(-1)-Enumerations of arrowed Gelfand-Tsetlin patterns

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1) Arrowed Gelfand-Tsetlin pattern

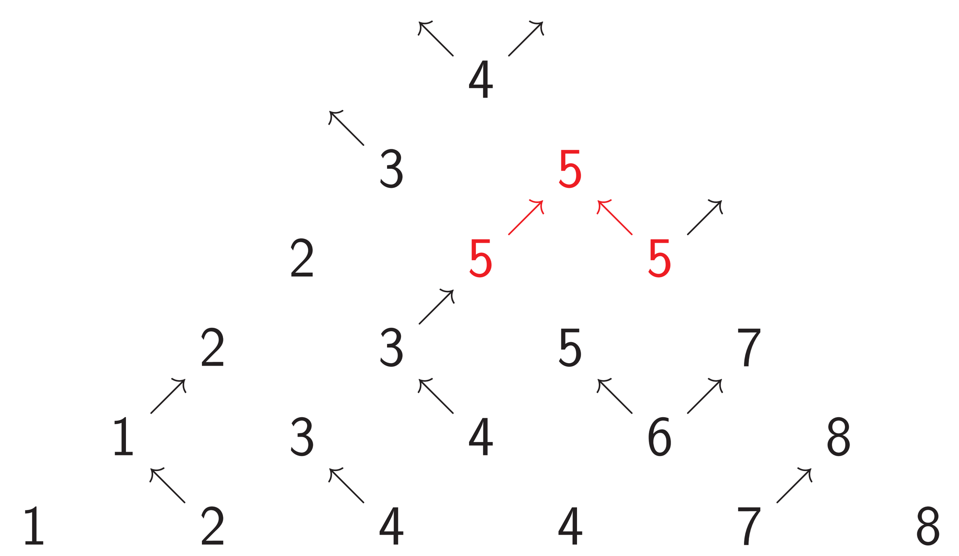
An arrowed Gelfand-Tsetlin patter (AGT) is a triangular array of integers of the form

$$\begin{array}{ccccccc} & & & a_{1,1} & & & \\ & & a_{2,1} & & a_{2,2} & & \\ & ⋰ & & & & ⋱ & \\ & a_{n-1,1} & & \cdots & & a_{n-1,n-1} & \\ a_{n,1} & & a_{n,2} & & \cdots & & a_{n,n} \end{array} \qquad \begin{array}{ccc} & a_{i,j} & \\ \leq & & \leq \\ a_{i+1,j} & \leq & a_{i+1,j+1} \end{array}$$

together with a decoration of the entries by the symbols $\emptyset, \nwarrow, \nearrow, \nwarrow\!\!\!\nearrow$ such that

$$a_{i+1,j} = a_{i,j} \text{ and } a_{i+1,j} \text{ is decorated by } \nearrow \text{ or } \nwarrow\!\!\!\nearrow,$$
$$\Leftrightarrow$$
$$a_{i+1,j+1} = a_{i,j} \text{ and } a_{i+1,j+1} \text{ is decorated by } \nwarrow \text{ or } \nwarrow\!\!\!\nearrow.$$

Example



The above AGT has weight $-t^{10}u^4v^4w^3x_1^4x_2^3x_3^5x_4^6x_5^6x_6^2$; we marked the special little triangle in red.

2) Weighted enumeration of AGTs

We call the following local configurations special little triangle

The sign $\operatorname{sgn}(A)$ of an AGT A is

$$(-1)^{\# \text{ of special little triangles in } A}.$$

We define the weight of A as

$$W(A) := \operatorname{sgn}(A) t^{\#\emptyset} u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\!\nearrow} \prod_{i=1}^{n} x_i^{\sum_{j=1}^{i} a_{i,j} - \sum_{j=1}^{i-1} a_{i-1,j} + \#\nearrow \text{ in row } i - \#\nwarrow \text{ in row } i}.$$

Theorem([3, Thm 3.2]) Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition and set $\mathbf{x} = (x_1, \ldots, x_n)$. The weighted enumeration $\mathcal{A}_\lambda(t,u,v,w;\mathbf{x})$ of all AGTs with bottom row $(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$ is given by

$$\mathcal{A}_\lambda(t,u,v,w;\mathbf{x}) = \prod_{i=1}^{n} \left(ux_i + vx_i^{-1} + w + t\right)$$
$$\times \prod_{1 \le i < j \le n} \left(t\,\mathrm{id} + uE_{\lambda_j} + vE_{\lambda_i}^{-1} + wE_{\lambda_j}E_{\lambda_i}^{-1}\right) s_\lambda(\mathbf{x}).$$

3) Some special cases

$$\mathcal{A}_\lambda(1,0,0,0;\mathbf{x}) = s_\lambda(\mathbf{x}),$$
$$\mathcal{A}_{(n,n-1,\ldots,1)}(0,1,1,-1;\mathbf{1}) = \# n \times n \text{ ASMs},$$
$$\mathcal{A}_{(2n,2n-2,\ldots,2)}(0,1,1,-1;\mathbf{1}) = \#(2n+1) \times (2n+1) \text{ VSASMs},$$
$$\mathcal{A}_{(n-1,n-2,\ldots,0)}(1,1,1,-1;\mathbf{1}) = \# \text{ configurations of the 20-vertex model}.$$

4A) The main theorem for $w = -1$

Theorem([4, Thm 1]) For positive integers n, m we have

$$\sum_{0 \le \lambda_n < \lambda_{n-1} < \ldots < \lambda_1 \le m} \mathcal{A}_\lambda(1,1,1,-1;\mathbf{1}) = 2^n \prod_{i=1}^{n} \frac{(m-n+3i+1)_{i-1}(m-n+i+1)_i}{\left(\frac{m-n+i+2}{2}\right)_{i-1}(i)_i}.$$

In the case $m = n-1$, which corresponds to having AGTs with bottom row $(0, 1, , \ldots, n-1)$ the above becomes

$$2^{\binom{n}{2}} \prod_{i=0}^{n-1} \frac{(4i+2)!}{(n+2i+1)!},$$

which appears in recent work by Di Francesco [1], when divided by 2^n.

5A) A signless interpretation for $w = -1$

Proposition([4, Prop 3]) For λ a partition, $\mathcal{A}_\lambda(1,1,1,-1;\mathbf{1})$ counts the number of Gelfand-Tsetlin patterns with bottom row $(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$ such that

- only the bottom row can contain three equal entries
- with weight 2^r where r is the number of entries which are not equal to their north-east or north-west neighbour.

6) Proof sketch

1. Use a generalisation of a bounded Littlewood identity [2, Cor 1.2] to rewrite the operator formula as a determinant.
2. Guess a LU decomposition for both $w = 0$ and $w = -1$.
3. Use Sister Celine's algorithm and creative telescoping to prove the triple sum identity obtained in the LU decomposition.

4B) The main theorem for $w = 0$

Theorem([4, Thm 2]) For positive integers n, m we have

$$\sum_{0 \le \lambda_n < \lambda_{n-1} < \ldots < \lambda_1 \le m} \mathcal{A}_\lambda(1,1,1,0;\mathbf{1}) = 3^{\binom{n+1}{2}} \prod_{i=1}^{n} \frac{(2n+m+2-3i)_i}{(i)_i}.$$

5B) A signless interpretation for $w = 0$

Proposition([4, Prop 4]) For λ a partition, $\mathcal{A}_\lambda(1,1,1,0;\mathbf{1})$ counts the number of arrowed Gelfand-Tsetlin patterns with bottom row $(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$ without the $\nwarrow\!\!\!\nearrow$ decoration such that

- each entry appears at most twice in each row,
- an entry can only 'point' at another entry with a different value,
- for two equal entries in one row, one must be decorated with $\nearrow$ or $\nwarrow$.

7) Literature

[1] P. Di Francesco. Twenty vertex model and domino tilings of the Aztec triangle. *Electron. J. Combin.*, 28(4): Paper No. 4.38, 50, 2021.

[2] I. Fischer. Bounded Littlewood identity related to alternating sign matrices. *arXiv: 2301.00175*, 2022.

[3] I. Fischer and F. Schreier-Aigner, The relation between alternating sign matrices and descending plane partitions: $n+3$ pairs of equivalent statistics. *Adv. Math.*, 413:108831, 47, 2023.

[4] I. Fischer and F. Schreier-Aigner, (-1)-enumerations of arrowed Gelfand–Tsetlin patterns. *arXiv: 2302.04164*, 2023.

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