Alternating sign matrices and totally symmetric plane partitions

Florian Aigner

joint work with I. Fischer

Arbeitsgemeinschaft Diskrete Mathematik

Outline

1. A multivariate generating function for ASMs
   - Alternating sign matrices (ASMs)
   - Plane partitions
   - Connecting both via symmetric polynomials
   - Proof sketch
2. A weighted generating function for d-DPPs
   - Extending the family of symmetric polynomials
   - Lozenge Tilings
   - Proof sketch
3. On the antisymmetriser to determinant lemma

Alternating sign matrices

Definition (Mills-Robbins-Rumsey)

An alternating sign matrix (ASM) of size n is an $n \times n$ matrix with entries $1, 0, -1$, such that

- all row- and column-sums are equal to 1,
- in each row and column, the non-zero entries alternate.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Enumeration of ASMs

The enumeration formula for ASMs was conjectured by Robbins and Rumsey in the early 1980s.

Theorem (Zeilberger, 1996)

The number of ASMs of size n is given by

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Enumeration of ASMs

The enumeration formula for ASMs was conjectured by Robbins and Rumsey in the early 1980s.

Theorem (Zeilberger, 1996)

The number of ASMs of size n is given by

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Further proofs were found by

- Kuperberg in 1996 using the six-vertex model approach,
- Fischer in 2007 using her *operator formula* (a short version of this paper appeared in 2016),
- A. in 2021 which is also based on the operator formula.

Monotone Triangles

A monotone triangle (MT) is an array of integers of the form

$$\begin{array}{ccccccc} & & & m_{1,1} & & & \\ & & m_{2,1} & & m_{2,2} & & \\ & \cdots & & & & \cdots & \\ m_{n,1} & & \cdots & \cdots & \cdots & & m_{n,n} \end{array}$$

$$\begin{array}{ccc} & m_{i,j} & \\ \leq & & \leq \\ m_{i+1,j} & < & m_{i+1,j+1} \end{array}$$

ASMs are in bijection to MTs with bottom row $(1, 2, \ldots, n)$.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Monotone Triangles

A monotone triangle (MT) is an array of integers of the form

$$\begin{array}{ccccccc} & & & m_{1,1} & & & \\ & & m_{2,1} & & m_{2,2} & & \\ & \iddots & & & & \ddots & \\ m_{n,1} & & \cdots & \cdots & \cdots & & m_{n,n} \end{array}$$

$$\begin{array}{ccc} & m_{i,j} & \\ \leq & & \leq \\ m_{i+1,j} & < & m_{i+1,j+1} \end{array}$$

ASMs are in bijection to MTs with bottom row $(1, 2, \ldots, n)$.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

4

Monotone Triangles

A monotone triangle (MT) is an array of integers of the form

$$\begin{array}{ccccccc} & & & m_{1,1} & & & \\ & & m_{2,1} & & m_{2,2} & & \\ & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ m_{n,1} & & \cdots & \cdots & \cdots & & m_{n,n} \end{array}$$

$$\begin{array}{ccc} & m_{i,j} & \\ \leq & & \leq \\ m_{i+1,j} & < & m_{i+1,j+1} \end{array}$$

ASMs are in bijection to MTs with bottom row $(1, 2, \ldots, n)$.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccc} & 4 & \\ 3 & & 4 \end{array}$$

Monotone Triangles

A monotone triangle (MT) is an array of integers of the form

$$\begin{array}{ccccccc} & & & m_{1,1} & & & \\ & & m_{2,1} & & m_{2,2} & & \\ & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ m_{n,1} & & \cdots & \cdots & \cdots & & m_{n,n} \end{array}$$

$$\begin{array}{ccc} & m_{i,i} & \\ \rotatebox{45}{$\le$} & & \rotatebox{-45}{$\ge$} \\ m_{i+1,j} & < & m_{i+1,j+1} \end{array}$$

ASMs are in bijection to MTs with bottom row $(1, 2, \ldots, n)$.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccccc} & & 4 & & \\ & 3 & & 4 & \\ 1 & & 3 & & 5 \end{array}$$

Monotone Triangles

A monotone triangle (MT) is an array of integers of the form

$$\begin{array}{ccccccc} & & & m_{1,1} & & & \\ & & m_{2,1} & & m_{2,2} & & \\ & \iddots & & & & \ddots & \\ m_{n,1} & & \cdots & \cdots & \cdots & & m_{n,n} \end{array}$$

$$\begin{array}{ccc} & m_{i,i} & \\ \leq & & \leq \\ m_{i+1,j} & < & m_{i+1,j+1} \end{array}$$

ASMs are in bijection to MTs with bottom row $(1, 2, \ldots, n)$.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 3 & & 4 & & \\ & 1 & & 3 & & 5 & \\ 1 & & 2 & & 4 & & 5 \end{array}$$

Monotone Triangles

A monotone triangle (MT) is an array of integers of the form

$$
\begin{array}{ccccccc}
 & & & m_{1,1} & & & \\
 & & m_{2,1} & & m_{2,2} & & \\
 & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\
m_{n,1} & \cdots & \cdots & \cdots & \cdots & \cdots & m_{n,n}
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & m_{i,i} & & \\
 & \rotatebox{45}{$\le$} & & \rotatebox{-45}{$\le$} & \\
m_{i+1,j} & & < & & m_{i+1,j+1}
\end{array}
$$

ASMs are in bijection to MTs with bottom row $(1, 2, \ldots, n)$.

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & -1 & 1 \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\qquad
\begin{array}{ccccccccc}
 & & & & 4 & & & & \\
 & & & 3 & & 4 & & & \\
 & & 1 & & 3 & & 5 & & \\
 & 1 & & 2 & & 4 & & 5 & \\
1 & & 2 & & 3 & & 4 & & 5
\end{array}
$$

Weights for monotone triangles

Let $M = (m_{i,j})$ be a monotone triangle. We define

$$
\begin{aligned}
s_i(M) &= \#j : \ m_{i+1,j} < m_{i,j} < m_{i+1,j+1}, && \text{(special entries)}\\
l_i(M) &= \#j : m_{i,j} = m_{i+1,j}, && \text{(left-leaning entries)}\\
r_i(M) &= \#j : m_{i,j} = m_{i+1,j+1}, && \text{(right-leaning entries)}\\
\widetilde{d}_i(M) &= \sum_j (m_{i,j}) - \sum_j (m_{i-1,j}) - i.
\end{aligned}
$$

The weight $\omega(M)$ is defined as

$$
\omega(M) = \prod_{i=1}^{n} u^{r_i(M)} v^{l_i(M)} x_i^{\widetilde{d}_i(M) + 2r_{i-1}(M)} (v + wx_i + ux_i^2)^{s_{i-1}}.
$$

Example - MTs with bottom row $(1,2,3)$

$$\widetilde{d}_i(M) = \sum_j (m_{i,j}) - \sum_j (m_{i-1,j}) - i,$$

$$\omega(M) = \prod_{i=1}^{n} u^{r_i(M)} v^{l_i(M)} x_i^{\widetilde{d}_i(M) + 2r_{i-1}(M)} (v + wx_i + ux_i^2)^{s_{i-1}}.$$

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array}$$

$$v^3 \qquad uv^2x_1x_2 \qquad uv^2x_2x_3 \qquad uvx_1(v + wx_2 + ux_2^2)x_3$$

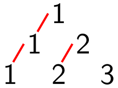
$$\begin{array}{ccccc} & & 3 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array}$$

$$u^2vx_1^2x_2x_3 \qquad u^2vx_1x_2x_3^2 \qquad u^3x_1^2x_2^2x_3^2$$

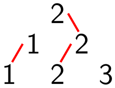
Example - MTs with bottom row $(1, 2, 3)$

$$\widetilde{d}_i(M) = \sum_j (m_{i,j}) - \sum_j (m_{i-1,j}) - i,$$

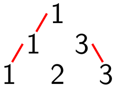
$$\omega(M) = \prod_{i=1}^{n} u^{r_i(M)} v^{l_i(M)} x_i^{\widetilde{d}_i(M) + 2r_{i-1}(M)} (v + w x_i + u x_i^2)^{s_{i-1}}.$$



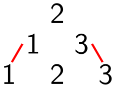
v^3



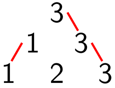
$uv^2 x_1 x_2$



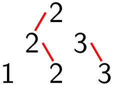
$uv^2 x_2 x_3$



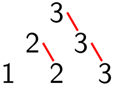
$uvx_1(v + wx_2 + ux_2^2)x_3$



$u^2 v x_1^2 x_2 x_3$



$u^2 v x_1 x_2 x_3^2$



$u^3 x_1^2 x_2^2 x_3^2$

A multivariate generating function for MTs

Denote by E_x denote the *shift operator* $E_x f(x) = f(x+1)$.

Theorem (A.-Fischer)

The multivariate generating function for monotone triangles with bottom row $(k_n, k_{n-1}, \ldots, k_1)$ *w.r.t. the weight* ω *is*

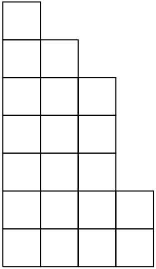
$$\sum_M \omega_M(\mathbf{x})$$

$$= \prod_{1 \le i < j \le n} \left(v E_{\lambda_j}^{-1} + w E_{\lambda_i} E_{\lambda_j}^{-1} + u E_{\lambda_i} \right) s_{(\lambda_n, \ldots, \lambda_1)}(\mathbf{x}) \Bigg|_{\lambda_i = k_i - 1},$$

where the sum is over all monotone triangles with bottom row $(k_n, k_{n-1}, \ldots, k_1)$.

Frobenius notation for partitions

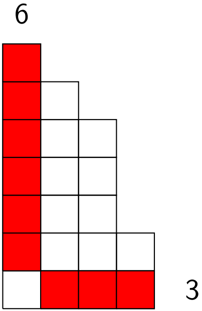
Let λ be a partition and l the length of the Durfee square. The Frobenius notation of λ is $(\lambda_1 - 1, \ldots, \lambda_l - l | \lambda'_1 - 1, \ldots, \lambda'_l - l)$.



$$\lambda = (4, 4, 3, 3, 3, 2, 1)$$

Frobenius notation for partitions

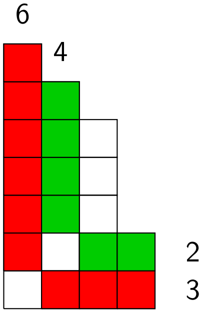
Let λ be a partition and l the length of the Durfee square. The Frobenius notation of λ is $(\lambda_1 - 1, \ldots, \lambda_l - l | \lambda'_1 - 1, \ldots, \lambda'_l - l)$.



$$\lambda = (4, 4, 3, 3, 3, 2, 1)$$
$$= (3, \quad |6, \quad)$$

Frobenius notation for partitions

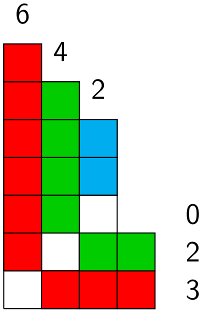
Let λ be a partition and l the length of the Durfee square. The Frobenius notation of λ is $(\lambda_1 - 1, \ldots, \lambda_l - l | \lambda'_1 - 1, \ldots, \lambda'_l - l)$.



$$\lambda = (4, 4, 3, 3, 3, 2, 1)$$
$$= (3, 2, \quad | 6, 4, \quad)$$

Frobenius notation for partitions

Let λ be a partition and l the length of the Durfee square. The Frobenius notation of λ is $(\lambda_1 - 1, \ldots, \lambda_l - l | \lambda'_1 - 1, \ldots, \lambda'_l - l)$.



$$\lambda = (4, 4, 3, 3, 3, 2, 1)$$
$$= (3, 2, 0 | 6, 4, 2)$$

Plane partitions

Definition (MacMahon)

A *plane partition* $\pi = (\pi_{i,j})$ inside an (a, b, c)-box is an array of non-negative integers

$$\begin{matrix} \pi_{1,1} & \pi_{1,2} & \cdots & \pi_{1,b} \\ \pi_{2,1} & \pi_{2,2} & \cdots & \pi_{2,b} \\ \vdots & \vdots & & \vdots \\ \pi_{a,1} & \pi_{a,2} & \cdots & \pi_{a,b} \end{matrix}$$

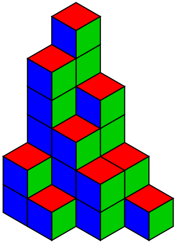
such that $\pi_{i,j} \leq c$ and all rows and columns are weakly decreasing.

Five times plane partitions

6	4	2	1
5	3	2	0
2	1	0	0

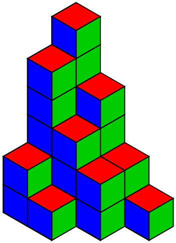
Five times plane partitions

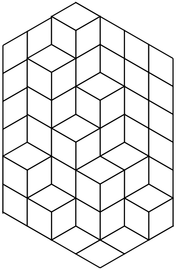
6	4	2	1
5	3	2	0
2	1	0	0



Five times plane partitions

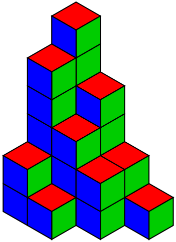
6	4	2	1
5	3	2	0
2	1	0	0

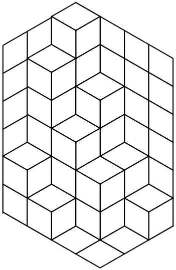


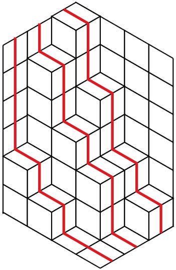


Five times plane partitions

6	4	2	1
5	3	2	0
2	1	0	0

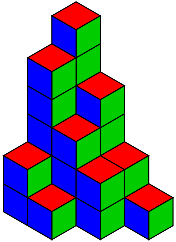


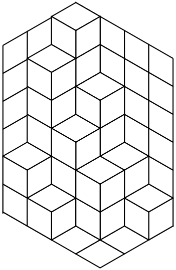


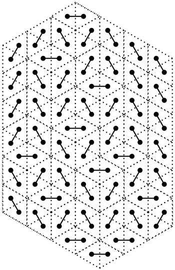


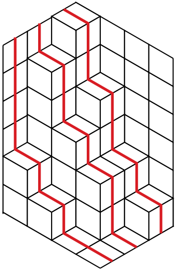
Five times plane partitions

6	4	2	1
5	3	2	0
2	1	0	0

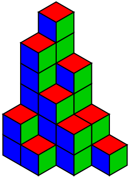




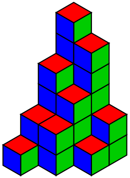




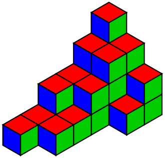
Symmetry operations on plane partitions



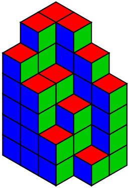
π



reflection / transposition



rotation



completion π^c

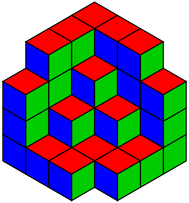
$$\pi^c_{i,j} := c - \pi_{a+1-i,b+1-j}$$

Totally symmetric plane partitions

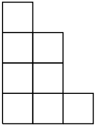
Denote by TSPP_n the set of totally symmetric plane partitions inside an (n, n, n)-box. For $T \in \mathrm{TSPP}_n$ define

$$\mathrm{diag}(T) = (T_{i,i})' = (a_1, \ldots, a_l | b_1, \ldots, b_l),$$
$$\pi(T) = (a_1, \ldots, a_l | b_1 {+1}, \ldots, b_l {+1}),$$
$$\omega_T(r, u, v, w) = r^l u^{\sum_{i=1}^{l}(a_i+1)} v^{\binom{n}{2} - \sum_{i=1}^{l}(b_i+1)} w^{\sum_{i=1}^{l}(b_i - a_i)}.$$

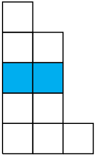
4	4	4	3
4	3	2	1
4	2	1	1
3	1	1	0



T



$\mathrm{diag}(T)$



$\pi(T)$

A multivariate generating function for ASMs

Theorem (A.-Fischer-Konvalinka-Nadeau-Tewari)

The multivariate generating function for ASMs w.r.t ω is

$$\sum_{M} \omega(M) = \sum_{T \in \mathrm{TSPP}_{n-1}} \omega_T(1, u, v, w) s_{\pi(T)}(\mathbf{x}),$$

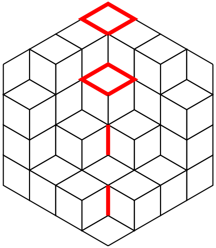
where the first sum is over all monotone triangles with bottom row $(1, 2, \ldots, n)$.

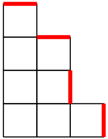
Proof sketch - right hand side

Proposition

Let $\lambda = (a_1, \dots, a_l | b_1, \dots, b_l)$. *The number of totally symmetric plane partitions* T *with* $\operatorname{diag}(T) = \lambda$ *is given by*

$$\det_{1 \le i,j \le l} \left(\binom{b_i}{a_j} \right).$$



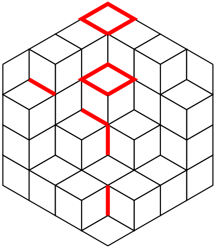


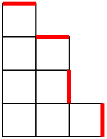
Proof sketch - right hand side

Proposition

Let $\lambda = (a_1, \ldots, a_l | b_1, \ldots, b_l)$. *The number of totally symmetric plane partitions* T *with* $\operatorname{diag}(T) = \lambda$ *is given by*

$$\det_{1 \le i,j \le l} \left(\binom{b_i}{a_j} \right).$$



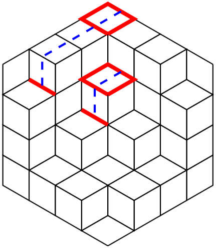


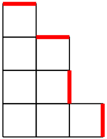
Proof sketch - right hand side

Proposition

Let $\lambda = (a_1, \ldots, a_l | b_1, \ldots, b_l)$. The number of totally symmetric plane partitions T with $\mathrm{diag}(T) = \lambda$ is given by

$$\det_{1 \le i,j \le l} \left(\binom{b_i}{a_j} \right).$$





Proof sketch - left hand side I

$$\sum_{M} \omega(M) = \prod_{1 \le i < j \le n} \left(uE_{\lambda_i} + wE_{\lambda_i} E_{\lambda_j}^{-1} + vE_{\lambda_j}^{-1} \right) s_{(\lambda_n, \ldots, \lambda_1)}(\mathbf{x}) \Big|_{\lambda_i = i-1}$$

Proof sketch - left hand side I

$$\sum_{M} \omega(M) = \prod_{1 \le i < j \le n} \left(u E_{\lambda_i} + w E_{\lambda_i} E_{\lambda_j}^{-1} + v E_{\lambda_j}^{-1} \right) s_{(\lambda_n, \ldots, \lambda_1)}(\mathbf{x}) \Bigg|_{\lambda_i = i-1}$$

$$= \prod_{1 \le i < j \le n} \left(u E_{\lambda_i} + w E_{\lambda_i} E_{\lambda_j}^{-1} + v E_{\lambda_j}^{-1} \right) \frac{\mathbf{ASym}_{\mathbf{x}} \left(\prod_{i=1}^{n} x_i^{\lambda_i + i - 1} \right)}{\prod_{1 \le i < j \le n} (x_j - x_i)} \Bigg|_{\lambda_i = i-1}$$

Proof sketch - left hand side I

$$\sum_{M} \omega(M) = \prod_{1 \le i < j \le n} \left(u E_{\lambda_i} + w E_{\lambda_i} E_{\lambda_j}^{-1} + v E_{\lambda_j}^{-1} \right) s_{(\lambda_n, \ldots, \lambda_1)}(\mathbf{x}) \Bigg|_{\lambda_i = i-1}$$

$$= \prod_{1 \le i < j \le n} \left(u E_{\lambda_i} + w E_{\lambda_i} E_{\lambda_j}^{-1} + v E_{\lambda_j}^{-1} \right) \frac{\mathbf{ASym}_{\mathbf{x}} \left(\prod_{i=1}^{n} x_i^{\lambda_i + i - 1} \right)}{\prod_{1 \le i < j \le n} (x_j - x_i)} \Bigg|_{\lambda_i = i-1}$$

$$= \frac{\mathbf{ASym}_{\mathbf{x}} \left[\prod_{1 \le i < j \le n} \left(u x_i + w x_i x_j^{-1} + v x_j^{-1} \right) \prod_{i=1}^{n} x_i^{\lambda_i + i - 1} \right]}{\prod_{1 \le i < j \le n} (x_j - x_i)} \Bigg|_{\lambda_i = i-1}$$

Proof sketch - left hand side I

$$\sum_{M} \omega(M) = \prod_{1 \le i < j \le n} \left(uE_{\lambda_i} + wE_{\lambda_i} E_{\lambda_j}^{-1} + vE_{\lambda_j}^{-1} \right) s_{(\lambda_n, \ldots, \lambda_1)}(\mathbf{x}) \Bigg|_{\lambda_i = i-1}$$

$$= \prod_{1 \le i < j \le n} \left(uE_{\lambda_i} + wE_{\lambda_i} E_{\lambda_j}^{-1} + vE_{\lambda_j}^{-1} \right) \frac{\mathbf{ASym_x} \left(\prod_{i=1}^{n} x_i^{\lambda_i + i - 1} \right)}{\prod_{1 \le i < j \le n} (x_j - x_i)} \Bigg|_{\lambda_i = i-1}$$

$$= \frac{\mathbf{ASym_x} \left[\prod_{1 \le i < j \le n} \left(ux_i + wx_i x_j^{-1} + vx_j^{-1} \right) \prod_{i=1}^{n} x_i^{2(i-1)} \right]}{\prod_{1 \le i < j \le n} (x_j - x_i)}$$

Proof sketch - left hand side II

Lemma (A.-Fischer)

Let $n \geq 1$, and $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n)$ be indeterminants. Then

$$\widehat{\mathbf{ASym}} \left(\prod_{1 \leq i \leq j \leq n} (y_j - x_i) \right) = \det_{1 \leq i,j \leq n} \left(y_i^j - x_i^j \right),$$

with $\widehat{\mathbf{ASym}}(f) = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}; y_{\sigma(1)}, \ldots, y_{\sigma(n)})$.

Proof sketch - left hand side II

Lemma (A.-Fischer)

Let $n \geq 1$, and $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n)$ be indeterminants. Then

$$\widehat{\mathbf{ASym}} \left(\prod_{1 \leq i \leq j \leq n} (y_j - x_i) \right) = \det_{1 \leq i,j \leq n} \left(y_i^j - x_i^j \right),$$

with $\widehat{\mathbf{ASym}}(f) = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}; y_{\sigma(1)}, \ldots, y_{\sigma(n)})$.

$$\sum_M \omega(M) = \frac{\det_{1 \leq i,j \leq n} \left(x_i^{n-j} p_j(x_i) \right)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)},$$

with $p_j(x) = \sum_{k=0}^{j-1} x^k (-w - ux)^k v^{j-k-1}$.

Proof sketch - left hand side III

For a sequence $L = (L_1, \ldots, L_n)$ of non-negative integers, we define

$$s_L(\mathbf{x}) := \frac{\det\limits_{1 \le i,j \le n} \left(x_i^{L_j+n-j} \right)}{\prod\limits_{1 \le i < j \le n} (x_i - x_j)} = \begin{cases} \operatorname{sgn}(\sigma) s_\lambda(\mathbf{x}) & L_j = \lambda_{\sigma(j)} + j - \sigma(j), \\ 0 & \text{otherwise,} \end{cases}$$

Proof sketch - left hand side III

For a sequence $L = (L_1, \ldots, L_n)$ of non-negative integers, we define

$$s_L(\mathbf{x}) := \frac{\det_{1 \le i,j \le n} \left(x_i^{L_j+n-j} \right)}{\prod_{1 \le i < j \le n} (x_i - x_j)} = \begin{cases} \operatorname{sgn}(\sigma) s_\lambda(\mathbf{x}) & L_j = \lambda_{\sigma(j)} + j - \sigma(j), \\ 0 & \text{otherwise,} \end{cases}$$

For $p_j(x) = \sum_{k \ge 0} a_{j,k} x^k$, we have

$$\frac{\det_{1 \le i,j \le n} \left(x_i^{n-j} p_j(x_i) \right)}{\prod_{1 \le i < j \le n} (x_i - x_j)} = \sum_{k_1, \ldots, k_n \ge 0} \left(\prod_{j=1}^n a_{j,k_j} \right) \frac{\det_{1 \le i,j \le n} \left(x_i^{k_j+n-j} \right)}{\prod_{1 \le i < j \le n} (x_i - x_j)}$$

Proof sketch - left hand side III

For a sequence $L = (L_1, \ldots, L_n)$ of non-negative integers, we define

$$s_L(\mathbf{x}) := \frac{\det\limits_{1 \le i,j \le n} \left(x_i^{L_j+n-j} \right)}{\prod\limits_{1 \le i < j \le n} (x_i - x_j)} = \begin{cases} \operatorname{sgn}(\sigma) s_\lambda(\mathbf{x}) & L_j = \lambda_{\sigma(j)} + j - \sigma(j), \\ 0 & \text{otherwise}, \end{cases}$$

For $p_j(x) = \sum_{k \ge 0} a_{j,k} x^k$, we have

$$\frac{\det\limits_{1 \le i,j \le n} \left(x_i^{n-j} p_j(x_i) \right)}{\prod\limits_{1 \le i < j \le n} (x_i - x_j)} = \sum_{k_1, \ldots, k_n \ge 0} \left(\prod_{j=1}^{n} a_{j,k_j} \right) \frac{\det\limits_{1 \le i,j \le n} \left(x_i^{k_j+n-j} \right)}{\prod\limits_{1 \le i < j \le n} (x_i - x_j)}$$

$$= \sum_{\lambda} s_\lambda(\mathbf{x}) \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} a_{j, \lambda_{\sigma(j)} + j - \sigma(j)} \right)$$

Proof sketch - left hand side III

For a sequence $L = (L_1, \ldots, L_n)$ of non-negative integers, we define

$$s_L(\mathbf{x}) := \frac{\det_{1 \le i,j \le n}\left(x_i^{L_j+n-j}\right)}{\prod_{1 \le i < j \le n}(x_i - x_j)} = \begin{cases} \operatorname{sgn}(\sigma) s_\lambda(\mathbf{x}) & L_j = \lambda_{\sigma(j)} + j - \sigma(j), \\ 0 & \text{otherwise,} \end{cases}$$

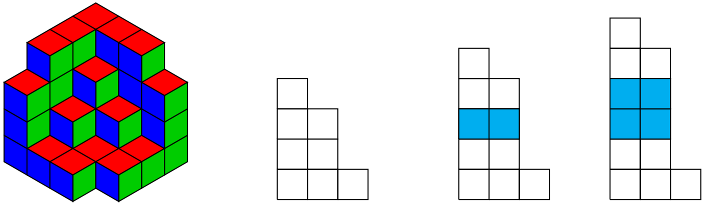
For $p_j(x) = \sum_{k \ge 0} a_{j,k} x^k$, we have

$$\frac{\det_{1 \le i,j \le n}\left(x_i^{n-j} p_j(x_i)\right)}{\prod_{1 \le i < j \le n}(x_i - x_j)} = \sum_{k_1, \ldots, k_n \ge 0} \left(\prod_{j=1}^{n} a_{j,k_j}\right) \frac{\det_{1 \le i,j \le n}\left(x_i^{k_j+n-j}\right)}{\prod_{1 \le i < j \le n}(x_i - x_j)}$$

$$= \sum_{\lambda} s_\lambda(\mathbf{x}) \det_{1 \le i,j \le n}\left(a_{j,\lambda_i+j-i}\right).$$

Extending the family of symmetric polynomials

For $T \in \mathrm{TSPP}_n$ with diagonal $\mathrm{diag}(T) = (a_1, \ldots, a_l | b_1, \ldots, b_l)$ define $\pi_k(T) = (a_1, \ldots, a_l | b_1 + k, \ldots, b_l + k)$.

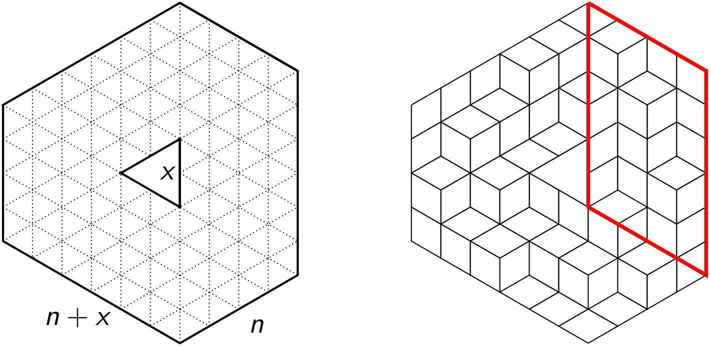


T $\qquad$ $\mathrm{diag}(T) = \pi_0(T)$ $\qquad$ $\pi_1(T)$ $\qquad$ $\pi_2(T)$

We define the symmetric polynomial in $\mathbf{x} = (x_1, \ldots, x_{n+k-1})$

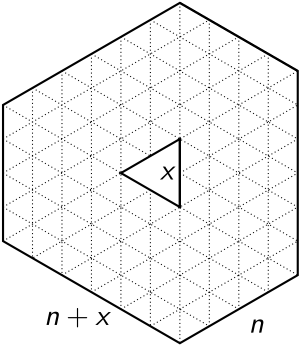
$$\mathcal{A}_{n,k}(r, u, v, w; \mathbf{x}) = \sum_{T \in \mathrm{TSPP}_{n-1}} \omega_T(r, u, v, w) s_{\pi_k(T)}(\mathbf{x}).$$

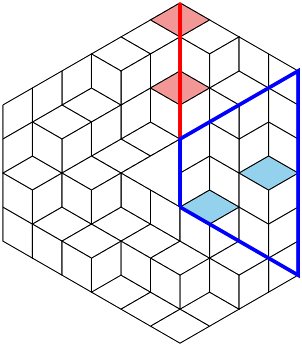
Cyclically symmetric lozenge tilings I



Denote by $\mathrm{CS}_{n,x}(r,t)$ the generating function of cyclically symmetric lozenge tilings in a cored hexagon with side lengths $(n, n+x, n, n+x, n, n+x)$ with respect to the weight

Cyclically symmetric lozenge tilings I





Denote by $\mathrm{CS}_{n,x}(r,t)$ the generating function of cyclically symmetric lozenge tilings in a cored hexagon with side lengths $(n, n+x, n, n+x, n, n+x)$ with respect to the weight

$$r^{\#\lozenge \text{ on the red line}} \; t^{\#\lozenge \text{ in the blue region}}$$

Cyclically symmetric lozenge tilings II

- **Krattenthaler 2006:** Cyclically symmetric lozenge tilings in a cored hexagon with side lengths $(n, n + x, n, n + x, n, n + x)$ are in bijection to $(2 - x)$-descending plane partitions (DPPs).
- **Andrews 1979:** introduced and enumerated d-DPPs.
- **Mills, Robbins and Robbins 1982:** conjectured ASMs to be equinumerous to 0-DPPs.
- **Behrend, Di Francesco and Zinn-Justin 2013:** showed that ASMs and 0-DPPs are equinumerous with respect to four statistics.

Three enumeration formulas

Remember, the symmetric polynomials $\mathcal{A}_{n,k}(r,u,v,w;\mathbf{x})$ were defined as

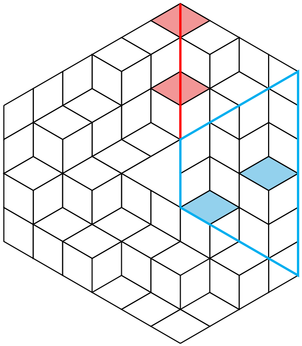
$$\mathcal{A}_{n,k}(r,u,v,w;\mathbf{x}) = \sum_{T \in \mathrm{TSPP}_{n-1}} \omega_T(r,u,v,w) s_{\pi_k(T)}(\mathbf{x}).$$

Theorem (A.-Fischer)

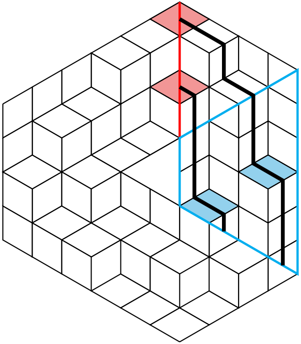
Let n be a positive integer and let $\mathbf{1} = (1,\ldots,1)$. Then,

$$\mathcal{A}_{n,0}(r,1,t,1;\mathbf{1}) = \mathrm{CS}_{n-1,0}(r,t+2),$$
$$\mathcal{A}_{n,k}(r,1,-1,1;\mathbf{1}) = \mathrm{CS}_{n-1,2k}(r,1),$$
$$\mathcal{A}_{n,k}(r,1,0,1;\mathbf{1}) = \mathrm{CS}_{n-1,k}(r,2).$$

Proof sketch - right hand side



Proof sketch - right hand side



$$\mathrm{CS}_{n,x}(r,t) \quad = \quad \det_{0\le i,j\le n-1}\left(\delta_{i,j} + r\sum_{l\ge 0}\binom{i}{l}\binom{j+x}{l+x}t^{j-l}\right).$$

Proof sketch - left hand side

Lemma (A.-Fischer)

Let $n \geq 2$ be an integer, then

$$\mathcal{A}_{n,k}(r,u,v,w;\mathbf{x}) = \det_{0 \leq i,j \leq n-2} \left((-1)^{j-i} v^{j+1} \binom{i}{j} + r u^{i+1} w^{j-i} s_{(i|j+k)}(\mathbf{x}) \right).$$

Proof sketch - left hand side

Lemma (A.-Fischer)

Let $n \geq 2$ be an integer, then

$$\mathcal{A}_{n,k}(r,u,v,w;\mathbf{x}) = \det_{0 \leq i,j \leq n-2} \left((-1)^{j-i} v^{j+1} \binom{i}{j} + r u^{i+1} w^{j-i} s_{(i|j+k)}(\mathbf{x}) \right).$$

Proof sketch

$$\mathcal{A}_{n,k}(r,u,v,w;\mathbf{x}) = \sum_{\lambda=(a_1,\ldots,a_l|b_1,\ldots,b_l)} s_\lambda(\mathbf{x}) \det_{1 \leq i,j \leq l} \left(\binom{b_i}{a_j} \right) \times r^l u^{\sum_i (a_i+1)} v^{\binom{n}{2} - \sum_i (b_i+1)} w^{\sum_i (b_i - a_i)}.$$

Proof sketch - left hand side

Lemma (A.-Fischer)

Let $n \geq 2$ be an integer, then

$$\mathcal{A}_{n,k}(r,u,v,w;\mathbf{x}) = \det_{0 \leq i,j \leq n-2} \left((-1)^{j-i} v^{j+1} \binom{i}{j} + r u^{i+1} w^{j-i} s_{(i|j+k)}(\mathbf{x}) \right).$$

Proof sketch

$$\mathcal{A}_{n,k}(r,u,v,w;\mathbf{x}) = \sum_{\lambda=(a_1,\ldots,a_l|b_1,\ldots,b_l)} s_\lambda(\mathbf{x}) \det_{1 \leq i,j \leq l} \left(\binom{b_i}{a_j} \right) \times r^l u^{\sum_i (a_i+1)} v^{\binom{n}{2} - \sum_i (b_i+1)} w^{\sum_i (b_i - a_i)}.$$

(Giambelli) $$s_{(a_1,\ldots,a_l|b_1,\ldots,b_l)}(\mathbf{x}) = \det_{1 \leq i,j \leq l} \left(s_{(a_i|b_j)}(\mathbf{x}) \right).$$

A combinatorial interpretation I

$$\widehat{\mathbf{ASym}}\left(\prod_{1\le i\le j\le n}(y_j - x_i)\right) = \det_{1\le i,j\le n}\left(y_i^j - x_i^j\right).$$

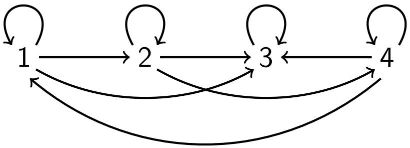
A combinatorial interpretation I

$$\widehat{\mathbf{ASym}}\left(\prod_{1\le i\le j\le n}(y_j+x_i)\right)=\det_{1\le i,j\le n}\left(y_i^j+(-1)^{j+1}x_i^j\right).$$

A combinatorial interpretation I

$$\widehat{\mathbf{ASym}} \left(\prod_{1 \le i \le j \le n} (y_j + x_i) \right) = \det_{1 \le i,j \le n} \left(y_i^j + (-1)^{j+1} x_i^j \right).$$

We consider orientations O of the complete graph on n vertices together with a loop from each vertex to itself,



$x_1^2 y_1 x_2^3 x_3 y_4^3$

with weight $w(O) = \prod_{i=1}^n x_i^{\#\{j \ge i : i \to j\}} y_i^{\#\{j \le i : j \leftarrow i\}}$.

A combinatorial interpretation II

We consider the subset $\mathcal{P}_n$ of orientations obtained recursively:

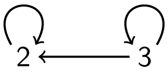
- start with a vertex and its loop.

x_2

A combinatorial interpretation II

We consider the subset $\mathcal{P}_n$ of orientations obtained recursively:

- start with a vertex and its loop.
- add a neighbouring vertex at the left (resp. right) with all new edges oriented to the right (resp. left).

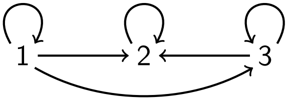


$x_2 y_3^2$

A combinatorial interpretation II

We consider the subset $\mathcal{P}_n$ of orientations obtained recursively:

- start with a vertex and its loop.
- add a neighbouring vertex at the left (resp. right) with all new edges oriented to the right (resp. left).

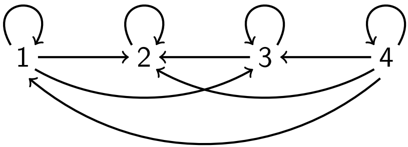


$x_1^3 x_2 y_3^2$

A combinatorial interpretation II

We consider the subset $\mathcal{P}_n$ of orientations obtained recursively:

- start with a vertex and its loop.
- add a neighbouring vertex at the left (resp. right) with all new edges oriented to the right (resp. left).



$x_1^3 x_2 y_3^2 y_4^4$

A combinatorial interpretation III

We claim $\widehat{\mathbf{ASym}}\left(\sum_{O \in \mathcal{P}_n} w(O)\right) = \det_{1 \le i,j \le n}\left(y_i^j + (-1)^{j+1} x_i^j\right)$:

A combinatorial interpretation III

We claim $\widehat{\mathbf{ASym}}\left(\sum_{O \in \mathcal{P}_n} w(O)\right) = \det_{1 \le i,j \le n}\left(y_i^j + (-1)^{j+1} x_i^j\right)$:

$$\sum_{O \in \mathcal{P}_n} w(O) = y_n^n \sum_{O \in \mathcal{P}_{n-1}} w(O)$$

$$+ \quad x_1^n (1,2,\ldots,n)\left(\sum_{O \in \mathcal{P}_{n-1}} w(O)\right),$$

where $(1,2,\ldots,n)$ denotes the long cycle that sends $i \mapsto i+1 \mod n$.

A combinatorial interpretation III

We claim $\widehat{\mathbf{ASym}}\left(\sum_{O \in \mathcal{P}_n} w(O)\right) = \det_{1 \le i,j \le n}\left(y_i^j + (-1)^{j+1} x_i^j\right)$:

$$\widehat{\mathbf{ASym}}\left(\sum_{O \in \mathcal{P}_n} w(O)\right) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma\left(y_n^n \sum_{O \in \mathcal{P}_{n-1}} w(O)\right)$$

$$+ \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma\left(x_1^n (1,2,\ldots,n)\left(\sum_{O \in \mathcal{P}_{n-1}} w(O)\right)\right),$$

where $(1,2,\ldots,n)$ denotes the long cycle that sends $i \mapsto i+1$ mod n.

A combinatorial interpretation III

We claim $\widehat{\mathbf{ASym}}\left(\sum_{O\in\mathcal{P}_n} w(O)\right) = \det_{1\le i,j\le n}\left(y_i^j + (-1)^{j+1}x_i^j\right)$:

$$\widehat{\mathbf{ASym}}\left(\sum_{O\in\mathcal{P}_n} w(O)\right) = \sum_{k=1}^{n} \sum_{\substack{\sigma\in S_n\\ \sigma(n)=k}} \operatorname{sgn}(\sigma)\sigma\left(y_n^n \sum_{O\in\mathcal{P}_{n-1}} w(O)\right)$$

$$+ \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\sigma\left(x_1^n(1,2,\ldots,n)\left(\sum_{O\in\mathcal{P}_{n-1}} w(O)\right)\right),$$

where $(1,2,\ldots,n)$ denotes the long cycle that sends $i\mapsto i+1$ mod n.

A combinatorial interpretation III

We claim $\widehat{\mathbf{ASym}}\left(\sum_{O \in \mathcal{P}_n} w(O)\right) = \det_{1 \le i,j \le n}\left(y_i^j + (-1)^{j+1} x_i^j\right)$:

$$\widehat{\mathbf{ASym}}\left(\sum_{O \in \mathcal{P}_n} w(O)\right) = \sum_{k=1}^{n} \sum_{\substack{\sigma \in S_n \\ \sigma(n)=k}} \operatorname{sgn}(\sigma)\sigma\left(y_n^n \sum_{O \in \mathcal{P}_{n-1}} w(O)\right)$$

$$+ \sum_{k=1}^{n} \sum_{\substack{\sigma \in S_n \\ \sigma(1)=k}} \operatorname{sgn}(\sigma)\sigma\left(x_1^n (1,2,\ldots,n)\left(\sum_{O \in \mathcal{P}_{n-1}} w(O)\right)\right),$$

where $(1,2,\ldots,n)$ denotes the long cycle that sends $i \mapsto i+1$ mod n.

A combinatorial interpretation III

We claim $\widehat{\textbf{ASym}}\left(\sum_{O\in\mathcal{P}_n} w(O)\right) = \det_{1\le i,j\le n}\left(y_i^j + (-1)^{j+1}x_i^j\right)$:

$$\widehat{\textbf{ASym}}\left(\sum_{O\in\mathcal{P}_n} w(O)\right)$$

$$= \sum_{k=1}^{n}(-1)^{n+k} y_k^n \det_{\substack{i\in\{1,\ldots,n\}\setminus\{k\}\\ 1\le j\le n-1}}\left(y_i^j + (-1)^{j+1}x_i^j\right)$$

$$+ \sum_{k=1}^{n}(-1)^{1+k} x_k^n \det_{\substack{i\in\{1,\ldots,n\}\setminus\{k\}\\ 1\le j\le n-1}}\left(y_i^j + (-1)^{j+1}x_i^j\right)$$

A combinatorial interpretation III

We claim $\widehat{\textbf{ASym}}\left(\sum_{O \in \mathcal{P}_n} w(O)\right) = \det_{1 \le i,j \le n}\left(y_i^j + (-1)^{j+1} x_i^j\right)$:

$$\widehat{\textbf{ASym}}\left(\sum_{O \in \mathcal{P}_n} w(O)\right)$$

$$= \sum_{k=1}^{n} (-1)^{n+k} \left(y_k^n + (-1)^{n+1} x_k^n\right) \det_{\substack{i \in \{1,\ldots,n\} \setminus \{k\} \\ 1 \le j \le n-1}} \left(y_i^j + (-1)^{j+1} x_i^j\right)$$

A combinatorial interpretation III

We claim $\widehat{\mathbf{ASym}}\left(\sum_{O\in\mathcal{P}_n} w(O)\right) = \det_{1\le i,j\le n}\left(y_i^j + (-1)^{j+1}x_i^j\right)$:

$$\widehat{\mathbf{ASym}}\left(\sum_{O\in\mathcal{P}_n} w(O)\right)$$

$$= \sum_{k=1}^{n}(-1)^{n+k}\left(y_k^n + (-1)^{n+1}x_k^n\right) \det_{\substack{i\in\{1,\ldots,n\}\setminus\{k\}\\ 1\le j\le n-1}}\left(y_i^j + (-1)^{j+1}x_i^j\right)$$

$$= \det_{1\le i,j\le n}\left(y_i^j + (-1)^{j+1}x_i^j\right).$$