i^{-1} + 1 + w + x_i\right) \frac{\det\limits_{1 \leq i,j \leq n} \left(x_i^{j-1} f_j(x_i) - x_i^{m+2n-j} f_j(x_i^{-1}) \right)}{\prod\limits_{i=1}^{n} (1 - x_i) \prod\limits_{1 \leq i < j \leq n} (1 - x_i x_j)(x_j - x_i)},$$

where $f_j(x) = (1+x)^{j-1}(1+wx)^{n-j}$.

For $m = 2\ell + 1$ we rewrite the above using the complete homogeneous symmetric polynomials h_k and set $x_1 = \ldots = x_n = 1$

...

A simple determinant

. . . and obtain

$$(3+w)^n 2^n \det_{1\leq i,j\leq n}\left(\sum_{p,q} w^{n-j-q}(-1)^j \times \binom{j-1}{p}\binom{n-j}{q}\binom{p-q-\ell+i-2}{2i-1}\right).$$

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$$\left. \times \binom{j-1}{p}\binom{n-j}{q}\binom{p-q-\ell+i-2}{2i-1}\right).$$

For $w=-1$ this can be simplified by using the Chu-Vandermonde identity

$$2^{2n}\det_{1\le i,j\le n}\left(\sum_{p}\binom{n-j}{p}\binom{\ell-p+i}{2i-j}\right).$$

Guessing the LU decomposition

Define $a_{i,j} = \sum_p \binom{n-j}{p}\binom{\ell-p+i}{2i-j}$ and

$$x_{i,j} = \begin{cases} (-1)^{i+1} \dfrac{(j)_j}{(2\ell-n+3j+2)_{j-1}(2\ell-n+i+2)_j} \\ \times \displaystyle\sum_t \Bigg(2^{2i-4t-n}(\ell-n/2+j/2+t+3/2)_{i-2t-1} & i \leq j, \\ \times \dfrac{(i-j-2t+1)_{2t}(i-2j+1)_{j-1-t}}{(1)_t(1)_{i-2t-1}} \Bigg) \\ 0 & \text{otherwise.} \end{cases}$$

Lemma

We have

$$\sum_{k=1}^{n} a_{i,k}x_{k,j} = \begin{cases} 1 & i = j, \\ 0 & i < j. \end{cases}$$

Next steps

- The expression in the sum can be simplified by using transformations for hypergeometric series (the package HYP by Krattenthaler was very useful for this!).
- It is then immediate that the expression is a polynomial in n and rational function in ℓ.

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- It is then immediate that the expression is a polynomial in n and rational function in ℓ.
- However, the triple sum can not be evaluated by summation rules for hypergeometric series (as far as we are aware of).
- We use two algorithms (described next) which help us to prove the Lemma.

Idea of Sister Celine's method

- Given a function $F(n) = \sum_k f(n,k)$ which we want to evaluate,
  - in our case: we want to show $F(n) = 0$ or $F(n) = 1$,
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  - $f(n,k)$ consists of Pochhammer symbols.
- Assume we can find a recursion for f of the form

$$\sum_{r,s} a_{r,s}(n) f(n-r, k-s) = 0,$$

then we obtain

$$0 = \sum_k \left(\sum_{r,s} a_{r,s}(n) f(n-r, k-s) \right) = \sum_{r,s} a_{r,s} F(n-r).$$

Basic idea of creative telescoping

- Given a function $F(n) = \sum_{k=i}^{j} f(n,k)$ which we want to evaluate.
- Assume we can find a recursion

$$a(n)f(n,k) + b(n)f(n+1,k) = g(n,k+1) - g(n,k),$$

- then we obtain for $F(n)$

$$a(n)F(n) + b(n)F(n+1) = g(n,j+1) - g(n,i).$$

Careful checking is necessary!

Let $f(n) = (n+1)_n$ and remember

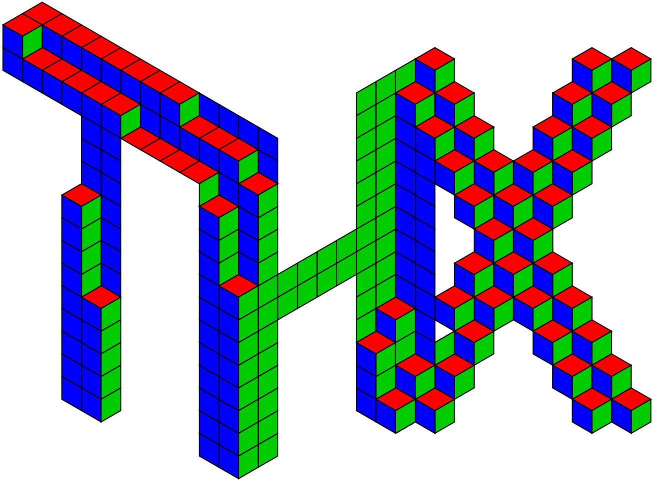
$$(x)_n = \begin{cases} (x)(x+1)\cdots(x+n-1) & n > 0, \\ 1 & n = 0, \\ \frac{1}{(x-1)(x-2)\cdots(x+n)} & n < 0. \end{cases}$$

The above algorithms will yield the recursion

$$f(n) = 2(2n-1)f(n-1),$$

which is however only true if $n \neq 0$.

More details on the proof.



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The triple sum

$$\sum_{\substack{s,t \\ 1 \le k \le j}} (-1)^{1+k+s+t} 2^{2-k-s} (k-n)_s (-r)_{k-1+2t} (j-t)_{j-t-1}$$

$$\times \frac{\left(\frac{-1-2i+4k-n-r+2s}{2}\right)_{2i-k-s} (2-2j+r)_{j-1-2t}}{(1)_{2i-k-s}(1)_s(1)_{k-1-2t}(1)_t}$$

$$= \begin{cases} \frac{(-r)_{2j-1}(-1+3j-r)_{j-1}}{(j)_j}, & 0 < i = j, \\ 0, & 0 < i < j \end{cases}$$

Call $f(n, r, i, j, k, s, t)$ the summand in the above sum ($\ell = \frac{n-r-3}{2}$).

Note that we can assume $t < j$.

A recursion for f

Using the computer, we found for $j \neq t$ the recursion

$$(3j - r - 2)(r + 1)_4 f(n, r, i, j, k, s, t) =$$
$$2(2j + 1)(2 - 2j + r)_2 f(n + 2, r + 4, i + 1, j + 1, k + 2, s, t + 1)$$
$$+ (j - r - 3) f(n + 2, r + 4, i + 1, j + 2, k + 2, s, t + 1),$$

which can be verified by hand.

Note that the polynomials in front of the f's do not depend on k, s, t.

Summing over k, s, t

Define

$$g(n,r,i,j,k) = \sum_{s,t} f(n,r,i,j,k,s,t),$$
$$h(n,r,i,j) = \sum_{1\le k\le j} g(n,r,i,j,k),$$

then the above implies

$$(3j-r-2)(r+1)_4 h(n,r,i,j) = 2(2j+1)(2j-r-3)_2$$
$$\times \Big(h(n+2,r+4,i+1,j+1) - g(n+2,r+4,i+1,j+1,j+2)\Big)$$
$$+ (j-r-3)h(n+2,r+4,i+1,j+2).$$

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then the above implies

$$(3j-r-2)(r+1)_4 h(n,r,i,j) = 2(2j+1)(2j-r-3)_2 \times \Big(h(n+2,r+4,i+1,j+1) - g(n+2,r+4,i+1,j+1,j+2)\Big) + (j-r-3)h(n+2,r+4,i+1,j+2).$$

We show in an extra step $g(n,r,i,j,j+1)=0$ which simplifies the above.

Summing over k, s, t

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We show in an extra step $g(n,r,i,j,j+1) = 0$ which simplifies the above.

An auxiliary result

We use creative telescoping to prove

$$h(n, r, i, j) = 0,$$

for $j \geq 2i$.

The involved polynomial coefficients have up to 1168 monomials.

Two inductions

For $i < j$ we need to show $h(n, r, i, j) = 0$. We use the previous recursion

$$\begin{aligned}(3j - r - 2)(r + 1)_4 h(n, r, i, j) &= \\ &2(2j + 1)(2j - r - 3)_2 h(n + 2, r + 4, i + 1, j + 1) \\ &\quad + (j - r - 3)h(n + 2, r + 4, i + 1, j + 2).\end{aligned}$$

and induction on i (starting with $i = 1$) and then $j - i$ (starting with $j \geq 2i$).

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- The base case follows from the auxiliary result.

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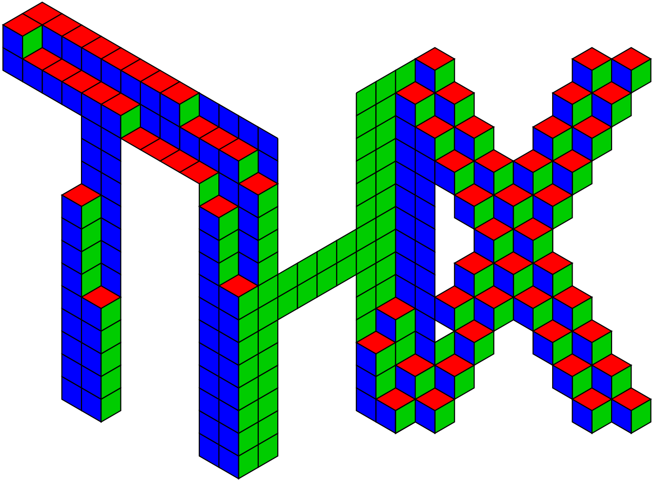
- The base case follows from the auxiliary result.
- The first induction hypothesis implies $h(n, r, i, j) = 0$.
- The second induction hypothesis implies $h(n + 2, r + 4, i + 1, j + 2) = 0$.
- We obtain the assertion since r is a variable.



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A signless formulation for $w = 0$

Theorem (Fischer – S.A.)

The (-1)-enumeration of AGT pattern with bottom row λ is equal to the weighted enumeration of GT pattern with bottom row λ where each entry appears at most twice in a row and entries are decorated by $\{\emptyset, \nearrow, \nwarrow\}$ such that the following is satisfied:

- *An entry may only point to entries with different values,*
- *two equal entries in a row are not allowed to be decorated both by $\emptyset$.*

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