Polynomiality phenomenon for FPLs and AST-trapezoids

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Noncrossing matchings

A noncrossing matching of size n consists of $2n$ aligned points which are connected by n noncrossing arches (lying above the points). We denote n nested arches by $()_n$ and n small arches by $()^n$.

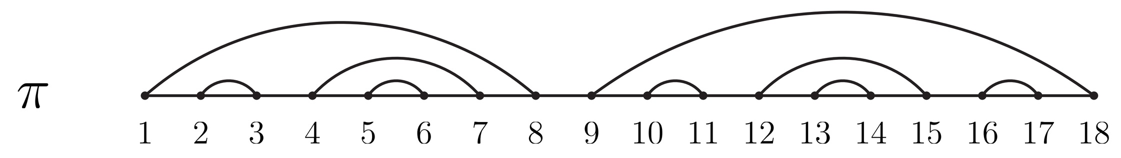
Centred catalan Sets

A centred Catalan set of size n is a n-subset of $\{-(n-1),\dots,n-1\}$ such that $|S\cap\{-i,-i+1,\dots,i\}|\geq i+1$ for all $0\leq i\leq n-1$. We define for a positive integer l

$$\mathfrak{s}_l(x)=\begin{cases}x+l & x>0,\\ 0 & x=0,\\ x-l & x<0.\end{cases}$$

A bijection between noncrossing matchings of size n and centred Catalan sets of size n

We assign to every noncrossing matching π of size n a centred Catalan set $S(\pi)$, where $S(\pi)$ contains 0, an integer $1\leq i\leq n-1$ iff $2i+1$ is a left-endpoint of an arc and an integer $-n+1\leq i\leq -1$ iff $-2i$ is a left-endpoint of an arc. We write $S\mapsto\pi(S)$ for the inverse map.



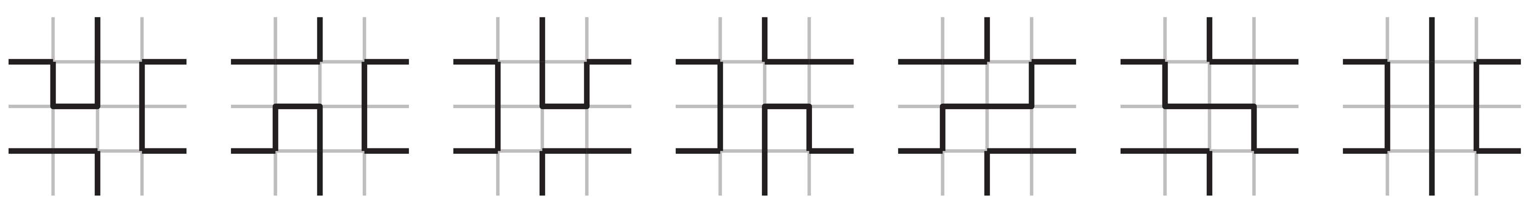
$\Leftrightarrow$ $S(\pi)=\{-8,-6,-5,-2,-1,0,2,4,6\}$

Fully packed loops

A FPL F of size n is a subgraph of the $n\times n$ grid with n external edges (they have only one incident vertex) satisfying the following.

- All vertices of the $n\times n$ grid have degree 2 in F.
- An FPL F contains every other external edge, beginning with the topmost at the left side.

Example (all FPLs of size 3).



AST-trapezoids

An (n,l)-AST-trapezoid is a configuration $(a_{i,j})_{1\leq i\leq n, i\leq j\leq 2(n+l)-i}$ with entries $-1,0,1$ satisfying the following.

- The non-zero entries alternate in all rows and columns.
- All rowsums are 1.
- The topmost non-zero entry is 1 for all columns.
- The central $(2l-1)$ columns have columnsum 0.

Example (all $(2,1)$-AST-trapezoids).

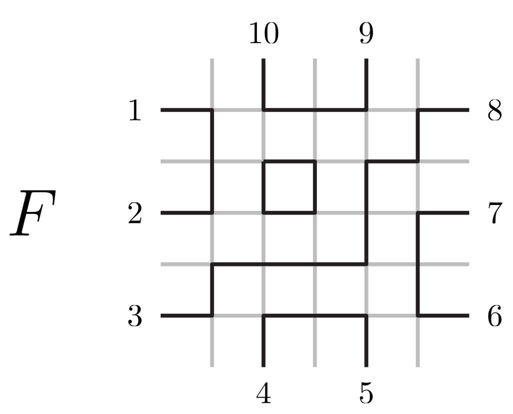
$\begin{array}{ccccc}1&0&0&0&0\\ &1&0&0&\end{array}$ $\begin{array}{ccccc}0&0&0&1&0\\ &1&0&0&\end{array}$ $\begin{array}{ccccc}0&0&0&0&1\\ &1&0&0&\end{array}$ $\begin{array}{ccccc}1&0&0&0&0\\ &0&0&1&\end{array}$ $\begin{array}{ccccc}0&1&0&0&0\\ &0&0&1&\end{array}$ $\begin{array}{ccccc}0&0&0&0&1\\ &0&0&1&\end{array}$ $\begin{array}{ccccc}0&0&1&0&0\\ &1&-1&1&\end{array}$

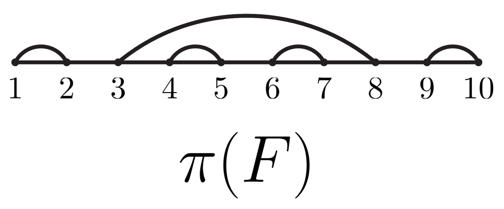
FPLs and AST-trapezoids

AST-trapezoids are a generalisation of alternating sign triangles which have been introduced by Ayyer, Behrend and Fischer.
Theorem(Ayyer, Behrend, Fischer; 2016). FPLs of size n and $(n-1,1)$-AST-trapezoids are equinumerous.

The link pattern

We assign to every FPL F a noncrossing matching $\pi(F)$, called its link pattern, by connecting the numbers i and j in $\pi(F)$ iff they are connected in F. Denote by A_π the number of FPLs F with $\pi(F)=\pi$.
Example.





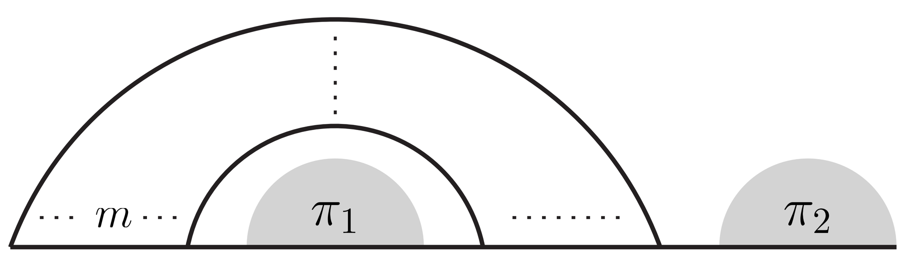
Associating CCSs to AST-trapezoids

We label the columns of an (n,l)-AST-trapezoid A form left to right with $-(n+l-1),\dots,n+l-1$. Denote by $S(A)$ the centred Catalan set of size n such that $\mathfrak{s}_l(S(A))$ is the set of columns with columnsum 1. Write $w_l(S)$ for the number of (n,l)-AST-trapezoids with $S(A)=S$.
Example.

$$A=\begin{array}{ccccccccccc}0&0&0&0&0&0&0&0&1&0&0\\ &0&0&0&0&1&0&0&0&0&\\ &&1&0&0&0&0&0&0&&\\ &&&1&0&-1&0&1&&&\end{array}\qquad S(A)=\{-2,-1,0,1,2\}$$

A polynomiality theorem for FPLs

Theorem(Zuber; Caselli, Krattenthaler, Lass, Nadeau; A.). Let π_1,π_2 be noncrossing matchings of size n_1 or n_2 respectively and let m be an integer. Then the number $A_{(\pi_1)_m\pi_2}$ is a polynomial function in m.



A splitting theorem

Theorem(A.). Let S_1,S_2 be centred Catalan sets of size n_1,n_2 respectively, then $w_l(S_1\cup\mathfrak{s}_{n_1-1}(S_2))=w_l(S_1)w_{n_1+l-1}(S_2)$.

A polynomiality theorem for ASTs

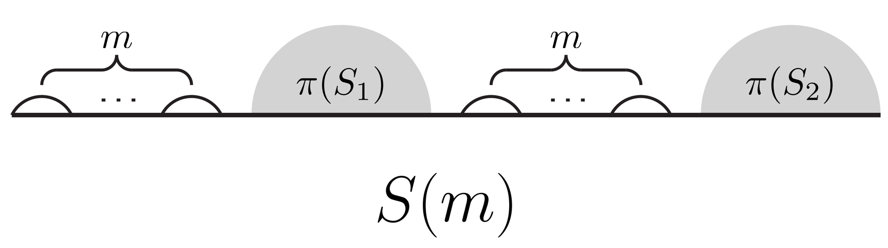
Theorem(A.). For a centred Catalan set S the number $w_l(S)$ is a polynomial function in l.

An analogy between the two theorems

Let S_1,S_2 be two centred Catalan sets of size n_1 or n_2 respectively and set

$$S(m)=\{1,2,\dots,m\}\cup\mathfrak{s}_m(S_1)\cup\{m+n_1,\dots,2m+n_1\}\cup\mathfrak{s}_{2m+n_1}(S_2).$$

Then $w_1(S(m))$ is a polynomial in m where the associated noncrossing matching of $S(m)$ is depicted on the right.



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