Fully packed loop configurations: polynomiality and nested arches

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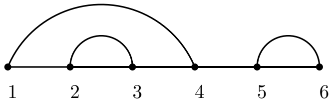
Outline

Noncrossing matchings

Definition

Let n be an integer. A noncrossing matching π of size n is a matching of the numbers $1, \ldots, 2n$ such that there exist no integers $1 \leq a < b < c < d \leq 2n$ for which π connects a, c and b, d.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \end{pmatrix} \quad \Leftrightarrow$$



- NC_n is the set of noncrossing matchings of size n.
- Its cardinality is given by $|NC_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$.

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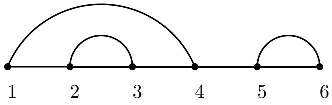

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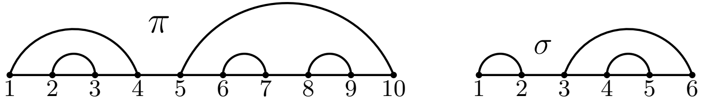
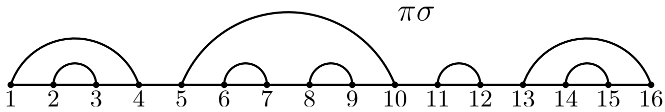
Notations I

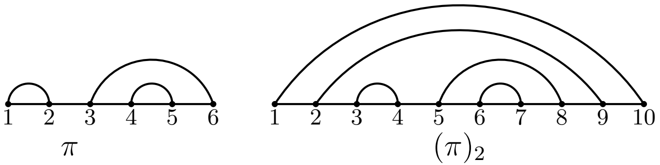
For noncrossing matchings $\pi \in NC_n, \sigma \in NC_{n'}$ denote by $\pi\sigma \in NC_{n+n'}$ the concatenation of π and σ.





Notations II

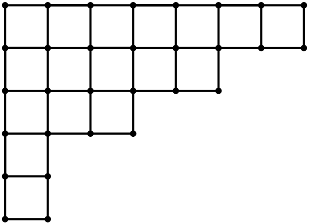
Let $m \in \mathbb{N}$. For $\pi \in NC_n$ write $(\pi)_m$ for the noncrossing matching which has m consecutive nested arches around π.



Young Diagrams

Definition

A Young Diagram λ is a finite collection of boxes, arranged in left-justified rows and weakly decreasing row-length from top to bottom.
Denote by $|\lambda|$ the number of boxes of λ.



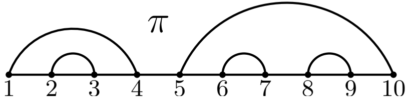
From noncrossing matchings to Young Diagrams

Let $\pi \in NC_n$. Denote by $\lambda(\pi)$ the Young Diagram obtained by the following algorithm:

- Draw a north-step if an arc is open.
- Draw a east-step if an arc is closed.
- The Young Diagram $\lambda(\pi)$ is the area between the above path and the path which consists out of n consecutive north-steps followed by n consecutive east-steps.

This yields an bijection between NC_n and the Young Diagrams with at most $n - i$ boxes in the i-th row from top.

Example



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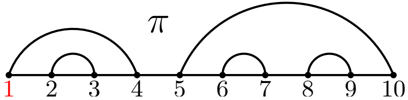
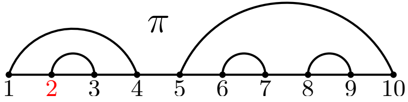
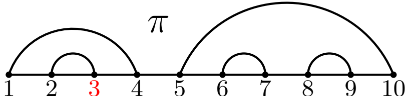
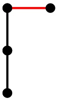




π
1
2
3
4
5
6
7
8
9
10

Example





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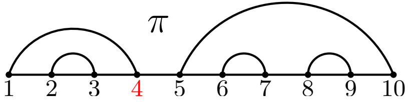

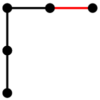

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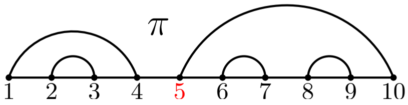

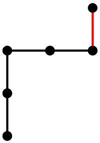

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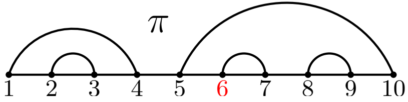

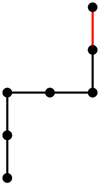

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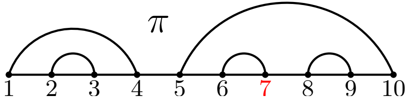

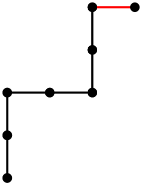

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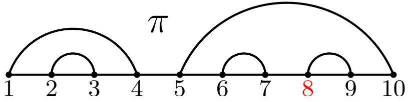

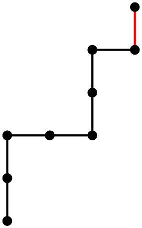

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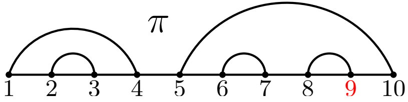

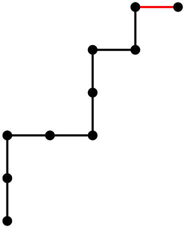

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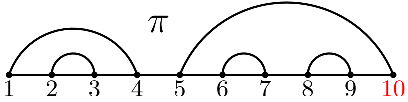

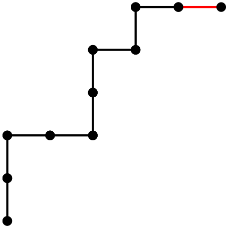

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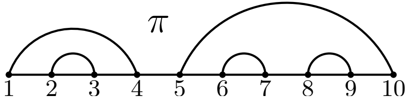

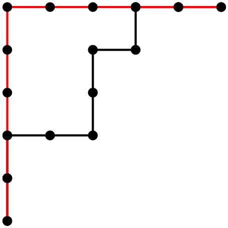

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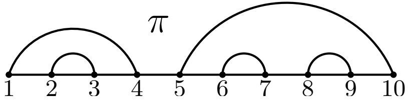
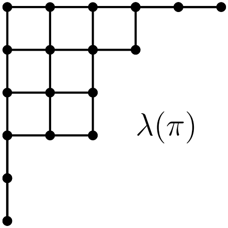


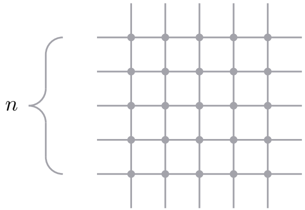


Fully packed Loop Configurations

Definition

A Fully packed Loop Configuration (FPL) F of size n is a subgraph of the $n \times n$ grid with n external edges on every side s.t.:

- F contains all vertices of the $n \times n$ grid and every vertex of F has degree 2.
- F contains every other external edge, beginning with the topmost at the left side.



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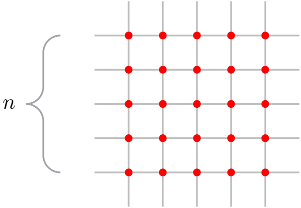

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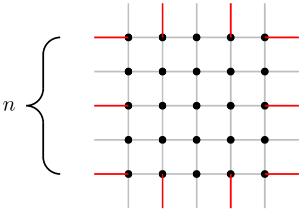

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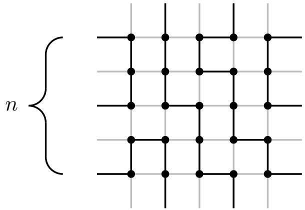
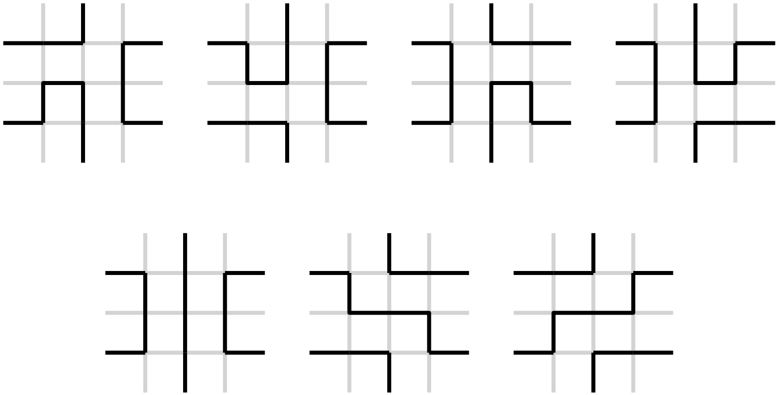


All FPLs of size 3



Number of FPLs

FPLs are in bijection with ASMs.

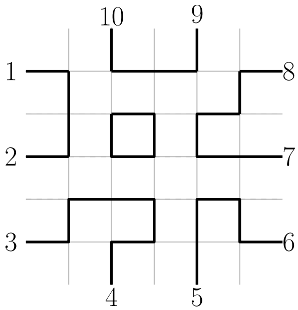
Theorem (Zeilberger, Kuperberg,...)

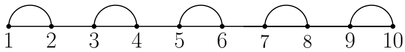
$$\#\textit{of FPLs of size } n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Link pattern

Definition

We number the external edges of a FPL F counter-clockwise with 1 up to $2n$. The link pattern $\pi(F)$ is the noncrossing matching given by $\pi(F)(i) = j$ iff the labels i and j are connected by a path in F.





The Conjecture

Definition

Let $\pi \in NC_n$. Denote by A_π the number of FPLs F with link pattern $\pi(F) = \pi$.

Conjecture (Zuber)

Let $\pi \in NC_n$, $\pi' \in NC_{n'}$ and $m \in \mathbb{N}$. Then $A_{(\pi)_m \pi'}$ is a polynomial in m of degree $|\lambda(\pi)| + |\lambda(\pi')|$ with leading coefficient $\frac{\dim(\lambda(\pi)) \dim(\lambda(\pi'))}{|\lambda(\pi)|! |\lambda(\pi')|!}$.

- The special case for π' the empty matching was proven by Caselli, Krattenthaler, Lass and Nadeau in 2004.
- The general case was also proven by them but only for large values of m.

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Wheel polynomials

Definition

Let n be an integer. A polynomial $p \in \mathbb{Q}(q)[z_1, \ldots, z_{2n}]$ is called wheel polynomial of order n if p is homogeneous of degree $n(n-1)$ and satisfies the wheel condition:

$$p(z_1, \ldots, z_{2n})_{|q^4 z_i = q^2 z_j = z_k} = 0,$$

for all $1 \leq i < j < k \leq 2n$.

Example

$$\prod_{1 \leq i < j \leq n} (q z_i - q^{-1} z_j)(q z_{n+i} - q^{-1} z_{n+j})$$

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Two families of operator

Definition

For $1 \leq k \leq 2n$ define the linear maps $S_k, D_k : \mathbb{Q}(q)[z_1, \ldots, z_{2n}] \longrightarrow \mathbb{Q}(q)[z_1, \ldots, z_{2n}]$ via

$$S_k(f)(z_1, \ldots, z_{2n}) := f(z_1, \ldots, z_{k-1}, z_{k+1}, z_k, z_{k+2}, \ldots, z_{2n}),$$

$$D_k(f)(z_1, \ldots, z_{2n}) := \frac{q z_k - q^{-1} z_{k+1}}{z_{k+1} - z_k}(S_k(f) - f).$$

Two families of operator

Denote by $W_n[z]$ the $\mathbb{Q}(q)$-vector space of Wheel polynomials of order n.

Lemma

- *$W_n[z]$ is closed under the action of D_k for all $1 \leq k \leq 2n$.*
- *For all $f, g \in \mathbb{Q}(q)[z_1, \ldots, z_{2n}]$ and $1 \leq k \leq 2n$ holds*

$$D_k(fg) = D_k(f)S_k(g) + f\,D_k(g).$$

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Partial order on NC_n

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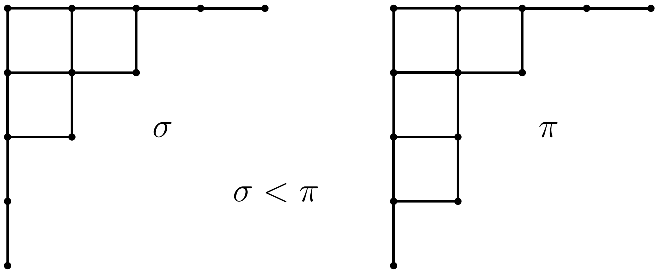
For $\sigma, \pi \in NC_n$ we say $\sigma \leq \pi$ iff $\lambda(\sigma)$ is contained in $\lambda(\pi)$.

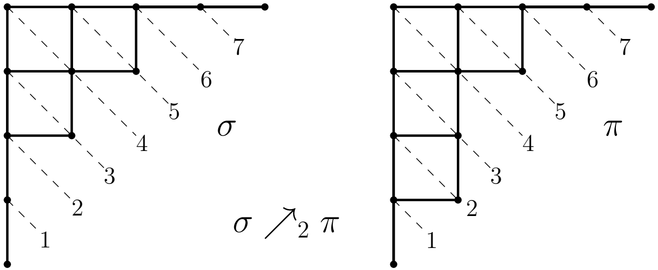


Partial order on NC_n

We can refine this notion:

Definition

For $\sigma, \pi \in NC_n$ we say $\sigma \nearrow_j \pi$ iff $\lambda(\pi)$ is obtained by adding a box to $\lambda(\sigma)$ on the j-th diagonal.



The vector space of wheel polynomials

Theorem (Zinn-Justin, Di Francesco)

There exists a $\mathbb{Q}(q)$*-basis* $\{\Psi_\pi \,|\pi \in NC_n\}$ *of* $W_n[z]$ *s.t.:*

- $\Psi_{()_n} = (q - q^{-1})^{-n(n-1)} \prod_{1 \leq i < j \leq n} (qz_i - q^{-1} z_j)(qz_{n+i} - q^{-1} z_{n+j}).$
- $\Psi_\pi(z) = D_j(\Psi_\sigma) - \sum_{\tau \in e_j^{-1}(\sigma) \setminus \{\sigma, \pi\}} \Psi_\tau$, *if* $\sigma \nearrow_j \pi$.
- $\Psi_{\rho(\pi)}(z_1, \ldots, z_{2n}) = \Psi_\pi(z_2, \ldots, z_{2n}, q^6 z_1).$
- *Set* $q = e^{\frac{2\pi i}{3}}$, *then* $\Psi_\pi(1, \ldots, 1) = A_\pi$ *holds for all* $\pi \in NC_n$.

The last statement follows by the Razumov-Stroganov (ex-)Conjecture, which was proven by Cantini and Sportiello.

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- $\Psi_{()_n} = (q - q^{-1})^{-n(n-1)} \prod_{1 \leq i < j \leq n} (q z_i - q^{-1} z_j)(q z_{n+i} - q^{-1} z_{n+j})$.
- $\Psi_\pi(z) = D_j(\Psi_\sigma) - \sum_{\tau \in e_j^{-1}(\sigma) \setminus \{\sigma, \pi\}} \Psi_\tau$, *if* $\sigma \nearrow_j \pi$.
- $\Psi_{\rho(\pi)}(z_1, \ldots, z_{2n}) = \Psi_\pi(z_2, \ldots, z_{2n}, q^6 z_1)$.
- *Set* $q = e^{\frac{2\pi i}{3}}$, *then* $\Psi_\pi(1, \ldots, 1) = A_\pi$ *holds for all* $\pi \in NC_n$.

The last statement follows by the Razumov-Stroganov (ex-)Conjecture, which was proven by Cantini and Sportiello.

The vector space of wheel polynomials

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"Explicit Basis for $W_3[z]$

Let $n = 3$. Then we obtain by the previous Theorem:

- $\Psi_{\emptyset} = (-3)^{-3} \prod_{1 \le i < j \le 3} (q z_i - q^{-1} z_j)(q z_{3+i} - q^{-1} z_{3+j})$,
- $\Psi_{(1)} = D_3(\Psi_{\emptyset})$,
- $\Psi_{(2)} = D_4(\Psi_{(1)}) - \Psi_{\emptyset} = D_4 \circ D_3(\Psi_{\emptyset}) - \Psi_{\emptyset}$,
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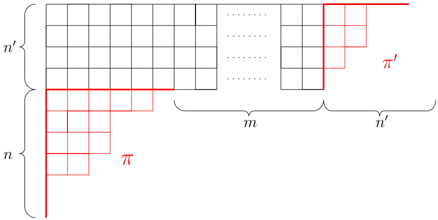
Remark on the order

Lemma

Let π, π' be noncrossing matchings of size n or n' respectively and m an integer. Then

$$e^{-1}_{2(n+m)+i}((\pi)_m \pi') = \{(\pi)_m \sigma | e_i(\sigma) = \pi'\},$$

for $2 \leq i \leq 2n' - 2$.



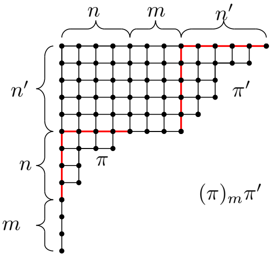
The Main Theorem

Theorem (A.)

Let $\pi \in NC_n$, $\pi' \in NC_{n'}$ and $m \in \mathbb{N}$. Then $A_{(\pi)_m \pi'}$ is a polynomial in m of degree $|\lambda(\pi)| + |\lambda(\pi')|$ with leading coefficient $\frac{\dim(\lambda(\pi)) \dim(\lambda(\pi'))}{|\lambda(\pi)|! |\lambda(\pi')|!}$.

The basic idea of the proof I

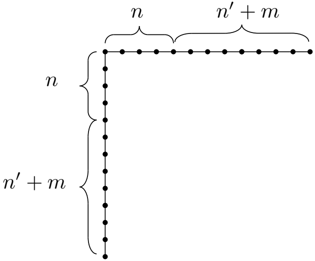
- $A_{(\pi)_m \pi'}$ is given by
- $A_{(\pi)_m \pi'} = \Psi_{(\pi)_m \pi'}(1, \ldots, 1)$. Therefore we will first calculate $\Psi_{(\pi)_m \pi'}$ in an "intelligent" way.



The basic idea of the proof II

We will calculate $\Psi_{(\pi)_m \pi'}$ in three steps:

- We calculate $\Psi_{(\pi)_{m+n'}}$ by the recursion starting from $\Psi_{()_{m+n+n'}}$.
- By rotating $(\pi)_{m+n'}$ n' times we obtain $\Psi_{\rho^{-n'}((\pi)_{m+n'})}(z_1, \ldots, z_{2(m+n+n')}) = \Psi_{(\pi)_{m+n'}}(z_{2(m+n')+1}, \ldots, z_{2(m+n+n')}, z_1, \ldots, z_{2(m+n)+n'})$.
- $\Psi_{(\pi)_m \pi'}$ can be obtained by the recursion from $\Psi_{\rho^{-n'}((\pi)_{m+n'})}$.



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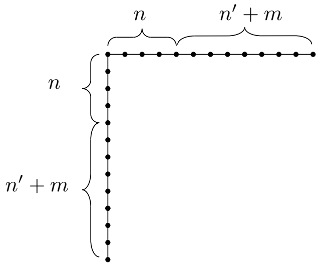

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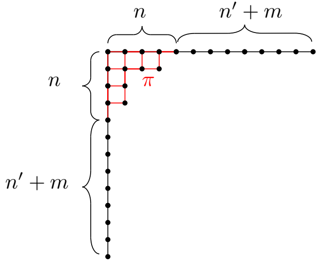

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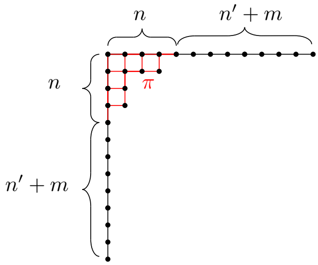

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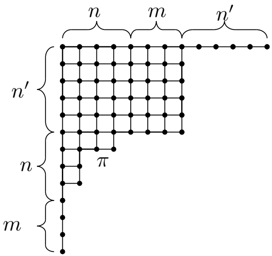

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We will calculate $\Psi_{(\pi)_m\pi'}$ in three steps:

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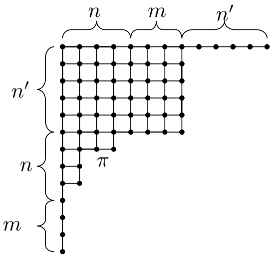

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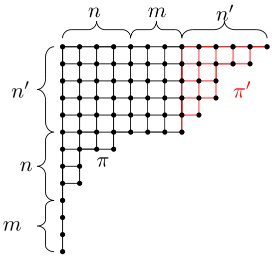

The basic idea of the proof III

$\Psi_{(\pi)_m \pi'}$ is a sum of products of the form

$$D_{i_1} \circ \cdots \circ D_{i_k} \left(\Psi_{()_{m+n+n'}} \right),$$

with:

- $k \leq |\lambda(\pi)| + |\lambda(\pi')|$,
- $i_j \in \{2, \ldots, n' - 2, m + n' + 2, \ldots, m + 2n + n' - 2, 2(m + n) + n' + 2, \ldots, 2(m + n + n')\}$ for $1 \leq j \leq k$.

Hence we want to show that the above term is for $z_1 = \ldots = z_{2(m+n+n')} = 1$ a polynomial in m of degree k.

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Notations

Remember

$$\Psi_{()_n} = (-3)^{-\binom{n}{2}} \prod_{1 \le i < j \le n} (qz_i - q^{-1}z_j)(qz_{n+i} - q^{-1}z_{n+j})$$

For $1 \le i \neq j \le 2n$ we define:

- $f(i,j) := \frac{qz_i - q^{-1}z_j}{q - q^{-1}}$,
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Some observations I

Lemma

For $1 \leq i, j, k \leq 2n$ *and* $i \neq j$ *holds*

- $D_k(f(i,j)) = \begin{cases} a \cdot f(k, k+1) & \{i,j\} \cap \{k, k+1\} \neq \emptyset \\ 0 & \textit{otherwise} \end{cases},$
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Let $1 \leq i, j, k \leq 2n$, $i \neq j$ and m be a positive integer. Then the following holds:

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A technical theorem

Theorem (A.)

Let $P = P(\alpha_{i,j}|\beta_i|\gamma_i)$ *, k an integer and* $i_1, \ldots, i_k \in \{1, \ldots, 2n\}$*. Then there exists a polynomial* $Q \in \mathbb{Q}(q)[y_1, \ldots, y_{2n(2n+1)}]$ *of degree at most k such that*

$$(D_{i_1} \circ \cdots \circ D_{i_k})(P)_{|z_1 = \ldots = z_{2n} = 1} = Q((\alpha_{i,j}), (\beta_i), (\gamma_i)).$$

Proof-sketch of the technical theorem

We proof this by induction on k in the following steps:

- $D_{i_k} P = \sum_{s \in S} a_s P_s$, for S finite, $a_s \in \{\pm 1, \pm q, \pm q^{-1}\}$ and $P_s = P(\alpha_{i,j;s} | \beta_{i;s} | \gamma_{i;s})$ for all $s \in S$.
- a_s and P_s depend piecewise linear in $\alpha_{i,j}, \beta_i, \gamma_i$.
- Finally use

$$D_{i_1} \circ \cdots \circ D_{i_{k-1}} \left(\sum_{s \in S} a_s P_s \right) = \sum_{s \in S} a_s D_{i_1} \circ \cdots \circ D_{i_{k-1}} (P_s).$$

An example

We calculate $D_1(P(\alpha_{i,j}|\beta_i|\gamma_i))$ for $n = 1$ explicitly:

$$D_1 \left(f(1,2)^{\alpha_{1,2}} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1} g(2)^{\beta_2} h(1)^{\gamma_1} h(2)^{\gamma_2} \right)$$

An example

$$D_1 \left(f(1,2)^{\alpha_{1,2}} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1} g(2)^{\beta_2} h(1)^{\gamma_1} h(2)^{\gamma_2} \right)$$
$$= - \sum_{t=1}^{\alpha_{1,2}} f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t+1} f(2,1)^{t-1} g(1)^{\beta_2} g(2)^{\beta_1} h(1)^{\gamma_2} h(2)^{\gamma_1}$$

An example

$$
\begin{aligned}
& D_1 \left(f(1,2)^{\alpha_{1,2}} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1} g(2)^{\beta_2} h(1)^{\gamma_1} h(2)^{\gamma_2} \right) \\
&= - \sum_{t=1}^{\alpha_{1,2}} f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t+1} f(2,1)^{t-1} g(1)^{\beta_2} g(2)^{\beta_1} h(1)^{\gamma_2} h(2)^{\gamma_1} \\
&\quad + \sum_{t=1}^{\alpha_{2,1}} f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t+1} f(2,1)^{t-1} g(1)^{\beta_2} g(2)^{\beta_1} h(1)^{\gamma_2} h(2)^{\gamma_1}
\end{aligned}
$$

An example

$$
\begin{aligned}
&D_1\left(f(1,2)^{\alpha_{1,2}} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1} g(2)^{\beta_2} h(1)^{\gamma_1} h(2)^{\gamma_2}\right) \\
&= -\sum_{t=1}^{\alpha_{1,2}} f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t+1} f(2,1)^{t-1} g(1)^{\beta_2} g(2)^{\beta_1} h(1)^{\gamma_2} h(2)^{\gamma_1} \\
&\quad + \sum_{t=1}^{\alpha_{2,1}} f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t+1} f(2,1)^{t-1} g(1)^{\beta_2} g(2)^{\beta_1} h(1)^{\gamma_2} h(2)^{\gamma_1} \\
&\quad - q^{-1} \sum_{t=1}^{\beta_1} f(1,2)^{\alpha_{1,2}+1} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1+\beta_2-t} g(2)^{t-1} h(1)^{\gamma_2} h(2)^{\gamma_1}
\end{aligned}
$$

An example

$$
\begin{aligned}
&D_1 \left(f(1,2)^{\alpha_{1,2}} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1} g(2)^{\beta_2} h(1)^{\gamma_1} h(2)^{\gamma_2} \right) \\
=& - \sum_{t=1}^{\alpha_{1,2}} f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t+1} f(2,1)^{t-1} g(1)^{\beta_2} g(2)^{\beta_1} h(1)^{\gamma_2} h(2)^{\gamma_1} \\
&+ \sum_{t=1}^{\alpha_{2,1}} f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t+1} f(2,1)^{t-1} g(1)^{\beta_2} g(2)^{\beta_1} h(1)^{\gamma_2} h(2)^{\gamma_1} \\
&- q^{-1} \sum_{t=1}^{\beta_1} f(1,2)^{\alpha_{1,2}+1} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1+\beta_2-t} g(2)^{t-1} h(1)^{\gamma_2} h(2)^{\gamma_1} \\
&+ q^{-1} \sum_{t=1}^{\beta_2} f(1,2)^{\alpha_{1,2}+1} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1+\beta_2-t} g(2)^{t-1} h(1)^{\gamma_2} h(2)^{\gamma_2}
\end{aligned}
$$

An example

$$
\begin{aligned}
&D_1\left(f(1,2)^{\alpha_{1,2}}f(2,1)^{\alpha_{2,1}}g(1)^{\beta_1}g(2)^{\beta_2}h(1)^{\gamma_1}h(2)^{\gamma_2}\right)\\
&=-\sum_{t=1}^{\alpha_{1,2}}f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t+1}f(2,1)^{t-1}g(1)^{\beta_2}g(2)^{\beta_1}h(1)^{\gamma_2}h(2)^{\gamma_1}\\
&\quad+\sum_{t=1}^{\alpha_{2,1}}f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t+1}f(2,1)^{t-1}g(1)^{\beta_2}g(2)^{\beta_1}h(1)^{\gamma_2}h(2)^{\gamma_1}\\
&\quad-q^{-1}\sum_{t=1}^{\beta_1}f(1,2)^{\alpha_{1,2}+1}f(2,1)^{\alpha_{2,1}}g(1)^{\beta_1+\beta_2-t}g(2)^{t-1}h(1)^{\gamma_2}h(2)^{\gamma_1}\\
&\quad+q^{-1}\sum_{t=1}^{\beta_2}f(1,2)^{\alpha_{1,2}+1}f(2,1)^{\alpha_{2,1}}g(1)^{\beta_1+\beta_2-t}g(2)^{t-1}h(1)^{\gamma_2}h(2)^{\gamma_2}\\
&\quad+q\sum_{t=1}^{\gamma_1}f(1,2)^{\alpha_{1,2}+1}f(2,1)^{\alpha_{2,1}}g(1)^{\beta_1}g(2)^{\beta_2}h(1)^{\gamma_1+\gamma_2-t}h(2)^{t-1}
\end{aligned}
$$

An example

$$
\begin{aligned}
= &- \sum_{t=1}^{\alpha_{1,2}} f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t+1} f(2,1)^{t-1} g(1)^{\beta_2} g(2)^{\beta_1} h(1)^{\gamma_2} h(2)^{\gamma_1} \\
&+ \sum_{t=1}^{\alpha_{2,1}} f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t+1} f(2,1)^{t-1} g(1)^{\beta_2} g(2)^{\beta_1} h(1)^{\gamma_2} h(2)^{\gamma_1} \\
&- q^{-1} \sum_{t=1}^{\beta_1} f(1,2)^{\alpha_{1,2}+1} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1+\beta_2-t} g(2)^{t-1} h(1)^{\gamma_2} h(2)^{\gamma_1} \\
&+ q^{-1} \sum_{t=1}^{\beta_2} f(1,2)^{\alpha_{1,2}+1} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1+\beta_2-t} g(2)^{t-1} h(1)^{\gamma_2} h(2)^{\gamma_2} \\
&+ q \sum_{t=1}^{\gamma_1} f(1,2)^{\alpha_{1,2}+1} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1} g(2)^{\beta_2} h(1)^{\gamma_1+\gamma_2-t} h(2)^{t-1} \\
&- q \sum_{t=1}^{\gamma_2} f(1,2)^{\alpha_{1,2}+1} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1} g(2)^{\beta_2} h(1)^{\gamma_1+\gamma_2-t} h(2)^{t-1}.
\end{aligned}
$$

An example

By setting $z_1 = z_2 = z_3 = z_4 = 1$ we obtain:

$$D_1 \left(f(1,2)^{\alpha_{1,2}} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1} g(2)^{\beta_2} h(1)^{\gamma_1} h(2)^{\gamma_2} \right)$$
$$= -\sum_{t=1}^{\alpha_{1,2}} 1 + \sum_{t=1}^{\alpha_{2,1}} 1 - q^{-1} \sum_{t=1}^{\beta_1} 1 + q^{-1} \sum_{t=1}^{\beta_2} 1 + q \sum_{t=1}^{\gamma_1} 1 - q \sum_{t=1}^{\gamma_2} 1$$
$$= -\alpha_{1,2} + \alpha_{2,1} + q^{-1}(\beta_2 - \beta_1) + q(\gamma_1 - \gamma_2).$$

Proof-sketch of the technical theorem

$$D_{i_k} P = D_{i_k} \left(\prod_{1 \le i \ne j \le 2n} f(i,j)^{\alpha_{i,j}} \prod_{i=1}^{2n} g(i)^{\beta_i} h(i)^{\gamma_i} \right) =$$

Proof-sketch of the technical theorem

$$D_{i_k} P = D_{i_k} \left(\prod_{1 \le i \ne j \le 2n} f(i,j)^{\alpha_{i,j}} \prod_{i=1}^{2n} g(i)^{\beta_i} h(i)^{\gamma_i} \right) =$$

$$= \sum_{1 \le i \ne j \le 2n} \prod_{\substack{1 \le i' \ne j' \le 2n \\ (i' < i) \vee (i' = i, j' < j)}} f(i',j')^{\alpha_{i',j'}} \times D_k(f(i,j)^{\alpha_{i,j}}) \times$$

$$\times S_k \left(\prod_{\substack{1 \le i' i \ne j' \le 2n \\ (i' > i) \vee (i' = i, j' > j)}} f(i',j')^{\alpha_{i',j'}} \prod_{i'=1}^{2n} g(i')^{\beta_{i'}} h(i')^{\gamma_{i'}} \right) + \dots$$

Proof-sketch of the technical theorem

$$\dots + \sum_{i=1}^{2n} \prod_{1 \le i' \ne j' \le 2n} f(i',j')^{\alpha_{i',j'}} \prod_{i'=1}^{i-1} g(i')^{\beta_{i'}} \times D_k(g(i)^{\beta_i}) \times$$

$$\times S_k \left(\prod_{i'=i+1}^{2n} g(i')^{\beta_{i'}} \prod_{i'=1}^{2n} h(i')^{\gamma_{i'}} \right) +$$

$$+ \sum_{i=1}^{2n} \prod_{1 \le i' \ne j' \le 2n} f(i',j')^{\alpha_{i',j'}} \prod_{i'=1}^{2n} g(i')^{\beta_{i'}} \prod_{i'=1}^{i-1} h(i')^{\gamma_{i'}} \times$$

$$\times D_k(h(i)^{\gamma_i}) \times S_k \left(\prod_{i'=i+1}^{2n} h(i')^{\gamma_{i'}} \right).$$

Proof-sketch of the Main Theorem

Remember

$\Psi_{(\pi)_m\pi'}$ is a sum of products of the form

$$D_{i_1} \circ \cdots \circ D_{i_k} \left(\Psi_{()_{m+n+n'}} \right),$$

- $k \leq |\lambda(\pi)| + |\lambda(\pi')|$,
- $i_j \in I := \{2, \ldots, n' - 2, m + n' + 2, \ldots, m + 2n + n' - 2, 2(m + n) + n' + 2, \ldots, 2(m + n + n')\}$ for $1 \leq j \leq k$.

$A_{(\pi)_m\pi'}(m) = \Psi_{(\pi)_m\pi'|z_1=\ldots=z_{2(m+n+n')}=1}$.

Proof-sketch of the Main Theorem

$$\Psi_{()_{m+n+n'}} = \prod_{1 \le i < j \le m+n+n'} f(i,j) f(m+n+n'+i, m+n+n'+j)$$

- The D_{i_j} with i_j as before act trivially on the variables $z_{n'+1}, \dots, z_{m+n'}, z_{m+2n+n'+1}, \dots, z_{2(m+n)+n'}$, hence we can set them equal 1.
- Therefore $\Psi_{()_{m+n+n'}}$ is of the form $P(\alpha_{i,j}|\beta_i|\gamma_i)$ with:

Proof-sketch of the Main Theorem

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- Therefore $\Psi_{()_{m+n+n'}}$ is of the form $P(\alpha_{i,j} | \beta_i | \gamma_i)$ with:

Proof-sketch of the Main Theorem

$$\alpha_{i,j} = \begin{cases} 1 & i,j \notin I, i < j, (j \leq (m+n+n') \text{ or } i > m+n+n') \\ 0 & \text{otherwise} \end{cases},$$

$$\beta_i = \begin{cases} m & i \in \{m+n'+1, \ldots, m+n+n', 2(m+n)+n'+1, \\ & \ldots, 2(m+n+n')\} \\ 0 & \text{otherwise} \end{cases},$$

$$\gamma_i = \begin{cases} m & i \in \{1, \ldots, n', m+n+n'+1, \ldots, m+2n+n'\} \\ 0 & \text{otherwise} \end{cases},$$

$$I = \{n'+1, \ldots, n'+m, m+2n+n'+1, \ldots, 2(m+n)+n'\}.$$