A graduate course on plane partitions

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Contents

Chapter 1

Introduction - Partitions and Dimensions

Partitions appear in many areas of mathematics in a natural way and can be seen as "one dimensional" objects since they are sequences of integers (with certain restrictions). At the end of the 19th century, MacMahon generalised partitions to arrays of integers (with certain restrictions), i.e., "two dimensional" objects, which he called plane partitions. The interest on plane partitions was awoken within the combinatorial community in the second half of the twentieth century and a rich theory on these objects was discovered. Nowadays, plane partitions are connected to many different areas within combinatorics, as well as to the theory of symmetric functions, representation theory and many more.

Further generalisation of partitions to "higher dimensional" objects on the other side are hardly studied and it seems impossible to obtain a similar intriguing and rich theory for them. While we do not attempt to explain why only the study of one and two dimensional partitions yield intriguing results, we will give a hint in this introduction from an enumerative point of view.

1.1 Partitions

Definition 1.1.1. *A* partition *of a non-negative integer n is a representation of n as a sum of positive integers*

$$n = \lambda_1 + \lambda_2 + \ldots + \lambda_k,$$

with $\lambda_1 \geq \ldots \geq \lambda_k > 0$. We write $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ and call λ_i the parts *of the partition.*

Another commonly used notation for partitions is to write $\lambda = a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_k^{\alpha_k}$ where $a_1 > a_2 > \ldots > a_k$ and the part a_i appears α_i times in λ for $1 \leq i \leq k$. Sometimes it is convenient to order the a_i in the opposite way, i.e., strictly increasing.

Example 1.1.2. *The partitions of 5 in the various notations are given in the following* *table.*

5	(5)	5
$4+1$	$(4,1)$	41
$3+2$	$(3,2)$	32
$3+1+1$	$(3,1,1)$	31^2
$2+2+1$	$(2,2,1)$	2^21
$2+1+1+1$	$(2,1,1,1)$	21^3
$1+1+1+1+1$	$(1,1,1,1,1)$	1^5

As usual for combinatorics, we are interested in the number of partitions of n, denoted by $p(n)$. There is no closed formula known which allows to express $p(n)$ as a function of n, however $p(n)$ has a very elegant generating function.

Theorem 1.1.3. *The generating function for the number of partitions $p(n)$ is given by*

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i}. \tag{1.1}$$

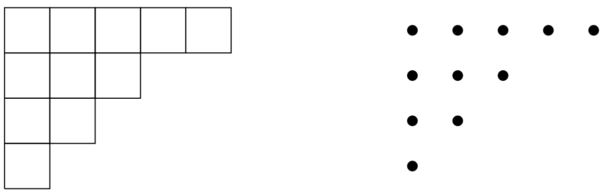
Proof. The proof consists of a series of simple transformations of sums and products.

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n &= \sum_{n=0}^{\infty}\sum_{\lambda\vdash n} q^n = \sum_{k=0}^{\infty}\sum_{\lambda=1^{\alpha_1}\cdots k^{\alpha_k}} \left(q^1\right)^{\alpha_1}\cdots\left(q^k\right)^{\alpha_k} \\ &= \sum_{\alpha_1,\alpha_2,\ldots}\prod_{i=1}^{\infty}\left(q^i\right)^{\alpha_i} = \prod_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(q^i\right)^j = \prod_{i=1}^{\infty}\frac{1}{1-q^i}. \qquad \square\end{aligned}$$

Partitions can also be regarded as geometric objects. For this we need the following definition.

Definition 1.1.4. *A* Young diagram *is a collection of boxes which are arranged in left-justified rows, where the rows have a weakly decreasing number of boxes from top to bottom in* English notation *or from bottom to top in* French notation[1]*.*

We assign to each partition a Young diagram, where the i-th part of a partition corresponds to the number of boxes in the i-th row of the diagram. For the rest of these notes, we identify partitions with their corresponding Young diagram. The boxes of the Young diagram are usually called *cells* and we will refer by the "cells of a partition" to the boxes of the Young diagram. Another form of visualisation is the *Ferrers diagram* for which the boxes of the Young diagram are replaced by dots. The following are the Young diagram (left) and Ferrers diagram (right) of the partition $(4, 3, 3, 1)$.



We want to finish this section by regarding a family of partitions closer. For two partitions λ, μ, we say that λ is *contained* in μ, denoted by $\lambda \subseteq \mu$, iff $\lambda_i \leq \mu_i$ for all i. We focus on partitions which are contained in the partition a^b, i.e., in an (a, b)-box. These

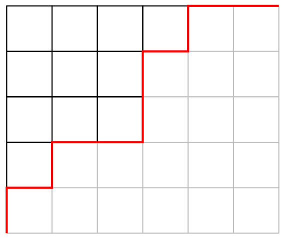


Figure 1.1: The partition $(4, 3, 3, 1)$ contained in a $(6, 5)$ box and its corresponding lattice path.

partitions are in bijection to lattice paths from $(0, 0)$ to (a, b) whose step set consists of north and east steps; for an example see Figure 1.1. Since there are no restrictions on these lattice paths, the number of partitions contained in a^b is the binomial coefficient $\binom{a+b}{a}$. It is not difficult to see that we can rewrite the binomial coefficient in the following form, which looks very similar to MacMahon's box formula (1.6),

$$\binom{a+b}{a} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{i+j}{i+j-1}. \tag{1.2}$$

We can refine this enumeration by taking the number of boxes of the Young diagram into account. For a variable q we define the q-analog $[n]_q$ of an integer n as

$$[n]_q := 1 + q + q^2 + \ldots + q^{n-1} = \frac{1-q^n}{1-q},$$

the q-analog of the factorial $[n]_q!$ as

$$[n]_q! := [n]_q [n-1]_q \ldots [1]_q,$$

and the q-binomial coefficient as

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{[m]_q!}{[n]_q! [m-n]_q!}.$$

It is an exercise for the reader to verify, that the q-binomial coefficient satisfies the recurrence

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \begin{bmatrix} m-1 \\ n-1 \end{bmatrix}_q + q^{m-n} \begin{bmatrix} m-1 \\ n \end{bmatrix}_q \tag{1.3}$$

We claim that the weighted enumeration of partitions in an (a, b)-box is given by

$$\sum_{\lambda} q^{|\lambda|} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q,$$

where the sum is over all partitions contained in an (a, b)-box. This can be proven by induction on the length of the associated lattice path. If the last step of the path is an east step, the weight of the partition is not changed by removing this step. Assume that the last step is a north step. Then the partition corresponding to the path with the last step deleted has a boxes less than our original partition. The assertion follows now by the induction hypothesis together with (1.3).

[1] In this note we will use English notation, which is natural when regarding plane partitions.

1.2 Partitions in two dimensions

By regarding partitions as one dimensional arrays of integers which are weakly decreasing from left to right, we can define *plane partitions* as arrays of integers, such that all rows and columns are partitions. A more useful and equivalent definition is the following one.

Definition 1.2.1. *A* plane partition $\pi = (\pi_{i,j})$ *is an array of positive integers*

$$\begin{array}{ccccc} \pi_{1,1} & \cdots & \cdots & \cdots & \pi_{1,\lambda_1} \\ \pi_{2,1} & \cdots & \cdots & \pi_{2,\lambda_2} & \\ \vdots & & \iddots & & \\ \pi_{n,1} & \cdots & \pi_{n,\lambda_n} & & \end{array}$$

with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ *such that the rows and columns are weakly decreasing, i.e.,* $\pi_{i,j} \geq \pi_{i+1,j}$ *and* $\pi_{i,j} \geq \pi_{i,j+1}$*. We say that* π *is a plane partition of* n *iff* $|\pi| := \sum_{i,j} \pi_{i,j} = n$ *.*

The following are three plane partitions of 20.

$$\begin{array}{cccc} 4 & 3 & 3 & 1 \\ 4 & 2 & 1 & \\ 2 & & & \end{array} \qquad \begin{array}{ccc} 4 & 3 & 1 \\ 4 & 2 & \\ 2 & 1 & \\ 1 & 1 & \\ 1 & & \end{array} \qquad \begin{array}{cc} 7 & 5 \\ 6 & 2 \end{array}$$

In various places it will be convenient to regard plane partitions as rectangular arrays of integers. We therefore allow 0 entries as well and identify two arrays if the only differ by 0 entries. For example we do not distinguish between the first plane partition listed above and the plane partition below.

$$\begin{array}{ccccc} 4 & 3 & 3 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{array}$$

In [13] MacMahon described the generating function for plane partitions.

Theorem 1.2.2. *The generating function for plane partitions is given by*

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{\infty} \frac{1}{(1-q^i)^i}, \tag{1.4}$$

where the sum is over all plane partitions.

We will present the proof of a generalisation due to Stanley in the next Section. Contrary to partitions, we can get a simple recursion using the above generating function. The following result can be found for example in [4, Theorem 1.2].

Corollary 1.2.3. *Let* $pp(n)$ *be the number of plane partitions of* n *and* $\sigma_2(n) = \sum_{i|n} i^2$ *the sum of the squares of the divisors of* n*. Then*

$$pp(n) = \frac{1}{n} \sum_{i=1}^{n} \sigma_2(i)\, pp(n-i). \tag{1.5}$$

Proof. By differentiating both sides of (1.4) and manipulating the sums, we obtain

$$\begin{aligned}\sum_{n\geq 1} n\, pp(n) q^{n-1} &= \sum_{i\geq 1} i^2 \frac{q^{i-1}}{1-q^i} \sum_{n\geq 0} pp(n) q^n \\ &= \sum_{i\geq 1} i^2 \left(q^{i-1} + q^{2i-1} + \dots\right) \sum_{n\geq 0} pp(n) q^n \\ &= \sum_{n\geq 0} \sum_{k=1}^{n} pp(n-k) q^{n-k} \sum_{i|k} i^2 q^{\frac{k}{i} i - 1}. \qquad \square\end{aligned}$$

The first few values for $pp(n)$ and $\sigma_2(n)$ are given in the next table.

n	0	1	2	3	4	5
$\sigma_2(n)$		1	5	10	21	26
$pp(n)$	1	1	3	6	13	24

An important feature of plane partitions is their graphical representation. We can visualise a plane partition π as stacks of unit cubes by putting $\pi_{i,j}$ unit cubes at the position (i, j), see Figure 1.2. We define a plane partition to fit into an (a, b, c)-box if it

$$\begin{matrix} 4 & 3 & 3 & 1 \\ 4 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \end{matrix}$$

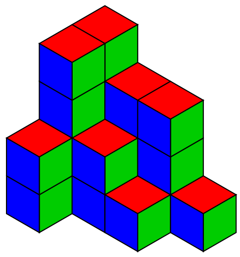


Figure 1.2: A plane partition and its graphical representation as stacks of cubes.

has at most a rows and b columns with non-zero entries and every entry is at most c. We call such a plane partition an (a, b, c)-boxed plane partition. The number of plane partitions inside a box is given by the famous formula by MacMahon.

Theorem 1.2.4 ([13]). *The number of (a, b, c)-plane partitions is*

$$\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}. \tag{1.6}$$

We will present a proof of the weighted version of the box formula in Section 2.2. It is very remarkable that not only the enumeration of boxed plane partitions, but also all of the ten symmetry classes have simple product formulas; more details will follow in the next chapter.

1.3 Higher-dimensional partitions

In the previous section we generalised partitions to two dimensional arrays of integers. Already MacMahon had the idea to add more dimensions and define multidimensional partitions.

Definition 1.3.1. *A d-dimensional partition π is an array of non-negative integers $\pi_{i_1,\dots,i_d}$ which is weakly decreasing when all but one coordinate are fixed, i.e., for all $1 \le k \le d$ holds*

$$\pi_{i_1,\dots,i_d} \ge \pi_{i_1,\dots,i_{k-1},i_k+1,i_{k+1},\dots,i_d}.$$

As before we identify two partitions iff all non-zero entries coincide. Comparing the enumeration formulas for 0, 1 and 2 dimensional partitions one could hope for simple formulas for any dimension d, however this is not true.

d	generating function	box formula
0	$\frac{1}{1-q}$	$\prod_{i=1}^{a} \frac{i+1}{i}$
1	$\prod_{i=1}^{\infty} \frac{1}{1-q^i}$	$\prod_{i=1}^{a}\prod_{j=1}^{b} \frac{i+j}{i+j-1}$
2	$\prod_{i=1}^{\infty} \frac{1}{(1-q^i)^i}$	$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$

For the rest of this section we restrict ourselves to 3-dimensional partitions which are also called *solid partitions*. Assume that the number of boxed solid partitions is given by a product formula. For given values for the box-dimensions, every factor in the product is contributing primes to the enumerator and denominator. Most of these primes will cancel out, but since we will obtain an integer there must be “many” prime numbers which survive. These primes can not be too large, e.g., for $d = 2$ they have to be smaller than $a + b + c - 1$. This means that iff we would have a product formula, where the denominator and numerator of the factors are linear or polynomial in the parameters, we can only have small primes appearing in the sequence. By writing a small program or comparing with the online Encylopedia of integer sequences (sequence A056932), one can show that the number of solid partitions inside an $(2, 2, 2, 3)$-box is 887. Since this is a prime, the box formula for d-dimensional partitions cannot be a product formula for $d \ge 3$.

Denote by $M_d(n)$ the number of d-dimensional partitions of n. MacMahon conjectured the following formula for the generating function

$$\sum_{n=0}^{\infty} M_d(n)q^n = \prod_{i=1}^{\infty}(1-q^i)^{-\binom{d+i-2}{d-1}}.$$

We can disprove this conjecture by the following arguments. First we expand the right hand side partially. Let μ_i be the coefficient of the above product when written as a series in q

$$\prod_{i=1}^{\infty}(1-q^i)^{-\binom{k+i-2}{k-1}} = \sum_{i=0}^{\infty} \mu_i q^i.$$

By using the Binomial Theorem we obtain

$$\mu_6 = 1 + 10d + 27\binom{d}{2} + 29\binom{d}{3} + 12\binom{d}{4} + \binom{d}{5}.$$

By going through all possibilities to form a d-dimensional partition we obtain

$$M_d(6) = 1 + 10d + 27\binom{d}{2} + 28\binom{d}{3} + 11\binom{d}{4} + \binom{d}{5},$$

for a detailed explanation see [2, pp. 191–196]. Also compare to the paper of Knuth [9].

Chapter 2

Enumeration of plane partitions

In the introduction we already saw two enumeration results for plane partitions, however without proofs. We start this chapter by deriving the generating function for plane partitions. To be more precise we are actually proving a refinement due to Stanley [15] called the *trace generating function*. This will be followed by a proof of *MacMahon's box formula* for the enumeration of plane partitions inside an (a, b, c)-box. Finally we will regard symmetry classes of plane partitions and prove the enumeration formulas for some of these. The proof techniques we will see are the *RSK algorithm*, non-intersecting lattice paths and the *Lindström-Gessel-Viennot Theorem*, determinant evaluations, symmetric functions and perfect matchings.

2.1 (Trace) generating function for plane partitions

Definition 2.1.1. *The* trace $\mathrm{tr}(\pi)$ *of a plane partition π is the sum of the parts on its main diagonal, i.e.,* $\mathrm{tr}(\pi) = \sum_i \pi_{i,i}$.

We prove the following refinement of the generating function for plane partitions which is due to Stanley.

Theorem 2.1.2. *The* trace generating function *of plane partitions with at most r rows and c columns is*

$$\sum_{\pi} t^{\mathrm{tr}(\pi)} q^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1}{1 - tq^{i+j-1}}, \tag{2.1}$$

where the sum is over all plane partitions with at most r rows and c columns.

We follow Stanley's original proof, which is based on the proof of Theorem 2 in [3]. The proof is using the fact that plane partitions have a remarkable similarity to *semistandard Young tableaux*, the *RSK*-algorithm as well as the *Frobenius notation* for partitions. Hence lets study these things first. For an alternative presentations of the proof of Theorem 2.1.2 see [15], [16, Section 7.20] or [12, Section 5].

Definition 2.1.3. *Let λ be a partition. A* semistandard Young tableau *(SSYT) of shape λ is a filling of the cells of λ with positive integers such that all rows are weakly increasing from left to right and the columns are strictly increasing from top to bottom*[1].

[1] In French notation the columns are strictly decreasing from bottom to top.

Next we display an SSYT of shape $(5, 4, 3, 1)$.

$$\begin{array}{lllll} 1 & 3 & 3 & 4 & 5 \\ 2 & 4 & 5 & 5 & \\ 4 & 5 & 7 & & \\ 6 & & & & \end{array}$$

The *Robinson-Schensted-Knuth algorithm* (RSK) is a bijection between pairs of SSYTs of the same shape and matrices with non-negative integer entries. The core ingredient of RSK is *row-insertion*. Inserting an integer k into a row is defined as follows.

- If all entries of the row are smaller than k, we add k to the end of the row ,
- otherwise we replace the left-most entry which is larger than k by k; the replaced entry is said to be *bumped*.

The row-insertion of k into an SSYT T, denoted by $T \leftarrow k$ is defined by the following process:

1. Insert k into the first (top) row of T.
2. If an entry is bumped, insert it in the next row.
3. Repeat step 2 until no entry is bumped.

Example 2.1.4. *The row-insertion of* 3 *in the SSYT from above is displayed next, where we attach the integer inserted in a row to the corresponding arrow and the newly inserted entries are marked in red.*

$$\begin{array}{lllll} 1 & 3 & 3 & 4 & 5 \\ 2 & 4 & 5 & 5 & \\ 4 & 5 & 7 & & \\ 6 & & & & \end{array} \quad \underset{3}{\rightarrow} \quad \begin{array}{lllll} 1 & 3 & 3 & \color{red}{3} & 5 \\ 2 & 4 & 5 & 5 & \\ 4 & 5 & 7 & & \\ 6 & & & & \end{array} \quad \underset{4}{\rightarrow} \quad \begin{array}{lllll} 1 & 3 & 3 & 3 & 5 \\ 2 & 4 & \color{red}{4} & 5 & \\ 4 & 5 & 7 & & \\ 6 & & & & \end{array}$$

$$\underset{5}{\rightarrow} \quad \begin{array}{lllll} 1 & 3 & 3 & 3 & 5 \\ 2 & 4 & 4 & 5 & \\ 4 & 5 & \color{red}{5} & & \\ 6 & & & & \end{array} \quad \underset{7}{\rightarrow} \quad \begin{array}{lllll} 1 & 3 & 3 & 3 & 5 \\ 2 & 4 & 4 & 5 & \\ 4 & 5 & 5 & & \\ 6 & \color{red}{7} & & & \end{array}$$

For every semistandard Young tableau T and integer k, the row-insertion of k into T yields again a semistandard Young tableau. First we note that inserting an integer into a weakly increasing row generates a weakly increasing row. Hence it suffices to prove that the columns of the result of the insertion procedure are strictly increasing. First we claim that a bumped entry z is inserted at position weakly to the left than its original position. Indeed, when reinserted, z replaces the left-most entry that is larger than z. If z would be reinserted at a position strictly to the right than its original position, then this would imply that the entry below z's original position is smaller than z which is a contradiction. Since we bumped the left-most z in the previous step, all entries in the previous row up to z's original position are now smaller than z. Hence, when reinserted, z is strictly larger than the entry above. Since z is replacing a larger entry when reinserted, it is for sure smaller than the entry below. This implies that the result of an insertion process is again a semistandard Young tableau.

We define a *biword* as a $2 \times n$ array of integers

$$\begin{matrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{matrix},$$

such that the top row is weakly increasing and $b_i \leq b_{i+1}$ if $a_i = a_{i+1}$. We associate to a matrix A with non-negative integer entries the unique biword ω_A which contains the column (i, j) exactly $a_{i,j}$ many times, see the next example. Given a biword, we construct a sequence of pairs of semistandard Young tableaux $(P_i, Q_i)_{0 \leq i \leq n}$ of the same shape. The first pair of tableaux is given by $(P_0, Q_0) = (\emptyset, \emptyset)$. For $i \geq 1$, the tableau P_i is obtained by row-inserting b_i into P_{i-1}; the tableau Q_i is obtained by adding the integer a_i to Q_{i-1} such that P_i and Q_i have the same shape. We call the tableaux P_i the *insertion tableaux* and Q_i *recording tableaux*. The RSK-algorithm is defined to map the matrix A to the pair (P_n, Q_n) of the described process. It is an exercise for the reader to verify that Q_n is a semistandard Young tableau; this follows also from Proposition **??**.

Example 2.1.5. *We regard the following matrix and its corresponding biword*

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix} \qquad \begin{matrix} 1 & 1 & 1 & 2 & 2 \\ 1 & 3 & 3 & 1 & 2 \end{matrix}.$$

Then the sequence of pair of tableaux appearing in the RSK-algorithm is

$$\begin{matrix} P_i: & \emptyset, & 1\,, & 1\;3\,, & 1\;3\;3\,, & \begin{matrix} 1 & 1 & 3 \\ 3 & & \end{matrix}, & \begin{matrix} 1 & 1 & 2 \\ 3 & 3 & \end{matrix}, \\ Q_i: & \emptyset, & 1\,, & 1\;1\,, & 1\;1\;1\,, & \begin{matrix} 1 & 1 & 1 \\ 2 & & \end{matrix}, & \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & \end{matrix}. \end{matrix}$$

The main theorem on the RSK-algorithm is the following.

Theorem 2.1.6. *The Robinson-Schensted-Knuth algorithm is a bijection between matrices $A = (a_{i,j})$ with non-negative entries and pairs of semistandard Young tableaux (P, Q) such that*

- *j appears in P exactly $\sum_i a_{i,j}$ times,*
- *i appears in Q exactly $\sum_j a_{i,j}$ times.*

Proof. The properties of the map follow immediately from the insertion process. Hence we only need to show that it is actually a bijection. We prove this by constructing an inverse map. We construct again a sequence of pair of SSYTs (P_k, Q_k) with $P_n = P$ and $Q_n = Q$. Denote by a_i the largest entry of Q_i. We obtain Q_{i-1} by deleting the the right-most occurrence of a_i. We construct P_{i-1} by deleting the same cell as we did for Q_{i-1} and denote its entry by z. We reinsert z in the row above by replacing the right-most entry of the above row which is smaller than z by z. We reinsert the replaced entry in the row above in the same way and continue until we reach the first row. Denote by b_i the entry which is replaced in the first row and hence can not be reinserted in a row above. It is clear that this process is exactly reversing the row-insertion and therefore allows us to recover the biword ω_A. $\square$

The final ingredient is the *modified Frobenius notation* for a partition. For a partition λ let l denote the length of its *Durfee square*, i.e., the largest i such that $\lambda_i \geq i$. The modified Frobenius notation[2] of λ is $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_l - l + 1 | \lambda'_1, \lambda'_2 - 1, \ldots, \lambda'_l - l + 1)$

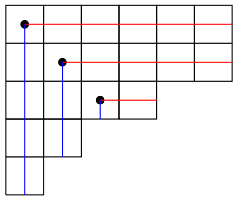


Figure 2.1: The Young diagram of the partition $(4,3,3,1)$, in modified Frobenius notation $(6,5,2|5,3,1)$. The red lines correspond to the first sequence in the modified Frobenius notation, the blue lines to the second one.

where λ' is the conjugate partition of λ. For an example see Figure 2.1.

We associate a plane partition π with at most r rows and c columns to an $c \times r$ matrix with non-negative entries by a sequence of bijections which we will demonstrate in the example of the plane partition

$$\begin{array}{cccc} 4 & 4 & 2 & 1 \\ 4 & 3 & 2 & \\ 2 & 1 & 1 & \\ 2 & & & \end{array}.$$

First we map π to the plane partition π' by conjugation the rows of π which are actually partitions. We obtain a plain partition with at most r rows whose entries are at most c. The statistics of our interest are

$$|\pi| = |\pi'|,$$
$$\operatorname{tr}(\pi) = |\{\pi'_{i,j} \geq i\}|.$$

In our example we obtain

$$\begin{array}{cccc} 4 & 3 & 2 & 2 \\ 3 & 3 & 2 & 1 \\ 3 & 1 & & \\ 1 & 1 & & \end{array}.$$

Next we rewrite π' as a pair (C_1, C_2) of plane partitions of the same shape using the modified Frobenius notation. The i-th column of C_1 (resp. C_2) is defined as the first (resp. second) sequence of the modified Frobenius notation of the i-th column of π'. We obtain for our example

$$\begin{array}{cccc} 4 & 3 & 2 & 2 \\ 2 & 2 & 1 & \\ 1 & & & \end{array}, \qquad \begin{array}{cccc} 4 & 4 & 2 & 2 \\ 2 & 1 & 1 & \\ 1 & & & \end{array}.$$

The entries of C_1 are at most c and the entries of C_2 at most r. The statistics are now given by

$$|\pi| = |C_1| + |C_2| - \#(\text{parts of } C_1),$$
$$\operatorname{tr}(\pi) = \#(\text{parts of } C_1).$$

[2] The Frobenius notation does not take the diagonal into account, i.e., its defined as $(\lambda_1 - 1, \lambda_2 - 2, \ldots, \lambda_l - l | \lambda'_1 - 1, \lambda'_2 - 2, \ldots, \lambda'_l - l)$.

Since the two sequences of the Frobenius notation are strictly decreasing, the rows of C_1, C_2 are strictly decreasing as well. Such plane partitions are called *column-strict* or *reverse semistandard Young tableaux* since the inequality condition for a SSYT are reversed.

We can now use the inverse of a variation of RSK to map the pair (C_1, C_2) to a matrix $A = (a_{i,j})$ with non-negative entries. In this variation we either inverse the role of $\leq$ and $\geq$ in the description of the algorithm, or alternatively RSK is applied to the biword

$$\begin{matrix} -a_n & -a_{n-1} & \cdots & -a_1 \\ -b_n & -b_{n-1} & \cdots & -b_1 \end{matrix},$$

i.e., reverse the order of the rows and introduce minus signs, and omit the signs in the result. Applying the reverse variation of RSK, where we write $\overline{x} := -x$ and mark the entries in red which are recovered in each step, yields

$$\begin{matrix} \overline{4} & \overline{3} & \overline{2} & {\color{red}\overline{2}} \\ \overline{2} & \overline{2} & \overline{1} & \\ \overline{1} & & & \end{matrix} \quad \begin{matrix} \overline{4} & \overline{4} & \overline{2} & \overline{2} \\ \overline{2} & \overline{1} & {\color{red}\overline{1}} & \\ \overline{1} & & & \end{matrix} \longrightarrow \begin{matrix} \overline{4} & {\color{red}\overline{3}} & \overline{2} & \overline{1} \\ \overline{2} & \overline{2} & & \\ \overline{1} & & & \end{matrix} \quad \begin{matrix} \overline{4} & \overline{4} & \overline{2} & \overline{2} \\ \overline{2} & {\color{red}\overline{1}} & & \\ \overline{1} & & & \end{matrix} \longrightarrow$$

$$\begin{matrix} {\color{red}\overline{4}} & \overline{2} & \overline{2} & \overline{1} \\ \overline{2} & & & \\ \overline{1} & & & \end{matrix} \quad \begin{matrix} \overline{4} & \overline{4} & \overline{2} & \overline{2} \\ \overline{2} & & & \\ {\color{red}\overline{1}} & & & \end{matrix} \longrightarrow \begin{matrix} \overline{2} & \overline{2} & \overline{2} & {\color{red}\overline{1}} \\ \overline{1} & & & \end{matrix} \quad \begin{matrix} \overline{4} & \overline{4} & \overline{2} & {\color{red}\overline{2}} \\ \overline{2} & & & \end{matrix} \longrightarrow$$

$$\begin{matrix} \overline{2} & \overline{2} & {\color{red}\overline{2}} \\ \overline{1} & & \end{matrix} \quad \begin{matrix} \overline{4} & \overline{4} & {\color{red}\overline{2}} \\ \overline{2} & & \end{matrix} \longrightarrow \begin{matrix} \overline{2} & {\color{red}\overline{2}} \\ \overline{1} & \end{matrix} \quad \begin{matrix} \overline{4} & \overline{4} \\ {\color{red}\overline{2}} & \end{matrix} \longrightarrow \overline{2} \; {\color{red}\overline{2}} \quad \overline{4} \; {\color{red}\overline{4}} \longrightarrow {\color{red}\overline{2}} \quad {\color{red}\overline{4}} \longrightarrow \emptyset \quad \emptyset \,.$$

We obtain the biword

$$\begin{matrix} \overline{4} & \overline{4} & \overline{2} & \overline{2} & \overline{2} & \overline{1} & \overline{1} & \overline{1} \\ \overline{2} & \overline{1} & \overline{2} & \overline{1} & \overline{1} & \overline{4} & \overline{3} & \overline{2} \end{matrix}$$

and hence the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Theorem 2.1.6 implies that A has c columns and r rows and that the statistics we are interested in are given by

$$|\pi| = \sum_{i,j} (i+j-1) a_{i,j},$$
$$\mathrm{tr}(\pi) = \sum_{i,j} a_{i,j}.$$

Summarising we obtain for the trace generating function

$$\sum_{\pi} t^{\mathrm{tr}(\pi)} q^{|\pi|} = \sum_{A=(a_{i,j})} t^{\sum_{i,j} a_{i,j}} q^{\sum_{i,j}(i+j-1)a_{i,j}} = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1}{1 - tq^{i+j-1}}.$$

where the first sum is over all plane partitions π with at most r rows and c columns and the second sum is over all $c \times r$ matrices A. This proves Theorem 2.1.2.

In [7, 8] Gansner generalises the trace for plane partitions and derives generating functions for plane partitions and reverse plane partitions..

2.2 Enumeration of boxed plane partition

We encounter one of the standard methods for enumerating plane partitions in the proof of MacMahon's box formula of Theorem 1.2.4. This method is used for the enumeration of many of the symmetry classes of plane partitions as well as related combinatorial objects. It boils down to the three steps:

1. Translate plane partitions (or the combinatorial object of choice) into families of non-intersecting lattice paths.

2. Use the Lindström-Gessel-Viennot Theorem to express the enumeration as a determinant.

3. Evaluate the determinant explicitly.

The first two steps of this method are usually straight forward. The third step however, can be quite difficult or even lead to an open problem. A good resource which lists many methods for evaluating determinants is the paper [11] by Krattenthaler.

In order to transform a plane partition inside an (a, b, c)-box into a family of non-intersecting lattice paths, we display the plane partition graphically together with the contour of the box (this time we omit the colours of the unit cubes), see Figure 2.2. The obtained figure is an hexagon. We draw paths from the middle of the tilings on the top left of the hexagon towards the bottom of the right of the hexagon, by always connecting the opposite sites of the following two tilings.

One could also imagine the configuration of unit cubes in a slightly tilted box where we spill water along the left boundary of the box and record the path the water is taking while running down to the other side of the box. Finally we transform these paths to

4	3	3	1
4	2	1	
2			

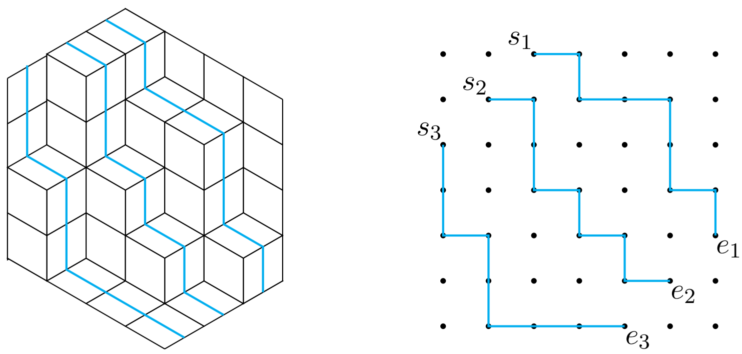


Figure 2.2: A $(3, 4, 4)$-boxed plane partition (left), its representation as stacks of unit cubes inside an $(3, 4, 4)$-box (middle) and its corresponding family of non-intersecting lattice paths (right).

paths in a grid and obtain a family of non-intersecting lattice paths. If the box had dimensions (a, b, c) then the starting points of the lattice paths have the form $s_i = (-i, -i)$ for $1 \le i \le a$ and the end points are $e_i = (b - i, -c - i)$ for $1 \le i \le a$.

The Lindström-Gessel-Viennot Theorem allows us to enumerate the previous described lattice paths. However we need some notations first. Let G be a finite directed graph without cycles. A *path* in G is a sequence of vertices $(v_1, \ldots, v_n)$ such that (v_i, v_{i+1}) is an edge in G for $1 \leq i < n$. If the graph has an edge weight ω, then the weight of a path $(v_1, \ldots, v_n)$ is

$$\prod_{i=1}^{n-1} \omega(v_i, v_{i+1}).$$

We call a family of paths in G *non-intersecting* if no pair of paths in the family has a vertex in common. The weight of a family of paths is the product over all path weights. For a family $\mathcal{P}$ of n paths with starting points $\{s_1, \ldots, s_n\}$ and end points $\{e_1, \ldots, e_n\}$ where the i-th path has starting point s_i and end point $e_{\sigma(i)}$, we define the sign $\operatorname{sgn}(\mathcal{P}) := \operatorname{sgn}(\sigma)$.

Theorem 2.2.1 (Lindström-Gessel-Viennot)**.** *Let $s_1, \ldots, s_n$ and $e_1, \ldots, e_n$ be two families of pairwise different vertices of a finite directed graph G with edge weight ω and denote by $P(i,j)$ the weighted enumeration of paths in G from s_i to e_j. The weighted, signed enumeration of families of non-intersecting paths from $s_1, \ldots, s_n$ to $e_1, \ldots, e_n$ is given by the determinant*

$$\sum_{\mathcal{P}} \operatorname{sgn}(\mathcal{P}) \omega(\mathcal{P}) = \det_{1 \leq i,j \leq n} \left(P(i,j) \right), \tag{2.2}$$

where the sum is over all families $\mathcal{P}$ of non-intersecting paths with starting points $\{s_1, \ldots, s_n\}$ and end points $\{e_1, \ldots, e_n\}$.

Proof. First we use Leibniz's formula to expand the determinant

$$\det_{1 \leq i,j \leq n} \left(P(i,j) \right) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} P(i, \sigma(i)). \tag{2.3}$$

The sum on the right hand side is now the weighted and signed sum over all families $\mathcal{P}$ of paths with starting points $\{s_1, \ldots, s_n\}$ and end points $\{e_1, \ldots, e_n\}$. It suffices to proof the existence of a weight preserving, sign inverting involution on the set of families of intersecting paths.

Let $(p_1, \ldots, p_n)$ be a family of intersecting paths where p_i starts at s_i and ends at $e_{\sigma(i)}$. The sign of this family is $\operatorname{sgn}(\sigma)$. Let (s,t) be the lexicographically smallest pair of indices such that p_s and p_t are intersecting. Denote by $p_{s,1}$ ($p_{t,1}$ resp.) the part of p_s (p_t resp.) before the last intersection of p_s and p_t and by $p_{s,2}$ ($p_{t,2}$ resp.) the rest of p_s (p_t resp.). We define $\hat{p}_s$ to be the concatenation of the paths $p_{s,1}$ and $p_{t,2}$ and the path $\hat{p}_t$ to be the concatenation of $p_{t,1}$ and $p_{s,2}$. The family $(p_1, \ldots, p_{s-1}, \hat{p}_s, p_{s+1}, \ldots, p_{t-1}, \hat{p}_t, p_{t+1}, \ldots, p_n)$ has the same weight as the original family and its sign is given by $\operatorname{sgn}((\sigma(s), \sigma(t))\sigma) = -\operatorname{sgn}(\sigma)$. Since the new family has the same intersections as the original one, the map

$$(p_1, \ldots, p_n) \mapsto (p_1, \ldots, p_{s-1}, \hat{p}_s, p_{s+1}, \ldots, p_{t-1}, \hat{p}_t, p_{t+1}, \ldots, p_n),$$

is an involution on the set of families of intersecting lattice paths which is weight preserving and sign reversing. Hence the families on intersecting paths in (2.3) cancel each other. □

We can now apply this theorem to our setting of (a,b,c)-boxed plane partitions. The starting points s_i and end points e_i are given by

$$s_i = (-i, -i), \qquad e_i = (b - i, -c - i),$$

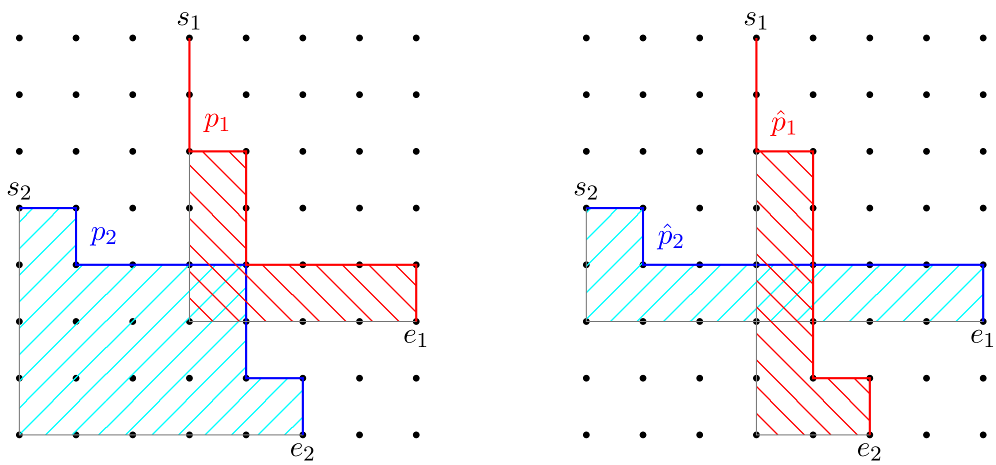


Figure 2.3: The paths p_1, p_2 (left) and $\hat{p}_1, \hat{p}_2$ (right) together with their area which is shaded according to the colour of the path.

for $1 \leq i \leq a$. For the unweighted case follows

$$P(i,j) = \binom{b+c}{b-j+i}.$$

The number of (a,b,c)-boxed plane partitions is therefore

$$\det_{1\leq i,j\leq a}\left(\binom{b+c}{b-j+i}\right). \tag{2.4}$$

We could now use for example the Desnanot-Jacobi Theorem to evaluate this determinant to prove Theorem 1.2.4. It is however not difficult to prove a generalisation.

Theorem 2.2.2. *The generating function of plane partitions inside an (a,b,c)-box is given by*

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}, \tag{2.5}$$

where the sum is over all (a,b,c)-boxed plane partitions π.

Proof. It turns out that we cannot directly use Theorem 2.2.1 since the weights we want can not be obtained by edge weights; however we can adapt the proof of it to our situation. Let $s_i = (-i,-i)$ and $e_i = (b-i,-c-i)$. We weight a path p from s_i to e_j by $q^{i(i-j)}$ times q to the area area(p) between the path and the minimal path from s_i to e_j, see Figure 2.3. For paths fro s_i to e_i this corresponds exactly to weight we want to have. Let p_k be two paths from s_{i_k} to e_{j_k} for $k=1,2$ which intersect and denote by $\hat{p}_k$ the paths where we swapped the endings as in the proof of the Lindström-Gessel-Viennot Theorem. For an example see Figure 2.3. We can immediately see that we have the following relation between the combined areas

$$\text{area}(p_1) + \text{area}(p_2) = \text{area}(\hat{p}_1) + \text{area}(\hat{p}_2) + (i_1 - i_2)(j_1 - j_2).$$

We obtain that the combined weight of p_1, p_2 is equal to the combined weight of $\hat{p}_1, \hat{p}_2$. This implies that the bijection in the proof of Theorem 2.2.1 is in our setting still weight preserving. The weighted enumeration of plane partitions inside an (a, b, c)-box is therefore given by the determinant

$$\det_{1\le i,j\le a}\left(q^{i(i-j)}\begin{bmatrix} b+c \\ b-j+i \end{bmatrix}_q\right). \tag{2.6}$$

The final step is to evaluate this determinant. By the definition of the q-binomial coefficient the determinant is equal to

$$\det_{1\le i,j\le a}\left(q^{i(i-j)}\frac{\prod_{k=1}^{b+c}(1-q^k)}{\prod_{k=1}^{b-j+i}(1-q^k)\prod_{k=1}^{c+j-i}(1-q^k)}\right).$$

By factoring out

$$\frac{\prod_{k=1}^{b+c}(1-q^k)}{\prod_{k=1}^{b+i-1}(1-q^k)\prod_{k=1}^{a+c-i}(1-q^k)},$$

from the i-th row for $1 \le i \le a$, we obtain

$$\prod_{i=1}^{a}\left(\frac{\prod_{k=1}^{b+c}(1-q^k)}{\prod_{k=1}^{b+i-1}(1-q^k)\prod_{k=1}^{a+c-i}(1-q^k)}\right)$$
$$\times \det_{1\le i,j\le a}\left(q^{i(i-j)}(1-q^{c+a-i})\cdots(1-q^{c-i+j+1})(1-q^{b+i-j+1})\cdots(1-q^{b+i-1})\right).$$

Next we rewrite the expressions in the determinant in the form $(q^i - q^*)$ and obtain for the determinant

$$\det_{1\le i,j\le a}\left((-1)^{j-1}q^{b(j-1)-\binom{j}{2}+i(i-a)}(q^i-q^{c+a})\cdots(q^i-q^{c+j+1})(q^i-q^{j-b-1})\cdots(q^i-q^{1-b})\right)$$
$$=\prod_{i=1}^{a}(-1)^{i-1}q^{i(b-a)-b+\binom{i+1}{2})}\det_{1\le i,j\le n}\left((q^i-q^{c+a})\cdots(q^i-q^{c+j+1})(q^i-q^{j-b-1})\cdots(q^i-q^{1-b})\right),$$

where we factored out $(-1)^{j-1}q^{b(j-1)-\binom{j}{2}}$ from the j-th column for $1 \le j \le n$ and $q^{i(i-a)}$ from the i-th row for $1 \le i \le n$. We use Lemma 2.2.3 for $x_i = q^i$, $a_j = -q^{c+j}$ and $b_j = -q^{-b+j-1}$ to evaluate the determinant and obtain

$$\prod_{1\le i<j\le a}(q^i-q^j)\prod_{2\le i\le j\le a}(-q^{-b+i-1}+q^{c+j})$$
$$=(-1)^{\binom{a}{2}}\prod_{1\le i<j\le a}(1-q^{j-i})\prod_{2\le i\le j\le a}(1-q^{b+c+j-i+1})\prod_{i=1}^{a}q^{i(a-i)}\prod_{i=2}^{a}q^{(-b+i-1)(a-i+1)}.$$

Combining the above, the generating function is equal to

$$(-1)^{2\binom{a}{2}} \prod_{i=2}^{a} q^{i(b-a)-b+\binom{i+1}{2})+i(a-i)+(-b+i-1)(a-i+1)}$$
$$\times \prod_{i=1}^{a} \left(\frac{\prod_{k=1}^{b+c}(1-q^k)}{\prod_{k=1}^{b+i-1}(1-q^k) \prod_{k=1}^{a+c-i}(1-q^k)} \right) \prod_{1\le i<j\le a} (1-q^{j-i}) \prod_{2\le i\le j\le a} (1-q^{b+c+j-i+1}).$$

It is an exercise to the reader to verify that the sign and the q power in front cancel and that the rest is equal to the assertion. $\square$

To complete the above proof, we show the following generalisation of the Vandermonde determinant by Krattenthaler [10, Lemma 2.2], see also [11, Lemma 3].

Lemma 2.2.3. *Let $x_1, \ldots, x_n$, $a_2, \ldots, a_n$ and $b_2, \ldots, b_n$ be indeterminates. Then holds*

$$\det_{1\le i,j\le n} \Big((x_i + a_n) \cdots (x_i + a_{j+1})(x_i + b_j) \cdots (x_i + b_2) \Big)$$
$$= \prod_{1\le i<j\le n} (x_i - x_j) \prod_{2\le i\le j\le n} (b_i - a_j). \quad (2.7)$$

Proof. The determinant is an alternating polynomial in the $x_1, \ldots, x_n$. It is therefore a product of the Vandermonde determinant and a symmetric function in the $x_1, \ldots, x_n$. When regarding as a function in x_1, both determinant as well as the Vandermonde determinant are a polynomial of degree $n-1$ in x_1. Hence the symmetric function is actually a constant, i.e. the determinant is equal to $C \prod_{1\le i<j\le n}(x_i - x_j)$. By setting $x_i = -a_i$ for $2 \le i \le n$ the matrix in the determinant becomes an upper triangular matrix. Comparing the determinant, which became the product of the entries on the main diagonal

$$\prod_{k=2}^{n}(x_1 + a_k) \prod_{2\le i<j\le n} (a_j - a_i) \prod_{2\le k\le i\le n} (b_k - a_i)$$

to the Vandermonde determinant we see that the constant C is the product $\prod_{2\le i\le j\le n}(b_i - a_j)$ which proves the claim. $\square$

2.3 Symmetries of plane partitions

We will consider three operations on plane partitions and regard the symmetry classes with respect to these operations. It is convenient to regard the plane partition as a lozenge tiling of a regular hexagon, which we have done implicitly in the previous section. As before we first interpret an (a, b, c)-boxed plane partition graphically as stacks of unit cubes. Next we forget about the shading of the unit cubes and also display the boundary of the (a, b, c) box itself, for an example see Figure 2.4. We have obtained a tiling of a regular hexagon with side lengths a, b, c, a, b, c by unit rhombi, also called *lozenges*. It is noteworthy that this was first realised in 1989 [6].

The first operation is the *reflection* $(\pi_{i,j}) \mapsto \pi^* = (\pi_{j,i})$. This corresponds to a reflection along the vertical axis for the lozenge tiling. The second operation is the *rotation* π^ρ which is defined for the lozenge tiling representation by an $120°$ rotation. The third

$$\begin{matrix} 4 & 3 & 3 & 1 \\ 4 & 2 & 1 & \\ 2 & & & \end{matrix}$$

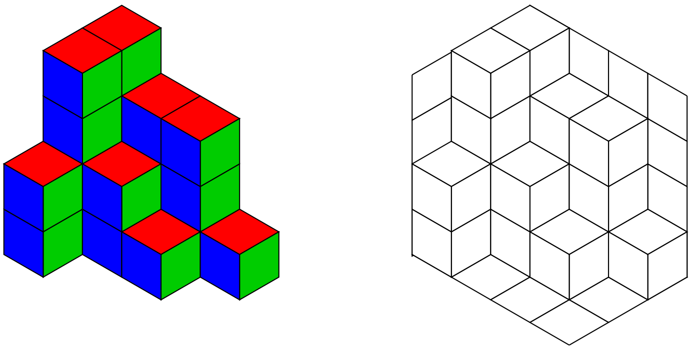


Figure 2.4: A $(3, 4, 4)$-boxed plane partition (left), its representation as stacks of unit cubes (middle) and the corresponding lozenge tiling of an hexagon with side lengths $3, 4, 4, 3, 4, 4$ (right).

Class	Abbreviation	Symmetry group
1)	PP	$\langle 1 \rangle$
2)	SPP	$\langle * \rangle$
3)	CSPP	$\langle \rho \rangle$
4)	TSPP	$\langle *, \rho \rangle$
5)	SCPP	$\langle c \rangle$
6)	TCPP	$\langle * \circ c \rangle$
7)	SSCPP	$\langle *, c \rangle$
8)	CSTCPP	$\langle \rho, * \circ c \rangle$
9)	CSSCPP	$\langle \rho, c \rangle$
10)	TSSCPP	$\langle *, \rho, c \rangle$

Table 2.1: The 10 symmetry classes of plane partitions. The abbreviations are: PP – plane partitions, S – symmetric, CS – cyclically symmetric, TS – totally symmetric, SC – self-complementary, TC – transpose-complementary.

operation is the *complement* π^c. Intuitively this corresponds to regarding the "thin air" in the box instead of the original plane partition. Formally it is defined by

$$\pi^c = (c - \pi_{a+1-i,b+1-j})_{i,j}.$$

In the lozenge tiling representation this corresponds to an $180°$ rotation.

Denote by $N_d(a, b, c)$ the enumeration of plane partitions of class d inside an (a, b, c)-box. It turns out that the enumeration formulas of the classes $5, 7, 9$ can be expressed by the formulas for the classes 1 and 10

$$\begin{aligned} N_5(a, b, 2c) &= N_1\left(\left\lceil \frac{a}{2} \right\rceil, \left\lfloor \frac{b}{2} \right\rfloor, c\right) N_1\left(\left\lfloor \frac{a}{2} \right\rfloor, \left\lceil \frac{b}{2} \right\rceil, c\right), \\ N_7(a, a, 2c) &= N_1\left(\left\lceil \frac{a}{2} \right\rceil, \left\lfloor \frac{a}{2} \right\rfloor, c\right), \\ N_9(2a, 2a, 2a) &= N_{10}(2a, 2a, 2a)^2. \end{aligned}$$

Further we have the system of 4 relations involving all classes

$$\begin{aligned} N_1(a,a,2b) &= N_2(a,a,2b)N_6(a,a,2b), \\ N_5(a,a,2b) &= N_7(a,a,2b)^2, \\ N_3(2a,2a,2a) &= N_4(2a,2a,2a)N_8(2a,2a,2a), \\ N_9(2a,2a,2a) &= N_10(2a,2a,2a)^2, \end{aligned}$$

which involves the enumeration formulas in the general case (up to substituting $a \mapsto \frac{a}{2}$ or $b \mapsto \frac{b}{2}$) for all classes except for the first and the fifth. The proof is non-bijective and uses *perfect matchings*, see [5].

In the following sections we will have a closer look at some of the symmetry classes.

2.4 Enumeration of symmetric plane partitions

This is presented by Benjamin.

2.5 Enumeration of cyclically symmetric plane partitions

We present in this section the unweighted enumeration of cyclically symmetric plane partitions. This was first proven by Andrews [1]. The proof for the weighted enumeration was found by Mills, Robbins and Rumsey [14]; their proof is also presented with more details in [4, Chapter 4].

Instead of proving the enumeration formula for cyclically symmetric plane partitions (CSPPs) we consider a generalisation of them. Remember that CSPPs are in bijection to cyclically symmetric[3] lozenge tilings of a regular hexagon. Instead we regard cyclically symmetric lozenge tilings of a cored hexagon, this is a hexagon with side lengths $n, n+x, n, n+x, n, n+x$ where an equilateral triangular with side length x is removed from the center, such that its corners point towards the shorter edges of the hexagon, see Figure 2.5.

As for plane partitions, we translate cyclically symmetric lozenge tilings of a cored hexagon into families of non-intersecting lattice paths but also to certain arrays of integers. But first a few definitions.

Definition 2.5.1. *A (shifted) plane partition is called* column strict*, if the columns are strictly decreasing. A* shifted plane partition *is an array of positive integers of the form*

$$\begin{array}{ccccccccc} \pi_{1,1} & \pi_{1,2} & \cdots & & \cdots & & \cdots & \cdots & \pi_{1,\lambda_1} \\ & \pi_{2,2} & \cdots & & \cdots & & \cdots & \pi_{2,\lambda_2} & \\ & & \ddots & & \cdots & & ⋰ & & \\ & & & \pi_{l,l} & \cdots & \pi_{l\lambda_k} & & & \end{array}$$

such that $(\lambda_1, \ldots, \lambda_l)$ is a partition and the columns and rows are weakly decreasing. A shifted plane partition has class k, iff $\pi_{i,i} = \lambda_i + k - i + 1$ for $1 \leq i \leq l$, i.e., the first entry in every row exceeds the number of entries in its row by exactly k.

[3] This means invariant under rotation by 120°.

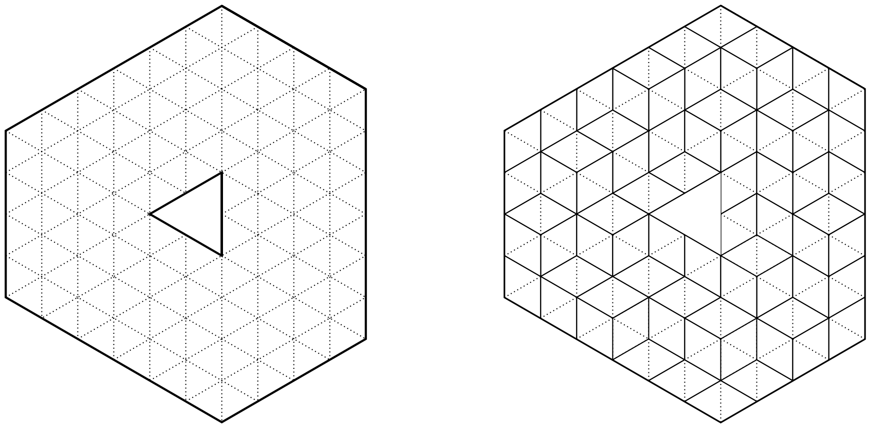
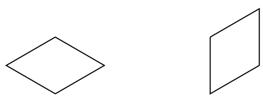


Figure 2.5: A cored hexagon with side lengths $4, 6, 4, 6, 4, 6$ (left) and a cyclically symmetric lozenge tiling of it (right).

For an example of a column strict shifted plane partition, or short CSSPP, of class 2, see Figure 2.6. If we subtract 1 of each entry of an CSSPP of class x and delete the 0 entries we obtain, what is called a $(2-x)$-*descending plane partitions* ($(2-x)$-DPPs). The 0-DPPs are also called descending plane partition (DPP).

In order to construct a family of non-intersecting lattice paths corresponding to an cyclically symmetric lozenge tiling, we regard a fundamental region of the lozenge tiling: we restrict ourselves to the area between the top corner and the two left-most corners of the hexagon and the bottom corner of the core. This is the region within the thick lines in Figure 2.6 (left). Because of the cyclically symmetry the lozenge tiling within this area determines the cyclically symmetric lozenge tiling of the whole cored hexagon uniquely. On the left and the bottom border of this region we may find lozenges which are only partially within this region; let us call them *partial*. The bottom border of the fundamental region is, when rotated by 120 degrees, contained in the left border. This implies that the number of partial lozenges on the left border is the same as those on the bottom border. We draw paths from the partial lozenges on the left border to those on the bottom border by connecting the opposite sides of the lozenges of the following form.



For an example see Figure 2.6 (left). Denote by I the set of positions of the partial lozenges along the bottom border of the fundamental region, where we number the possible positions from left to right with $1, \ldots, n$. In Figure 2.6 we have $I = \{2, 4\}$. The starting and end point of the corresponding family of non-intersecting lattice paths are $s_i = (1, x+i)$ and $e_i = (i, 1)$ for $i \in I$. The family of lattice paths induce a column strict shifted plane partition (CSSPP) of class x. The k-th row from top of the plane partition is equal to the y coordinate of the starting point of the k-th lattice path followed by the y coordinates of the horizontal steps, see Figure 2.6. It is easy to see that the CSSPP has at most n entries in the first row.

The following variation of the Lindström-Gessel-Viennot Theorem allows us to express the enumeration of cyclically symmetric lozenge tilings of a cored hexagon, or equivalently

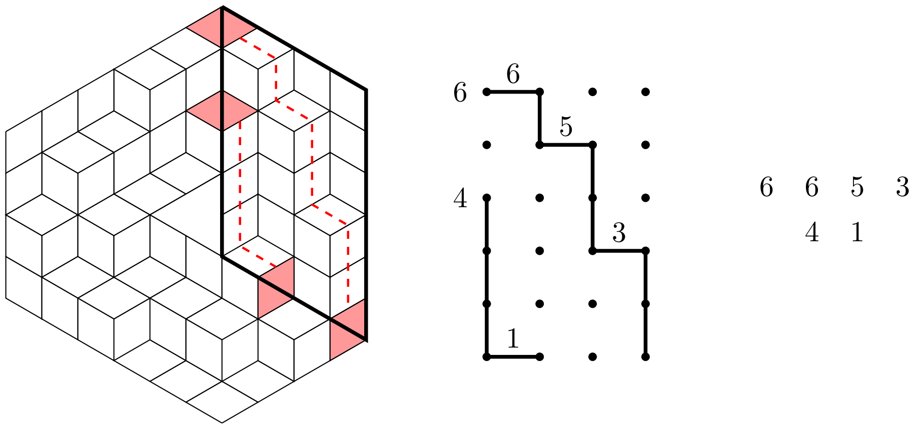


Figure 2.6: A cyclically symmetric lozenge tiling of a cored hexagon with side lengths 4, 6, 4, 6, 4, 6 (right), its representation as non-intersecting lattice paths (middle) and the corresponding CSSPP of class 2.

of CSSPPs of class x.

Lemma 2.5.2. *For starting points $s_1 \dots, s_n$ and end points $e_1, \dots, e_n$ denote by $P(i,j)$ the (weighted) count of lattice paths from s_i to e_j. The (weighted) enumeration of non-intersecting lattice paths from s_{i_1}, s_{i_k} to $e_{i_1}, \dots, e_{i_k}$ where we consider all subsets $\{i_1, \dots, i_k\}$ of $[n] = \{1, \dots, n\}$, is given by*

$$\det_{1\le i,j\le n}(\delta_{i,j} + P(i,j)). \tag{2.8}$$

Proof. By using the Leibniz formula and partially expanding the product, we obtain

$$\det_{1\le i,j\le n}(\delta_{i,j} + P(i,j)) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}\sigma \prod_{i=1}^{n} (\delta_{i,\sigma_i} + P(i,\sigma(i))) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}\sigma \sum_{I_\sigma} \prod_{i\in[n]\setminus I_\sigma} P(i,\sigma(i)),$$

where I_σ is the subset of the fix points of σ for which we have chosen $\delta_{i,\sigma(i)}$ in the product. By exchanging the sums, the above becomes

$$\sum_{I\subseteq[n]} \sum_{\sigma\in\mathfrak{S}_{[n]\setminus I}} \operatorname{sgn}(\sigma) \prod_{i\in[n]\setminus I} P(i,\sigma(i)) = \sum_{I\subseteq[n]} \det_{i,j\in\{1,\dots,n\}\setminus I} (P(i,j)),$$

which is by Theorem 2.2.1 equal to our claim. □

By the above Lemma the enumeration of CSSPPs of class x with at most n entries in the first row is equal to the determinant

$$\det_{1\le i,j\le n}\left(\delta_{i,j} + \binom{x+i+j-2}{j-1}\right). \tag{2.9}$$

Having a look at the literature, it turns out that the "easy" methods for evaluating determinants are not applicable to the above determinant directly. However we can first transform this determinant by a sequence of matrix manipulations using a family of lower triangular matrices and then use the Desnanot-Jacobi Theorem, which we will state later, to evaluate it. Let L be the family of matrices defined by

$$L_{n,x}(a,b) := \left(\binom{i+x}{j+x} a^{i+x} b^{j+x}\right)_{0\le i,j\le n-1}.$$

These matrices satisfy the following three properties

$$L_{n,0}(a_1,b_1)L_{n,x}(a_2,b_2)^T = \left(\sum_{l\geq 0}\binom{i}{l}\binom{j+x}{l+x}a_1^i a_2^{j+x} b_1^l b_2^{l+x}\right)_{0\leq i,j\leq n-1}, \tag{2.10}$$

$$\det\left(L_{n,x}(a,b)\right) = (ab)^{\binom{n}{2}+nx}, \tag{2.11}$$

$$L_{n,x}(a_1,b_1)L_{n,x}(a_2,b_2) = L_{n,x}\left(a_1(1+a_2b_1), \frac{a_2b_1b_2}{1+a_2b_1}\right), \tag{2.12}$$

where A^T denotes the transpose of a matrix A. Indeed, (2.10) is the definition of the matrix multiplication applied for the matrices L. Since the matrix L is an upper triangular matrix, its determinant is just the product of the entries on its diagonal, which implies (2.11). The (i,j)-th component of the left hand side of (2.12) is by matrix multiplication equal to

$$\sum_{l\geq 0}\binom{i+x}{l+x}\binom{l+x}{j+x}a_1^{i+x}(a_2b_1)^{l+x}b_2^{j+x} = a_1^{i+x}(a_2b_1b_2)^{j+x}\binom{i+x}{j+x}\sum_{l\geq 0}\binom{i-j}{l-j}(a_2b_1)^{l-j},$$

where we used the identity $\binom{i+x}{l+x}\binom{l+x}{j+x} = \binom{i+x}{j+x}\binom{i-j}{l-j}$. The sum is now by the binomial Theorem equal to $(1+a_2b_1)^(i-j)$ and we obtain the right hand side of (2.12).

By the Chu-Vandermonde identity, we have

$$\sum_{l\geq 0}\binom{i}{l}\binom{j+x}{l+x} = \binom{i+j+x}{j}.$$

Hence we can rewrite our determinant of interest by using (2.10) as

$$\det_{1\leq i,j\leq n-1}\left(\delta_{i,j}\binom{i+j+x}{j}\right) = \det\left(\mathrm{Id}_n + L_{n,0}(1,1)L_{n,x}(1,1)\right).$$

Denote by ζ_k the primitive k-th root of unity $e^{\frac{2\pi i}{k}}$. We multiply the matrix in the above determinant from left by $L_{n,0}(1,-\zeta_6^{-1})$ and from right by $L_{n,x}(1,-\zeta_6)^T$. By using (2.11) and (2.12) we obtain

$$(-\zeta_6)^{-nx}\det\left(L_{n,0}(1,-\zeta_6^{-1})L_{n,x}(1,-\zeta_6)^T + L_{n,0}(\zeta_6,\zeta_6)L_{n,x}(\zeta_6^{-1},\zeta_6^{-1})^T\right).$$

Using (2.10) and then the Chu-Vandermonde identity, the (i,j)-th entry of this matrix is equal to

$$\sum_{l\geq 0}\binom{i}{l}\binom{j+x}{l+x}\left((-\zeta_6)^x + \zeta_6^{i-j-2x}\right) = (-\zeta_6)^x\binom{i+j+x}{j}\left(1+\zeta_6^{i-j}\right).$$

Combining the above, we showed that the enumeration of CSSPP of class x with at most n entries in the first row is equal to

$$\det_{0\leq i,j\leq n-1}\left(\binom{i+j+x}{j}\left(1+\zeta_6^{i-j}\right)\right). \tag{2.13}$$

A very practical way to evaluate the determinant explicitly is the Desnanot-Jacobi Theorem, since it allows us to use induction.

Theorem 2.5.3 (Desnanot-Jacobi). *Let n be a positive integer, A an $n \times n$ matrix and denote by $A^{i_1,\dots,i_k}_{j_1,\dots,j_k}$ the matrix obtained by deleting the $i_1,\dots,i_k$-th rows and $j_1,\dots,j_k$-th columns of A. Then*

$$\det(A)\det\left(A^{1,n}_{1,n}\right) = \det\left(A^1_1\right)\det\left(A^n_n\right) - \det\left(A^1_n\right)\det\left(A^n_1\right). \tag{2.14}$$

Proof. We define A^* as the matrix

$$A^* = \begin{pmatrix} \det\left(A^1_1\right) & 0 & 0 & \cdots & 0 & (-1)^{n+1}\det\left(A^n_1\right) \\ -\det\left(A^1_2\right) & 1 & 0 & \cdots & 0 & (-1)^{n+2}\det\left(A^n_2\right) \\ \det\left(A^1_3\right) & 0 & 1 & \cdots & 0 & (-1)^{n+3}\det\left(A^n_3\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^n\det\left(A^1_{n-1}\right) & 0 & 0 & \cdots & 1 & -\det\left(A^n_{n-1}\right) \\ (-1)^{n+1}\det\left(A^1_n\right) & 0 & 0 & \cdots & 0 & \det\left(A^n_n\right) \end{pmatrix}$$

, which has determinant $\det(A^*) = \det\left(A^1_1\right)\det\left(A^n_n\right) - \det\left(A^n_1\right)\det\left(A^1_n\right)$. By Laplace formula the sum $\sum_{j=1}^n (-1)^{j+1} a_{i,j} \det\left(A^1_j\right)$ is equal to the determinant of the matrix where first row is replaced by the i-th row. Hence it is equal to 0 unless $i = 1$. Analogously we have $\sum_{j=1}^n (-1)^{j+n} a_{i,j} \det\left(A^n_j\right) = \delta_{i,n}$. Therefore we have

$$A \cdot A^* = \begin{pmatrix} \det(A) & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & 0 \\ 0 & a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & 0 \\ 0 & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} & \det(A) \end{pmatrix}.$$

The determinant of $A \cdot A^*$ is on the one side

$$\det(A)\left(\det\left(A^1_1\right)\det\left(A^n_n\right) - \det\left(A^n_1\right)\det\left(A^1_n\right)\right).$$

On the other side, by using Laplace formula we obtain for the determinant

$$\det(A)^2 \det\left(A^{1,n}_{1,n}\right).$$

Since $\det(A)$ is a non-zero polynomial in the entries of A we can divide both sides by it and obtain the assertion. $\square$

In order to apply the Desnanot-Jacobi Theorem we have to regard a slightly more general matrix. Set

$$D_{n,k}(x) = \left(\binom{i+j+x}{j}\left(1 + \zeta_6^{i-j+k}\right)\right)_{0 \le i,j \le n-1}.$$

Deleting the first and/or last row and the first and/or last column of the matrix $D_{n,k}(x)$, and taking determinants, gives the following expressions.

$$\begin{aligned} \det\left(D_{n,k}(x)^1_1\right) &= \det\left(D_{n-1,k}(x+2)\right)\binom{x+n}{n-1}, \\ \det\left(D_{n,k}(x)^n_n\right) &= \det\left(D_{n-1,k}(x)\right), \\ \det\left(D_{n,k}(x)^{1,n}_{1,n}\right) &= \det\left(D_{n-2,k}(x+2)\right)\binom{x+n-1}{n-2}, \\ \det\left(D_{n,k}(x)^1_n\right) &= \det\left(D_{n-1,k-1}(x+1)\right), \\ \det\left(D_{n,k}(x)^n_1\right) &= \det\left(D_{n-1,k+1}(x+1)\right)\binom{x+n-1}{n-1}. \end{aligned}$$

We explain the first of the above equations; the other follow analogously. By definition we have

$$\det\left(D_{n,k}(x)_1^1\right) = \det_{1\le i,j\le n-1}\left(\binom{i+j+x}{j}\left(1+\zeta_6^{i-j+k}\right)\right) = \det_{0\le i,j\le n-2}\left(\binom{i+j+x+2}{j+1}\left(1+\zeta_6^{i-j+k}\right)\right).$$

Since the i-th row is divisible by $(x+i+2)$ we factor it out for all $0 \le i \le n-2$. Further we factor out $(j+1)^{-1}$ for $0 \le j \le n-2$ and obtain

$$\prod_{i=0}^{n-2}\frac{x+i+2}{i+1}\det_{0\le i,j\le 2}\left(\binom{i+j+x+2}{j}\left(1+\zeta_6^{i-j+k}\right)\right) = \binom{n+x}{n-1}\det\left(D_{n-1,k}(x+2)_1^1\right).$$

By using a computer algebra program and computing the determinant (2.13) for small values of n one can guess the following formula for $k \cong 1$ modulo 6

$$\begin{aligned}\det\left(D_{n,6k+1}(x)\right) =&(\zeta_6)^{-2n}2^{\lfloor\frac{n+1}{2}\rfloor\lfloor\frac{n+2}{2}\rfloor-\lfloor\frac{n}{2}\rfloor}\prod_{i=1}^{\lfloor\frac{n+1}{2}\rfloor}\frac{(i-1)!}{(n-i)!}\\ &\times\prod_{i\ge0}\left(\frac{x}{2}+3i+2\right)_{\lfloor\frac{n-4i-1}{2}\rfloor}\left(\frac{x}{2}+3i+2\right)_{\lfloor\frac{n-4i-2}{2}\rfloor}\\ &\times\prod_{i\ge0}\left(\frac{x}{2}+2\left\lfloor\frac{n}{2}\right\rfloor-i+\frac{1}{2}\right)_{\lfloor\frac{n-4i}{2}\rfloor}\left(\frac{x}{2}+2\left\lfloor\frac{n-1}{2}\right\rfloor-i+\frac{3}{2}\right)_{\lfloor\frac{n-4i-3}{2}\rfloor},\end{aligned} \tag{2.15}$$

where $(x)_j := x(x+1)\cdots(x+j-1)$. The other cases have similar formulas, compare with [**?**, Theorem 4.2]. These formula can now be proven by induction and the Desnanot-Jacobi Theorem. This is in principal not difficult, but the calculations can be very time consuming.

In particular, by setting $x = 0$ in (2.15), we obtain the following Theorem.

Theorem 2.5.4. *The number of cyclically symmetric plane partitions inside an (n,n,n)-box is equal to*

$$\prod_{i=1}^{n}\frac{3i-1}{3i-2}\prod_{j=i}^{n}\frac{n+i+j-1}{2i+j-1}. \tag{2.16}$$

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