A determinantal expression for the Q-enumeration of ASMs

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Outline

1. A brief History

2. The Q-enumeration of ASMs

3. Results, Connections and Outlook

Alternating sign matrices

An alternating sign matrix (or short ASM) of size n is an $n \times n$ matrix with entries $1, 0, -1$, such that

- all row- and column-sums are equal 1,
- the non-zero entries alternate.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

An enumeration formula

Conjecture (Mills-Robbins-Rumsey, 1983)

The number of ASMs of size n is given by

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

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Descending Plane Partitions

$$\begin{array}{ccccc} 6 & 6 & 5 & 3 & 1 \\ & 5 & 3 & & \\ & & 2 & & \end{array}$$

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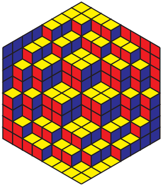
Totally Symmetric Self Complementary Plane Partitions



Proofs of the ASM Theorem

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- 1996: G. Kuperberg. “*Another proof of the alternating sign matrix conjecture*”.
- 2007: I. Fischer. “*A new proof of the refined alternating sign matrix theorem*”.
- 2016: I. Fischer. “*Short proof of the ASM theorem avoiding the six-vertex model*”.

A variation of the Operator formula

We define the forward difference $\overline{\Delta}_x$ as the operator

$$\overline{\Delta}_x f(x) = f(x+1) - f(x).$$

Theorem (Fischer, 2010)

The generating function of ASMs of size n with weight $Q^{\#\text{ of -1's}}$ *is*

$$\prod_{1 \le i < j \le n} \left(Q\,\mathrm{Id} + (Q-1)\overline{\Delta}_{x_i} + \overline{\Delta}_{x_j} + \overline{\Delta}_{x_i}\overline{\Delta}_{x_j} \right) \prod_{1 \le i < j \le n} \frac{x_j - x_i}{j - i} \Bigg|_{x_i = i}.$$

A Q-enumeration formula

Theorem (A. 2018)

The generating function of ASMs of size n with weight $(q^{-1}+2+q)^{\#\text{ of -1's}}$ is

$$\det_{1\le i,j\le n}\left(\binom{i+j-2}{j-1}\frac{1-(-q)^{1+j-i}}{1+q}\right).$$

Sketch of the proof

Set $Q = q^{-1} + 2 + q$.

1. Start with the operator formula.

$$\prod_{1 \le i < j \le n} \left(Q \operatorname{Id} + (Q-1)\overline{\Delta}_{x_i} + \overline{\Delta}_{x_j} + \overline{\Delta}_{x_i} \overline{\Delta}_{x_j} \right) \prod_{1 \le i < j \le n} \frac{x_j - x_i}{j - i} \Bigg|_{x_i = i}$$

Sketch of the proof

Set $Q = q^{-1} + 2 + q$.

1. Start with the operator formula.
2. Rewrite it to a constant term formula.

$$\mathrm{CT}_{x_1,\ldots,x_n} \frac{\mathcal{AS}_{x_1,\ldots,x_n}\left(\prod_{i=1}^{n}(1+x_i)^i \prod_{1\le i<j\le n}(Q+(Q-1)x_i+x_j+x_ix_j)\right)}{\prod_{1\le i<j\le n}(x_j-x_i)}$$

Sketch of the proof

Set $Q = q^{-1} + 2 + q$.

1. Start with the operator formula.
2. Rewrite it to a constant term formula.
3. Use a general Lemma (Fonseca, Zinn-Justin, '08; Fischer, '18) which transforms the antisymmetriser into a determinant.

$$\mathrm{CT}_{x_1,\ldots,x_n}\left((-1)^{\frac{n(n+1)}{2}} q^n (q-q^{-1})^{\frac{n(n+3)}{2}} \prod_{i=1}^{n} (1+x_i)^{n+1}(x_i+1+q)^{-2}\right.$$
$$\times \lim_{y_1,\ldots,y_n\to 1} \det_{1\le i,j\le n}\left(\frac{1}{\left(y_j - \frac{x_i+1+q^{-1}}{x_i+1+q}\right)\left(y_j - q^2\frac{x_i+1+q^{-1}}{x_i+1+q}\right)}\right)$$
$$\left.\times \prod_{1\le i<j\le n} (x_j-x_i)^{-1}(y_j-y_i)^{-1}\right)$$

Sketch of the proof

Set $Q = q^{-1} + 2 + q$.

1. Start with the operator formula.
2. Rewrite it to a constant term formula.
3. Use a general Lemma (Fonseca, Zinn-Justin, '08; Fischer, '18) which transforms the antisymmetriser into a determinant.
4. Use algebraic manipulations and a trick (Behrend, Di Franceso,Zinn-Justin, 2012) to obtain the wanted formula.

$$\det_{1 \leq i,j \leq n} \left(\binom{i+j-2}{j-1} \frac{1-(-q)^{1+j-i}}{1+q} \right)$$

A more general determinant

Instead of the previous determinant we consider

$$d_{n,k}(x,q) := \det_{1 \le i,j \le n} \left(\binom{x+i+j-2}{j-1} \frac{1-(-q)^{k+j-i}}{1+q} \right),$$

with $k \in \mathbb{Z}$ and x is a variable.

The weighted enumeration of ASMs is $d_{n,1}(0,q)$.

The Condensation method

Theorem (Desnanot-Jacobi, Condensation method)

Let n be a positive integer and A an $n \times n$ matrix, then holds

$$\det A \det A_{1,n}^{1,n} = \det A_1^1 \det A_n^n - \det A_1^n \det A_n^1,$$

where A is an $n \times n$ matrix and $A_{j_1,\cdots,j_k}^{i_1,\cdots,i_k}$ denotes the submatrix of A in which the $i_1, \cdots, i_k$-th rows and $j_1, \cdots, j_k$-th columns are omitted.

What do we obtain for general q?

The determinant $d_{n,k}(x,q)$ has the form

$$d_{n,k}(x,q) = q^{c_q(n,k)} p_{n,k}(x) f_{n,k}(x,q),$$

with

$$p_{n,k}(x) = \prod_{i=1}^{n-1} \frac{\left\lfloor \frac{i}{2} \right\rfloor !}{i!} \prod_{i=0}^{\left\lfloor \frac{n-|k|-1}{2} \right\rfloor} (x + |k| + 2i + 1),$$

$$c_q(n,k) = \begin{cases} 0 & k > 0, n \leq k, \\ n\,k & k < 0, n \leq -k, \\ -\sum_{i=1}^{n-k} \left\lfloor \frac{i}{2} \right\rfloor & \text{otherwise,} \end{cases}$$

and $f_{n,k}(x,q)$ being a polynomial in x and q which is given recursively.

What do we obtain for general q?

Theorem (Kuperberg 1996, A. 2018)

Denote by $A_n(Q)$ the Q-enumeration of ASMs of size n. Then there exists polynomials $p_n(Q)$ such that

$$A_{2n}(Q) = 2p_{2n}(Q)p_{2n+1}(Q),$$
$$A_{2n+1}(Q) = p_{2n+1}(Q)p_{2n+2}(Q).$$

This was conjectured by Mills-Robbins-Rumsey for $A_n(Q)$ and by Fischer for the evaluation of the determinant.

Various specialisations

0-enumeration: $q = -1$ (primitive second root of unity),

1-enumeration: $q = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ (primitive third root of unity),

2-enumeration: $q = \pm i$ (primitive fourth root of unity),

3-enumeration: $q = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ (primitive sixth root of unity),

4-enumeration: $q = 1$ (primitive first root of unity).

Various specialisations

Theorem (A.)

The 0-enumeration case:

$$d_{n,1}(x,-1) = \left(2\left\lfloor\frac{n+1}{2}\right\rfloor - 1\right)!!\prod_{i=1}^{\left\lfloor\frac{n}{2}\right\rfloor}(x+2i).$$

Various specialisations

Theorem (A.)

The 0-enumeration case:

$$d_{n,1}(x,-1) = \left(2\left\lfloor \frac{n+1}{2} \right\rfloor - 1\right)!! \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (x+2i).$$

Corollary (A.)

There are $n!$ permutations of size n.

Various specialisations

Theorem (A.)

The 1*-enumeration case: let* q *be a primitive third root of unity.*

$$d_{n,6k+1}(x,q) = 2^{\lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor} \prod_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(i-1)!}{(n-i)!}$$

$$\times \prod_{i\geq 0} \left(\frac{x}{2}+3i+1\right)_{\lfloor \frac{n-4i}{2} \rfloor} \left(\frac{x}{2}+3i+3\right)_{\lfloor \frac{n-4i-3}{2} \rfloor}$$

$$\times \prod_{i\geq 0} \left(\frac{x}{2}+n-i+\frac{1}{2}\right)_{\lfloor \frac{n-4i-1}{2} \rfloor} \left(\frac{x}{2}+n-i-\frac{1}{2}\right)_{\lfloor \frac{n-4i-2}{2} \rfloor},$$

where $(a)_i := a(a+1)\cdots(a+i-1)$.

Corollary (A.)

This implies the enumeration formula of ASMs.

Various specialisations

Theorem (A.)

The 2-enumeration case: let q be a primitive fourth root of unity.

$$d_{n,4k+1}(x,q) = 2^{\left\lfloor \frac{n}{2} \right\rfloor} \prod_{i=1}^{n-1} \frac{4^{\left\lfloor \frac{i}{2} \right\rfloor} \left\lfloor \frac{i}{2} \right\rfloor !}{i!} \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left(\frac{x}{2} + i \right)_{n-2i+1}.$$

Corollary (A.)

This implies the 2-enumeration formula of ASMs.

Various specialisations

Theorem (A.)

The 3-enumeration case: let q be a primitive sixth root of unity.

$$d_{n,3k+1}(x,q) = c(n) \prod_{i=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} (x+2+3i)_{n-1-2i},$$

with

$$c(n) = \begin{cases} 3^{\frac{(n-2)n}{4}} \prod_{i=0}^{n-1} \frac{\left\lfloor \frac{i}{2} \right\rfloor !}{i!} & n \text{ is even}, \\ 3^{\frac{(n-1)^2}{4}} \prod_{i=0}^{n-1} \frac{\left\lfloor \frac{i}{2} \right\rfloor !}{i!} & \text{otherwise}. \end{cases}$$

Corollary (A.)

This implies the 3-enumeration formula of ASMs.

Various specialisations

Theorem (A.)

The 4-enumeration case: $q = 1$.

$$d_{n,2k+1}(x,1) = \prod_{i=2}^{\lfloor \frac{n}{2} \rfloor} (2i-1)^{-(n+1-2i)} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (x+2i) p_n(x) p_{n-1}(x),$$

with

$$p_1(x) = 1, \quad p_3(x) = 2x+5, \quad p_{2n}(x) = p_{2n-1}(x+2),$$

$$p_{2n+1}(x) = \Big((x+2n+1)(x+2n+2) p_{2n-1}(x) p_{2n-1}(x+4) - \\ -(x+1)(x+2) p_{2n-1}(x+2)^2 \Big) \left(2n p_{2n-3}(x+4) \right)^{-1}.$$

Connection to another determinant

Let q be a sixth root of unity, then holds

$$d_{n,3-k}(x,q^2) = q^{-n} \det_{1\leq i,j\leq n}\left(\binom{x+i+j-2}{j-1} + q^k \delta_{i,j}\right).$$

Theorem (Ciucu-Eisenkölbl-Krattenthaler-Zare, 2001)

The above determinant counts weighted cyclically symmetric lozenge tilings of a hexagon with a triangular hole of size x.

Outlook

general x :

$$d_{n,k}(x,q)$$

specialising x :

Outlook

general x :

$$\begin{tikzcd} d_{n,k}(x,q) \arrow[d] \\ \text{ASMs} \end{tikzcd}$$

specialising x :

ASMs Alternating Sign Matrices

Outlook

general x :

$$\begin{tikzcd} d_{n,k}(x,q) \arrow[r, phantom, "\leftrightarrow"] \arrow[d] & \det_{1\leq i,j\leq n}\left(\binom{x+i+j-2}{j-1}+q^k\delta_{i,j}\right) \\ \text{ASMs} & \end{tikzcd}$$

specialising x : ASMs

ASMs Alternating Sign Matrices

Outlook

general x :

specialising x :

$$\begin{tikzcd} d_{n,k}(x,q) \arrow[r, leftrightarrow] \arrow[d] & \det_{1\le i,j\le n}\left(\binom{x+i+j-2}{j-1}+q^k\delta_{i,j}\right) \arrow[d] \\ \text{ASMs} & \text{DPPs} \end{tikzcd}$$

ASMs	Alternating Sign Matrices
DPPs	Descending Plane Partitions

Outlook

$$\begin{tikzcd}
\text{general } x: & d_{n,k}(x,q) \arrow[r, leftrightarrow] \arrow[d] & \det_{1\le i,j\le n}\left(\binom{x+i+j-2}{j-1}+q^k\delta_{i,j}\right) \arrow[d] \arrow[dr] & \\
\text{specialising } x: & \text{ASMs} & \text{DPPs} & \text{ASTs}
\end{tikzcd}$$

ASMs	Alternating Sign Matrices
DPPs	Descending Plane Partitions
ASTs	Alternating Sign Triangles

Outlook

$$\begin{tikzcd}
\text{general } x : & & d\text{-DPPs} & \\
& d_{n,k}(x,q) \arrow[d] \arrow[r, leftrightarrow] & \det_{1 \leq i,j \leq n} \left(\binom{x+i+j-2}{j-1} + q^k \delta_{i,j} \right) \arrow[u] \arrow[d] \arrow[dr] & \\
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ASMs	Alternating Sign Matrices
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d-DPPs	*d*-Descending Plane Partitions

Outlook

$$\begin{tikzcd}
\text{general } x : & & & d\text{-DPPs} & \text{ASts} \\
& d_{n,k}(x,q) \arrow[d] \arrow[r, leftrightarrow] & {} & \det_{1\le i,j\le n}\left(\binom{x+i+j-2}{j-1}+q^k\delta_{i,j}\right) \arrow[u] \arrow[d] \arrow[ur] \arrow[dr] & \\
\text{specialising } x : & \text{ASMs} & & \text{DPPs} & \text{ASTs} \arrow[uu]
\end{tikzcd}$$

ASMs	Alternating Sign Matrices
DPPs	Descending Plane Partitions
ASTs	Alternating Sign Triangles
d-DPPs	*d*-Descending Plane Partitions
ASts	Alternating Sign trapezoids

Outlook

$$\begin{tikzcd}
\text{general } x : & ? & & d\text{-DPPs} & \text{ASts} \\
 & & & & \nearrow \\
 & d_{n,k}(x,q) \arrow[uu] \arrow[dd] & \leftrightarrow & \det_{1 \leq i,j \leq n} \left(\binom{x+i+j-2}{j-1} + q^k \delta_{i,j} \right) \arrow[uu] \arrow[dd] & \uparrow \\
 & & & & \searrow \\
\text{specialising } x : & \text{ASMs} & & \text{DPPs} & \text{ASTs}
\end{tikzcd}$$

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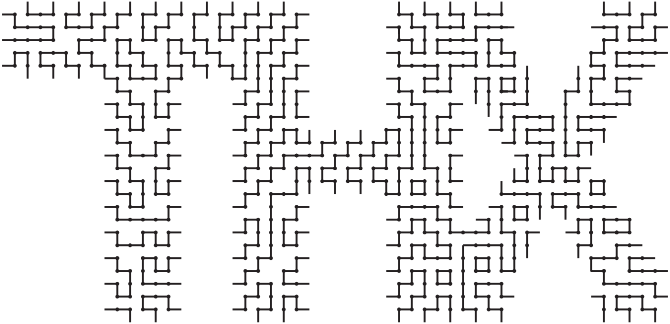

Since I will finish my PhD in spring I am looking for a Post-Doc :)