MASTERARBEIT

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"Height functions and the ring of adeles over a global field"

Verfasser

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Introduction

The intention of this thesis is to prove a generalization of a statement of *Roger Godement* presented in his article [3, p. 202] in the *Séminaire Nicolas Bourbaki* but given without proof. This statement originally lists six properties of *height functions* on the $\mathbb{A}_{\mathbb{Q}}$-module $\mathbb{A}_{\mathbb{Q}}^n$ where $\mathbb{A}_{\mathbb{Q}}$ is the *adele ring* over $\mathbb{Q}$ and n a natural number. It can be generalized for height functions on the $\mathbb{A}_K$-module $\mathbb{A}_K^n$ where K is a *global field*, i. e. an algebraic number field or an algebraic field extension of the rational function field in one variable with finite field of constants, and $\mathbb{A}_K$ is the adele ring over K. Therefore the aim of this work is to introduce the needed concepts and prove this general result in an exact and succinct way. The thesis is written in a way that basic knowledge in algebra and algebraic number theory should be sufficient for an understanding.

One motivation for looking at height functions is diophantine geometry where they are a fundamental tool. Their intention is to measure the „size“ of points, in our case, elements of $\mathbb{A}_K^n$ for a global field K. In order to get a useful notion of a size, on the one hand, height functions should reflect the arithmetic and geometric nature of points. To illustrate the first we want to consider the height function $H_{\mathbb{Q}}$ on $\mathbb{A}_{\mathbb{Q}}$ which is given on the rational numbers $\frac{a}{b}$ for $a, b \in \mathbb{Z}$ and $b \neq 0$ by

$$H_{\mathbb{Q}}(\frac{a}{b}) = \max\{|a|, |b|\},$$

where $|a| = \operatorname{sign}(a) \cdot a$ is the normal absolute value on $\mathbb{Q}$. Then the rational number $\frac{1}{2}$ has height $H_{\mathbb{Q}}(\frac{1}{2}) = 2$. However the number $\frac{1001}{2000}$, which is „more complicated“ than $\frac{1}{2}$, has height 2000. On the other hand, a function measuring sizes is motivated by the following finiteness property: For every real number $B > 0$ there are up to multiplication by a scalar only finitely many points with size smaller than B. For height functions on $\mathbb{A}_K^n$, the above property is fulfilled for points in $K^n \subseteq \mathbb{A}_K^n$.

Another application for height functions is reduction theory for reductive algebraic groups over $\mathbb{Q}$ or, more generally, over a global field. In [3], given the base field $\mathbb{Q}$, Godement uses height functions to construct fundamental domains and to deal with reduction theory in the adelic setting.

The first chapter gives an introduction into the theory of *valuations* beginning with the basic definitions and properties of *archimedean-* and *non-archimedean absolute values* and *valuations*. Since we need to apply the theory of valuations in later chapters solely in the case of global fields we focus in the given examples on valuations and absolute values over these fields. An important result in this section is Theorem 1.1.5 - often called *Ostrowski's Theorem* - which describes up to equivalence all absolute values on $\mathbb{Q}$ or $\mathbb{F}_q(x)$ respectively.

In the next section we introduce the notion of a *complete field* with respect to an absolute value. Theorem 1.2.2 states that every complete field with respect to an archimedean absolute value is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.

With this result at hand the rest of this section deals with the structure of complete fields with respect to a non-archimedean absolute value. However, Theorem 1.2.9, which states that all *norms* on a vector space over a complete field are equivalent, is general.

The third section is about extensions of absolute values on finite field extensions. These extensions are fully characterized by Theorem 1.3.4. In quite a few results we require that the considered field extension is separable. In some situations we can avoid this requirement by using Theorem 1.3.3. An essential concept introduced in this section are *places* of global fields, which are equivalence classes of absolute values. With the notion of *normed absolute values* $| \ |_{\mathfrak{p}}$ corresponding to a place $\mathfrak{p}$ of a global field K we obtain Theorem 1.3.12, which states

$$\prod_{\mathfrak{p}} |x|_{\mathfrak{p}} = 1,$$

where $x \in K$ and the product is over all places $\mathfrak{p}$ of K.

The last section is about *local fields.* These are finite field extensions of the completions of global fields with respect to a non-archimedean absolute value.

In the second chapter we introduce the *adele ring* $\mathbb{A}_K$ and the *idele group* $\mathbb{I}_K$ over a global field K. Both of them are defined using the *restricted direct product* of locally compact topological groups respectively rings which is introduced in the first section. For a global field K we can embed K in a natural way into its adele ring $\mathbb{A}_K$. In Theorem 2.3.1 we show that the image $\iota(K)$ of K under this embedding ι is a discrete and cocompact subring of $\mathbb{A}_K$, i. e., the quotient space $\mathbb{A}_K/\iota(K)$ is compact.

Finally we work towards Theorem 2.3.5 the so-called *strong approximation* of the adele ring over a global field. It states that every global field K embeds „almost" dense into its adele ring $\mathbb{A}_K$.

The final chapter concerns height functions. In the first section we deal with height functions on the projective space $\mathbb{P}^n(\overline{K})$ for a global field K and an integer n. To be more precise we only introduce the *absolute multiplicative / logarithmic height function* on $\mathbb{P}^n(\overline{K})$. With methods from algebraic geometry one can obtain more height functions (cf. [1], [4]), but this will not be treated in this thesis. The main result in this section is Theorem 3.1.6. It states that for all real numbers $B, D > 0$ there exist only finitely many points $P \in \mathbb{P}^n(\overline{K})$ with $[L : K] \leq D$ and $L|K$ separable for the smallest field L with $P \in \mathbb{P}^n(L)$, such that the height of P is limited by B.

In order to define height functions on $\mathbb{A}_K^n$ for a global field K in a simple way we introduce in the second section the notion of *adelic norms* on $\mathbb{A}_K^n$. These are families of norms $\| \ \|_{\mathfrak{p}}$ on the $K_{\mathfrak{p}}$-vector space $K_{\mathfrak{p}}^n$ for all places $\mathfrak{p}$ of K, where $K_{\mathfrak{p}}$ is the completion of K with respect to $\mathfrak{p}$, such that $\| \ \|_{\mathfrak{p}}$ is for almost all places the supremum norm on $K_{\mathfrak{p}}$. We obtain for every adelic norm $\mathcal{F}$ a height function $h_{\mathcal{F}} : \mathbb{A}_K^n \to \mathbb{R}$ corresponding to $\mathcal{F}$. Our main result, Theorem 3.2.7, states six properties of such height functions which are roughly speaking the following:

The first property states that all height functions on $\mathbb{A}_K^n$ are equivalent, where equivalence for height functions is defined analogously as for norms on vector spaces. Secondly, multiplying the argument of a height function by an scalar $t \in \mathbb{A}_K^*$ is equal to multiplying the value of the height function by the norm $|t|_{\mathbb{A}_K}$ of t. The third states that the image of a null sequence under a height function is a null sequence. Conversely, by the fourth property, if the image of a sequence of primitive elements under a height function is a null sequence, we can multiply every element of the sequence by a non zero scalar and obtain by this a null sequence. The fifth property is the analogue of Theorem 3.1.6, which is described above, in the adelic setting. The last property states that for every height function h and every compact subset $M \subseteq GL_n(\mathbb{A}_K^n)$ of the group of $\mathbb{A}_K$-automorphism of $\mathbb{A}_K^n$ the maps $h \circ (x \mapsto g \cdot x)$ for $g \in M$ are bounded from above and below by multiples of h with some constants, depending only on M and h.

Since the requirement for a family of norms $\| \|_{\mathfrak{p}}$ on the $K_{\mathfrak{p}}$-vector space $K_{\mathfrak{p}}^n$ to be an adelic norm on $\mathbb{A}_K^n$ is very restrictive we want to consider a generalisation of adelic norms such that their corresponding height functions also enjoy Theorem 3.2.7. We give at the end of this thesis a possible definition for a *general adelic norm* on $\mathbb{A}_K^n$ and their corresponding *general height functions*. Finally we proof Theorem 3.2.10 which is the generalisation of Theorem 2.2.5 to general height functions.

Last but not least I want to express my gratitude to my advisor Professor Joachim Schwermer for this interesting topic, his guidance and support during my studies and also to my family and friends without whom my studies would not be possible in that way.

1. Valuation theory

This chapter gives a brief introduction to the theory of valuations. It covers the basic definitions and provides all Theorems which are important for the further understanding of this thesis. The structure and notation are closely following [6]. Further [5] was taken as basis for valuations on function fields. One can also find in [2] a good introduction into the theory of valuations.

1.1. Absolute values and valuations

Definition 1.1.1 *An* **absolute value** $|\ |$ *on a field* K *is a map*

$$|\ | : K \to \mathbb{R},$$

such that for all $x, y \in K$ *holds:*

(i) $|x| \geq 0$ *and* $|x| = 0$ *iff* $x = 0$,
(ii) $|xy| = |x||y|$,
(iii) $|x + y| \leq |x| + |y|$.

Let $|\ |$ be an absolute value on a field K. We can define a metric $d : K \times K \to \mathbb{R}$ on K via

$$d(x, y) = |x - y|, \quad \textit{for all } x, y \in K.$$

Hence we obtain on every field K with an absolute value $|\ |$ a topology induced by the absolute value. The field K together with this topology is automatically a topological field, as one can prove easily. On every field K there exists a ***trivial absolute value*** $|\ |$ given by

$$|x| = \begin{cases} 1 & x \neq 0, \\ 0 & x = 0, \end{cases}$$

for $x \in K$. The trivial absolute value induces the discrete topology on K. We assume for the rest of the thesis that every absolute value is non-trivial.

Definition 1.1.2 *Two absolute values* $|\ |_1$ *and* $|\ |_2$ *on a field* K *are called* **equivalent** *if they define the same topology on* K.

Theorem 1.1.3 *Let* $|\ |_1$ *and* $|\ |_2$ *be two absolute values on a field* K. *Then they are equivalent iff there exists a real number* $s > 0$ *such that*

$$|x|_1 = |x|_2^s$$

for all $x \in K$.

Proof. See [6, p. 122]. □

Examples 1.1.4 (i) For the fields $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ we have the ordinary absolute value $|\ |_\infty$ given by

$$|x|_\infty := \sqrt{x\overline{x}},$$

for $x \in \mathbb{C}$, where $\overline{x}$ is the complex conjugate of x.

(ii) Let $p \in \mathbb{Z}$ be a prime number. Then we define the absolute value $|\ |_p$ on $\mathbb{Q}$ via

$$|x|_p := p^{-\alpha},$$

where $x = p^\alpha \frac{a}{b}$ with $\alpha, a, b \in \mathbb{Z}$, $b \neq 0$ and a, b coprime to p. As a consequence of the previous Theorem the absolute values $|\ |_p, |\ |_q, |\ |_\infty$ are pairwise not equivalent for all primes p, q with $p \neq q$.

(iii) Let $\mathbb{F}_q$ be the finite field of cardinality q. Denote by $\mathbb{F}_q(x) := \mathrm{Quot}(\mathbb{F}_q[x])$ the rational function field over $\mathbb{F}_q$. Since $\mathbb{F}_q[x]$ is a principal ideal domain we can write every element $r \in \mathbb{F}_q(x)$ as

$$r = \prod_p p^{\nu_p(r)},$$

where the product is over all irreducible polynomials $p \in \mathbb{F}_q[x]$, $\nu_p(r) \in \mathbb{Z}$ and $\nu_p(r) = 0$ for almost all p. For every irreducible polynomial $p \in \mathbb{F}_q(x)$ there is an absolute value $|\ |_p$ given by

$$|r|_p := q^{-\deg(p)\cdot\nu_p(r)},$$

for $r \in \mathbb{F}_q(x)$.

Further we obtain another absolute value on $\mathbb{F}_q(x)$ denoted by $|\ |_\infty$:

$$|\frac{f}{g}|_\infty = q^{\deg(f)-\deg(g)},$$

for $f, g \in \mathbb{F}_q[x]$ and $g \neq 0$. By using Theorem 1.1.3 one can easily show that the above described absolute values on $\mathbb{F}_q(x)$ are pairwise not equivalent.

The following Theorem is fundamental in the theory of valuations.

Theorem 1.1.5 (Ostrowski) *(i) Every absolute value on $\mathbb{Q}$ is equivalent to either $|\ |_\infty$ or $|\ |_p$ for a prime number p.*

(ii) Every absolute value on the rational function field $\mathbb{F}_q(x)$ over the finite field $\mathbb{F}_q$ is equivalent to either $|\ |_\infty$ or $|\ |_p$ for an irreducible polynomial $p \in \mathbb{F}_q[x]$.

Proof. See [6, p. 124] and [5, p. 105]. □

Definition 1.1.6 *An absolute value $|\ |$ on a field K is called* **non-archimedean** *if it satisfies $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. Otherwise the absolute value is called* **archimedean**.

By the above Theorem we see that up to equivalence the only archimedean absolute value on $\mathbb{Q}$ is $|\ |_\infty$ and every non-archimedean is equivalent to $|\ |_p$

for an appropriate prime number p. Further every absolute value on the function field $\mathbb{F}_q(x)$ over the finite field $\mathbb{F}_q$ is non-archimedean.

Definition 1.1.7 *A* **valuation** v *on a field* K *is a map* $v : K \to \mathbb{R} \cup \{\infty\}$ *with the following properties:*

(i) $v(x) = \infty$ *iff* $x = 0$*,*
(ii) $v(xy) = v(x) + v(y)$ *for all* $x, y \in K$
(iii) $v(x + y) \geq \min\{v(x), v(y)\}$ *for all* $x, y \in K$*.*

For a valuation v on a field K and a real number $q > 1$ we obtain a non-archimedean absolute value $| \ |_v$ corresponding to v by setting $|x|_v = q^{-v(x)}$ for every $x \in K$. Vice-versa we get for every non-archimedean absolute value $| \ |$ a valuation v via $v(x) = -\log(|x|)$ for all $x \in K$. If we set $q = e$ this defines a $1-1$ correspondence between the set of non-archimedean absolute values on K and the set of valuations on K.
Let $q > 1$ be fixed. We call two valuations v, w ***equivalent*** if the corresponding absolute values $|x|_v = q^{-v(x)}$ and $|x|_w = q^{-w(x)}$ are equivalent. The definition of equivalence of valuations is independent of the choice of the real number q:
Let v, w be equivalent valuations with respect to q and $q' > 1$ another real number. By Theorem 1.1.3 there exists a real number $s > 0$ such that $|x|_v = |x|_w^s$. Denote by $|x|'_v = q'^{-v(x)}$ and $|x|'_w = q'^{-w(x)}$ the absolute values corresponding to v, w with respect to q'. Then we obtain for every $x \in K$:

$$|x|'_v = q'^{-v(x)} = q^{-v(x)\frac{\log(q')}{\log(q)}} = |x|_v^{\frac{\log(q')}{\log(q)}} = |x|_w^{s\frac{\log(q')}{\log(q)}} =$$
$$= q^{-w(x)s\frac{\log(q')}{\log(q)}} = q'^{-sw(x)} = |x|'^{\,s}_w.$$

Hence the absolute values $| \ |'_v, | \ |'_w$ are equivalent iff the absolute values $| \ |_v, | \ |_w$ are equivalent.

As a consequence of Theorem 1.1.3 the valuations v and w are equivalent iff there exists a real number $s > 0$ with $v = s \cdot w$.

Theorem 1.1.8 *Let* v *be a valuation on the field* K*. Then the ring*

$$\mathcal{O}_v = \{x \in K : v(x) \geq 0\} = \{x \in K : |x|_v \leq 1\},$$

is an integral domain with quotient field K*. Its unit group is given by*

$$\mathcal{O}_v^* = \{x \in K : v(x) = 0\} = \{x \in K : |x|_v = 1\},$$

and the ideal

$$\mathfrak{p}_v = \{x \in K : v(x) > 0\} = \{x \in K : |x|_v < 1\},$$

is the only maximal ideal in $\mathcal{O}_v$*.*

Proof. Since $\mathcal{O}_v$ is a subring of K it is an integral domain. The quotient field of $\mathcal{O}_v$ is by definition a subfield of K. On the other hand for every $x \in K$ we have either $v(x) \geq 0$ which is equivalent to $x \in \mathcal{O}_v$ or we have $v(x) < 0$

which is equivalent to $\frac{1}{x} \in \mathcal{O}_v$. In the second case x is an element of the quotient field of $\mathcal{O}_v$ which implies $K \subseteq \mathrm{Quot}(\mathcal{O}_v)$. The second claim is an obvious consequence of the definition of $\mathcal{O}_v$.

The ideal $\mathfrak{p}_v$ is a maximal ideal of $\mathcal{O}_v$ since it is an ideal and $\mathcal{O}_v \setminus \mathfrak{p}_v = \mathcal{O}_v^*$. Assume there is a maximal ideal $\mathfrak{m} \neq \mathfrak{p}_v$ of $\mathcal{O}_v$. Then there exists an element $x \in \mathcal{O}_v \setminus \mathfrak{m}$ with $v(x) > 0$. Let $\mathfrak{m}'$ be the ideal generated by $\mathfrak{m}$ and x. Since $\mathfrak{m}$ is maximal we must have $\mathfrak{m}' = \mathcal{O}_v$. Hence there exist elements $y \in \mathfrak{m}$ and $\lambda \in \mathcal{O}_v$ with $y + \lambda x \in \mathcal{O}_v^*$ which is equivalent to $v(y + \lambda x) = 0$. However we have

$$v(y + \lambda x) \geq \min\{v(y), v(\lambda x)\} \geq \min\{v(y), v(\lambda) + v(x)\} > 0. \qquad \square$$

The ring $\mathcal{O}_v$ is called ***valuation ring*** and the field $\mathcal{O}_v/\mathfrak{p}_v$ is its ***residue field***. It is obvious to see that the ring $\mathcal{O}_v$, the group $\mathcal{O}_v^*$ and the prime ideal $\mathfrak{p}_v$ are invariant under equivalence of valuations.

Definition 1.1.9 *A valuation v on a field K is called* **discrete**, *if $v(K^*) = s\mathbb{Z}$ for an appropriate real number $s > 0$. A discrete valuation is called* **normed** *if $s = 1$.*

Let v be a discrete valuation on a field K and π an element of the valuation ring $\mathcal{O}_v$ with $v(\pi) = s$. Then every element $x \in K^*$ can be written in the form $x = u\pi^m$ for a unit $u \in \mathcal{O}_v^*$ and $m \in \mathbb{Z}$:
Since v is discrete we have $v(x) = sm$ with $m \in \mathbb{Z}$. This implies $v(x\pi^{-m}) = 0$ which is equivalent to $x\pi^{-m} \in \mathcal{O}_v^*$.
The element π is a prime element of $\mathcal{O}_v$ and every prime element of $\mathcal{O}_v$ is by the above consideration of the form $u\pi$ for $u \in \mathcal{O}_v^*$.

Examples 1.1.10 (i) Let p be a prime number. The valuation v_p on $\mathbb{Q}$ given by $v_p(p^n \frac{a}{b}) = n$ for $a, b, n \in \mathbb{Z}$ and a, b coprime to p is normed. The absolute value $|\ |_p$ can be written as

$$|x|_p = p^{-v_p(x)},$$

for all $x \in \mathbb{Q}$. Since every valuation corresponds to a non-archimedean absolute value, Theorem 1.1.5 implies that the valuations v_p for prime numbers p are up to equivalence all valuations on $\mathbb{Q}$.

(ii) Let $\mathbb{F}_q(x)$ be the rational function field over the finite field $\mathbb{F}_q$. For every irreducible polynomial $p \in \mathbb{F}_q[x]$ there is a valuation v_p given by

$$v_p(r) = \nu_p(r)$$

for every $r \in \mathbb{F}_q(x)$ where $r = \prod_p p^{\nu_p(r)}$ is the unique factorisation of r into irreducible polynomials $p \in \mathbb{F}_q[x]$. We can express the absolute value $|\ |_p$ through

$$|r|_p = q^{-\deg(p) v_p(r)}$$

for all $r \in \mathbb{F}_q(x)$.
We define $\deg(r) := \deg(f) - \deg(g)$ for $r = \frac{f}{g} \in \mathbb{F}_q(x)$ with $f, g \in \mathbb{F}_q[x]$ and $g \neq 0$. Let v_∞ be the valuation on $\mathbb{F}_q(x)$ given by

$$v_\infty(r) = -\deg(r)$$

for all $r \in \mathbb{F}_q(x)$. By Theorem 1.1.5 the valuations v_p for irreducible polynomials $p \in F_q[x]$ and v_∞ are up to equivalence all valuations on $\mathbb{F}_q(x)$.

Theorem 1.1.11 *Let $\mathcal{O}_v$ be the valuation ring of a discrete valuation v on a field K. Then $\mathcal{O}_v$ is a principal ideal domain.*

Proof. Let $I \neq (0)$ be an ideal of $\mathcal{O}_v$ and $\pi \in \mathcal{O}_v$ be a prime element. Every element of $\mathcal{O}_v \setminus \{0\}$ can be written as $u\pi^n$ for an integer n and a unit $u \in \mathcal{O}_v^*$. Let m be the minimal integer such that $(\pi^m) \subseteq I$. For every element $x = u\pi^n \in I$ holds $n \geq m$ as a consequence of the minimality of m. This implies $I = (\pi^m)$. □

1.2. Complete fields

Definition 1.2.1 *Let $|\ |$ be an absolute value on a field K. A sequence $(x_n)_{n\in\mathbb{N}}$ in K is called a* **Cauchy-sequence** *with respect to $|\ |$ if there exists for every $\epsilon > 0$ an integer N, such that $|x_n - x_m| < \epsilon$ for all $m, n \geq N$. A sequence (x_n) has a* **limit** *$x \in K$ iff there exists for all $\epsilon > 0$ an integer N, such that $|x_n - x| < \epsilon$ for all $n \geq N$. If a sequence has a limit, the limit is obviously unique.*
The field K is called **complete** *if every Cauchy-sequence in K converges to a limit in K.*

Let K be a field with an absolute value $|\ |$. Then there exists a complete field $\hat{K}$ with respect to an absolute value $|\ |'$ together with an embedding $\iota : K \to \hat{K}$ of fields, such that $|\iota(x)|' = |x|$ for all $x \in K$ and the embedding of K into $\hat{K}$ is dense. The field $\hat{K}$ is up to isomorphism unique and is called the completion of K with respect to $|\ |$:

We construct the completion $\hat{K}$ by „adding“ the needed limits to K. Let R be the ring of all Cauchy sequences (x_n) in K with respect to $|\ |$. The addition and multiplication on R are defined component-wise. Denote by $\mathfrak{m}$ the maximal ideal of all null sequences in R

$$\mathfrak{m} = \{(y_n) \in R : \lim_{n\to\infty} y_n = 0\}.$$

We set $\hat{K} := R/\mathfrak{m}$. The field K embeds into $\hat{K}$ via $\iota : x \mapsto (x, x, x, \cdots) + \mathfrak{m}$. For a Cauchy sequence (x_n) in K the sequence $(|x_n|)$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete with respect to $|\ |_\infty$ the limit $\lim_{n\to\infty} |x_n|$ exists. For $(y_n) \in \mathfrak{m}$ we obtain

$$\lim_{n\to\infty} |x_n| \leq \lim_{n\to\infty} (|x_n - y_n| + |y_n|) = \lim_{n\to\infty} |x_n - y_n| \leq$$
$$\leq \lim_{n\to\infty} (|x_n| + |y_n|) = \lim_{n\to\infty} |x_n|.$$

Hence $|(x_n) + \mathfrak{m}|' := \lim_{n\to\infty} |x_n|$ is a well-defined extension of the absolute value onto $\hat{K}$. Since $\hat{K}$ is constructed in such a way that $(x_n) + \mathfrak{m}$ is the limit of the Cauchy sequence $(\iota(x_n))$ we obtain for any extension $|\ |''$ of $|\ |$ onto $\hat{K}$:

$$\lim_{n\to\infty} (|x_n| - |(x_n) + \mathfrak{m}|'') = 0.$$

This implies that the extensions $|\ |'', |\ |'$ are equal and hence the extension of $|\ |$ onto $\hat{K}$ is unique.

For every $(x_n) + \mathfrak{m} \in \hat{K}$ and $\epsilon > 0$ there exists an integer N with $|\iota(x_N) - (x_n) + \mathfrak{m}|' < \epsilon$. Hence K is a dense subfield of $\hat{K}$.

Let (a_n) be a Cauchy sequence in $\hat{K}$. Since K embeds dense into $\hat{K}$ there exists for every $n \in \mathbb{N}$ an element $x_n \in K$ such that $|(x_n + \mathfrak{m}) - a_n|' < \frac{1}{n}$. The sequence (x_n) is then a Cauchy sequence in K and it follows that $(x_n) + \mathfrak{m}$ is the limit of (a_n). This proves the completeness of $\hat{K}$.

The uniqueness of the completion $\hat{K}$ up to isomorphism is straight forward.

We denote in the following the completion of the absolute value $|\ |$ onto $\hat{K}$ also with $|\ |$. Let $|\ |'$ be an absolute value on K which is equivalent to $|\ |$. An element $x \in K$ is limit of a sequence (x_n) with respect to $|\ |$ iff it is limit with respect to $|\ |'$. Furthermore a sequence (x_n) in K is a Cauchy sequence with respect to $|\ |$ iff it it is a Cauchy sequence with respect to $|\ |'$. Hence the completion $\hat{K}$ of K is independent under equivalence of absolute values.

Theorem 1.2.2 (Ostrowski) *Let K be a complete field with respect to an archimedean absolute value $|\ |$. Then there exists an isomorphism of fields σ from K to $\mathbb{R}$ or $\mathbb{C}$, such that*

$$|x| = |\sigma(x)|_\infty^s,$$

for a real number $s > 0$.

Proof. See [6, p. 130]. □

The above Theorem states that the fields $\mathbb{R}$ and $\mathbb{C}$ are up to isomorphisms the only fields which are complete with respect to an archimedean absolute value. Therefore the rest of the section deals mainly with fields which are complete with respect to a non-archimedean absolute value.

Let v be a valuation on a field K and let $|x|_v = q^{-v(x)}$ be an absolute value corresponding to v which is automatically non-archimedean. Let $\hat{K}$ be the completion of K with respect to v, i. e., the completion of K with respect to the absolute value $|\ |_v$. Since the extension of $|\ |_v$ onto $\hat{K}$ is also non-archimedean there exists a unique extension of v onto $\hat{K}$ which is also be denoted by v. It is given by

$$v(x) = \lim_{n\to\infty} v(x_n),$$

for $x = (x_n) + \mathfrak{m} \in \hat{K}$ with $x_n \in K$. This limit exists in $\mathbb{R} \cup \{\infty\}$ since

$$\lim_{n\to\infty} v(x_n) = \lim_{n\to\infty} -\frac{\log(|x_n|_v)}{\log(q)} = -\frac{\log(\lim_{n\to\infty} |x_n|_v)}{\log(q)} = \frac{-\log(|x|_v)}{\log(q)}.$$

If v is a discrete valuation on K then the extension onto $\hat{K}$ is also a discrete valuation.

Theorem 1.2.3 *Let v be valuation on K and $\hat{K}$ the completion of K with respect to v. Denote by $\mathcal{O}_v, \mathfrak{p}_v$ and $\hat{\mathcal{O}}_v, \hat{\mathfrak{p}}_v$ the valuation rings and their maximal ideal in K or $\hat{K}$ respectively. Then*

$$\hat{\mathcal{O}}_v/\hat{\mathfrak{p}}_v \cong \mathcal{O}_v/\mathfrak{p}_v$$

are isomorphic as topological rings. For a discrete valuation v one obtains further

$$\hat{\mathcal{O}}_v/\hat{\mathfrak{p}}_v^m \cong \mathcal{O}_v/\mathfrak{p}_v^m, \quad m \geq 1.$$

Proof. Since we can embed the valuation ring $\mathcal{O}_v$ into $\hat{\mathcal{O}}_v$ we want to identify $\mathcal{O}_v$ with its embedding. Let $\varphi : \mathcal{O}_v \to \hat{\mathcal{O}}_v/\hat{\mathfrak{p}}_v$ be the projection onto $\hat{\mathcal{O}}_v/\hat{\mathfrak{p}}_v$. The kernel of φ is obviously $\mathfrak{p}_v$. Let $x = (x_n) + \mathfrak{m}$ be an arbitrary element of $\hat{\mathcal{O}}_v$ with $x_n \in K$ for all $n \in \mathbb{N}$. Since $v(x) = \lim_{n\to\infty} v(x_n) \geq 0$ there exists an integer N_0 such that $x_m \in \mathcal{O}_v$ for all $m \geq N_0$. Further there exists an integer $N_1 > N_0$ such that $v(x - x_m) > 0$ for $m \geq N_1$ which is equivalent to $x_m \equiv x \mod \hat{\mathfrak{p}}_v$ for $m \geq N_1$. Hence φ is a surjection and induces an isomorphism of rings

$$\varphi' : \mathcal{O}_v/\mathfrak{p}_v \to \hat{\mathcal{O}}_v/\hat{\mathfrak{p}}_v.$$

Since both spaces are equipped with the discrete topology φ' is also a homeomorphism.

Now let v be discrete, without loss of generality v is also normed. Denote by $\varphi : \mathcal{O}_v \to \hat{\mathcal{O}}_v/\hat{\mathfrak{p}}_v^m$ the projection onto $\hat{\mathcal{O}}_v/\hat{\mathfrak{p}}_v^m$. Analogous to above there exists an integer N such that $x_k \in \mathcal{O}_v$ and $v(x - x_k) > m$ for all $k \geq N$, i. e., $x \equiv x_k \mod \hat{\mathfrak{p}}_v^m$ for $k \geq N$. Hence φ is a surjection with kernel $\mathfrak{p}_v^m$ and induces an isomorphism of rings

$$\varphi' : \mathcal{O}_v/\mathfrak{p}_v^m \longrightarrow \hat{\mathcal{O}}_v/\hat{\mathfrak{p}}_v^m.$$

Since the open neighbourhoods of 0 are of the form $\mathfrak{p}_v^k/\mathfrak{p}_v^m$ or $\hat{\mathfrak{p}}_v^k/\hat{\mathfrak{p}}_v^m$ respectively for $k \leq m$ the map φ' is also a homeomorphism. □

Theorem 1.2.4 *Let K be a field with a discrete valuation v, $R \subseteq \mathcal{O}_v$ be a set of representatives of $\mathcal{O}_v/\mathfrak{p}_v$ with $0 \in R$ and π a prime element of $\mathcal{O}_v$. Then there exists for every $x \in \hat{K}^*$ a unique representation as a converging series*

$$x = \pi^m \sum_{i\geq 0} a_i \pi^i$$

with $m \in \mathbb{Z}$, $a_i \in R$ and $a_0 \neq 0$.

Proof. Since π is also a prime element of the valuation ring $\hat{o}_v$ of $\hat{K}$ and v is a discrete valuation on $\hat{K}$ every element $x \in \hat{K}$ can be written as $x = \pi^m u$ for $m \in \mathbb{Z}$ and $u \in \hat{o}_v^*$. As a consequence of $\hat{o}_v/\hat{\mathfrak{p}}_v \cong o_v/\mathfrak{p}_v$ there is a unique representative $a_0 \in R$ with $a_0 \neq 0$ and $x = \pi^m(a_0 + \pi r_1)$ with $r_1 \in \hat{o}_v$. Assume that we have already $x = \pi^m(a_0 + a_1\pi + \cdots + a_n\pi^n + \pi^{n+1}r_{n+1})$ with unique $a_0, \cdots, a_n \in R$ and $r_{n+1} \in \hat{o}_v$. Since we can write $r_{n+1} = a_{n+1} + \pi r_{n+2}$ uniquely with $a_{n+1} \in R$ and $r_{n+2} \in \hat{o}_v$ we obtain

$$x = \pi^m(a_0 + a_1\pi + \cdots + a_n\pi^n + a_{n+1}\pi^{n+1} + \pi^{n+2}r_{n+2}).$$

Hence we can find a unique series $\pi^m \sum_{n=0}^{\infty} a_n\pi^n$ with $a_n \in R$, $a_0 \neq 0$. Since $x - \pi^m \sum_{i=0}^{N} a_i\pi^i \in \mathfrak{p}_v^{N+1}$ for every integer N the infinite series is equal x. $\square$

Examples 1.2.5 (i) Let p be a prime number. We denote by $\mathbb{Q}_p$ the unique completion of $\mathbb{Q}$ with respect to the absolute value $|\ |_p$. By the above Theorem every element $x \in \mathbb{Q}_p$ can be written as

$$x = p^m \sum_{i \geq 0} a_i p^i,$$

for $m \in \mathbb{Z}$, $a_i \in \{0, 1, \cdots, p-1\}$ and $a_0 \neq 0$. On the other hand there exists for every $m \in \mathbb{Z}$ and every sequence (a_i) with $a_i \in \{0, 1, \cdots, p-1\}$ and $a_0 \neq 0$ an element $x \in \mathbb{Q}_p$ with $x = p^m \sum_{i \geq 0} a_i p^i$, namely the limit of the sequence $(p^m \sum_{i=0}^{n} a_i p^i)_{n \in \mathbb{N}}$. The valuation ring of $\mathbb{Q}_p$, denoted as $\mathbb{Z}_p$, contains all elements which can written as a formal power series in p and coefficients in $\{0, 1, \cdots, p-1\}$. It is easy to see that $\mathbb{Z}_p$ is the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$.

(ii) Let $\mathbb{F}_q(x)$ be the rational function field over the finite field $\mathbb{F}_q$. Every element r in the completion of $\mathbb{F}_q(x)$ with respect to $|\ |_\infty$ can be written as

$$r = x^m \sum_{i \geq 0} a_i \frac{1}{x^i},$$

for $m \in \mathbb{Z}$, $a_i \in \mathbb{F}_q$ and $a_0 \neq 0$. Analogous to above every formal Laurent series in $\frac{1}{x}$ with coefficients in $\mathbb{F}_q$ represents an element of the completion of $\mathbb{F}_q(x)$ with respect to $|\ |_\infty$. Therefore the completion of $\mathbb{F}_q(x)$ with respect to $|\ |_\infty$ is isomorphic to the field of formal Laurent series $\mathbb{F}_q((\frac{1}{x}))$ in $\frac{1}{x}$ with coefficients in $\mathbb{F}_q$.

Definition 1.2.6 *Let $\{R_n\}_{n \in \mathbb{N}}$ be a family of topological rings and $\varphi_n : R_{n+1} \to R_n$ a homomorphism of topological rings for $n \in \mathbb{N}$. The ring*

$$\varprojlim_n R_n := \{(x_n) \in \prod_n R_n : \varphi_n(x_{n+1}) = x_n\},$$

with component-wise addition and multiplication together with the subset topology of $\prod_n R_n$ is called the **inverse limit** *of the R_n.*

Theorem 1.2.7 *Let K be a complete field with respect to a discrete valuation v, o_v be its valuation ring and $\mathfrak{p}_v$ the maximal ideal of o_v.*

Then the map $\varphi : \mathcal{O}_v \to \varprojlim_n \mathcal{O}_v/\mathfrak{p}_v^n$ *given by*

$$\varphi : x \mapsto (x \mod \mathfrak{p}_v^n)_{n\in\mathbb{N}},$$

is an isomorphism of topological rings.

Proof. The map φ is injective since its kernel is $\bigcap_{n\in\mathbb{N}} \mathfrak{p}_v^n = (0)$. Let $R \subseteq \mathcal{O}_v$ be a set of representatives of $\mathcal{O}_v/\mathfrak{p}_v$ with $0 \in R$ and π a prime element in $\mathcal{O}_v$. As a consequence of Theorem 1.2.4 every element $x \in \mathcal{O}_v/\mathfrak{p}_v^n$ for arbitrary $n \geq 1$ is of the form

$$x = \sum_{i=1}^{n} a_i \pi^i + \mathfrak{p}_v^n.$$

Hence every element $s \in \varprojlim_n \mathcal{O}_v/\mathfrak{p}_v^n$ can be written as $s = (\sum_{i=1}^n a_i\pi^i)_{n\in\mathbb{N}}$ for a sequence $a_i \in R$. The series $\sum_{i\geq 1} a_i\pi^i$ converges in $\mathcal{O}_v$ and its image under φ is s.
Any neighbourhood of 0 in $\varprojlim_n \mathcal{O}_v/\mathfrak{p}_v^n$ is of the form

$$\left(\prod_{n\leq m} (\mathfrak{p}_v^{k_n}/\mathfrak{p}_v^n) \prod_{n>m} (\mathcal{O}_v/\mathfrak{p}_v^n) \right) \cap \varprojlim_n \mathcal{O}_v/\mathfrak{p}_v^n,$$

with $k_n \leq n$ and $m \in \mathbb{N}$. If we set $k = \max_{n\leq m} k_n$ the pre-image of the above set under φ is $\mathfrak{p}_v^k$. Since every neighbourhood of $0 \in \mathcal{O}_v$ is of the form $\mathfrak{p}_v^m$ the map φ is a homeomorphism. □

Definition 1.2.8 *Let* K *be a field with an absolute value* $|\ |$ *and* V *a finite dimensional vector space over* K*. A* **norm** *on* V *is a map* $\|\ \| : V \to \mathbb{R}$*, such that*

(i) $\|x\| \geq 0$ *for all* $x \in V$ *and* $\|x\| = 0$ *iff* $x = 0$,
(ii) $\|\lambda x\| = |\lambda|\|x\|$ *for all* $x \in V$ *and all* $\lambda \in K$,
(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in V$.

Two norms $\|\ \|_1$ *and* $\|\ \|_2$ *on* V *are called* **equivalent** *if there exist real numbers* $c_1, c_2 > 0$*, such that*

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1,$$

for all $x \in V$.

Theorem 1.2.9 *Let* K *be a field which is complete with respect to an absolute value* $|\ |$ *on* K *and* V *a finite dimensional vector space over* K*. Then all norms on* V *are equivalent.*

Proof. We prove this by induction on the dimension $\dim_K(V) = n$. For $n = 1$ the statement is trivial. Let $\{b_1, \cdots, b_n\}$ be a K-basis of V and set $\|\sum_{1\leq i\leq n} \lambda_i b_i\|_\infty = \max\{|\lambda_1|, \cdots, |\lambda_n|\}$. Then it suffices to show that every norm $\|\ \|$ on V is equivalent to $\|\ \|_\infty$. By setting $c_2 = \sum_{1\leq i\leq n} \|b_i\|$ we obtain for all $x \in V$

$$\|x\| \leq c_2\|x\|_\infty.$$

We set

$$V_i = \bigoplus_{1\le j\le n, j\ne i} K b_j.$$

There exists a real number $\delta > 0$ such that $\|x + b_i\| > \delta$ for $1 \le i \le n$ and all $x \in V_i$. Assume the converse:
Then there exists an index i and a sequence (y_n) in V_i such that $\lim_{n\to\infty} \|y_n + b_i\| = 0$. This implies

$$\|y_n + b_i - (y_m + b_i)\| = \|y_n - y_m\| \to 0, \ \textit{for } n, m \to \infty.$$

Hence the sequence (y_n) is a Cauchy sequence with respect to $\| \ \|_{|V_i}$. Since by induction every norm on V_i is equivalent (y_n) is also a Cauchy sequence witch respect to $\| \ \|_{\infty|V_i}$. Therefore the components of (y_n) with respect to the basis $\{b_1, \cdots, b_n\}$ are Cauchy sequences in K. The component-wise limit $y \in V_i$ of (y_n) is also the limit of (y_n) with respect to $\| \ \|_{\infty|V_i}$. Hence we obtain $\lim_{n\to\infty} \|y_n + b_i\| = \|y + b_i\| \ne 0$ since $y + b_i \ne 0$.

Let $x = \sum_{1\le i\le n} \lambda_i b_i \in V \setminus \{0\}$ be arbitrary and j an index such that $|\lambda_j| = \|x\|_\infty$. Then we have

$$\|\lambda_j^{-1} x\| = \| \sum_{1\le i\le n, i\ne j} \lambda_i b_i + b_j\| > \delta.$$

This inequality implies $\|x\| > \delta \|x\|_\infty$. □

1.3. Global fields and extensions of valuations

We observe in this section the behaviour of absolute values and valuations under field extensions. This allows us to develop the concept of valuations on global fields, which play an important role in number theory.

Theorem 1.3.1 *Let K be a complete field with respect to an absolute value $| \ |$ and $L|K$ an algebraic field extension. Then there is a unique extension of the absolute value $| \ |$ onto L given by*

$$|x| = \sqrt[n]{|N_{L|K}(x)|}$$

for all $x \in L$ where $n = [L : K]$. The field L is then complete with respect to the extension $| \ |$.

Proof. See [6, p. 137] □

Definition 1.3.2 *A* **global field** *K is a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(x)$ for a prime number $p \in \mathbb{Z}$. A* **place** *$\mathfrak{p}$ of a global field K is an equivalence class of absolute values on K. We call $\mathfrak{p}$* **infinite** *if the restriction of a representative $| \ | \in \mathfrak{p}$ onto $\mathbb{Q}$ or $\mathbb{F}_p(x)$ respectively is equivalent to the absolute value $| \ |_\infty$ and otherwise* **finite** *and use the notations $\mathfrak{p}|\infty$ or $\mathfrak{p} \nmid \infty$ respectively. For* $\mathrm{char}(K) = 0$ *an infinite place $\mathfrak{p}$ is called* **real** *or* **complex** *if the completion* *of K associated to $\mathfrak{p}$ is $\mathbb{R}$ or $\mathbb{C}$ respectively. The set of all places of K is denoted by V_K.*

Remark Let K be a finite field extension of the rational function field $\mathbb{F}_q(x)$ with $\mathbb{F}_q$ finite. Then we have to keep in mind that $K|\mathbb{F}_q(x)$ is in general not a separable extension - in [2] separability of $K|\mathbb{F}_q(x)$ is a required property for K to be a global field. For many of the Theorems concerning valuations on field extensions separability is a needed requirement. Since we are mainly interested on valuations on K and not their behaviour on the field extension $K|\mathbb{F}_q(x)$ the following Theorem is very helpful.

Theorem 1.3.3 *Let K be a finite extension of the rational function field $\mathbb{F}_p(x)$. Then there exist two elements $s, t \in K$ with*

$$K = \mathbb{F}_p(s, t),$$

such that $K|\mathbb{F}_p(s)$ is a finite separable field extension and $\mathbb{F}_p(s)|\mathbb{F}_p(x)$ is purely inseparable. The element s is called **separating element** *for K.*

Proof. See [8, p. 144]. □

Let K be a global field and $\mathfrak{p}$ a place of K. We denote by $K_\mathfrak{p}$ the completion of K with respect to the place $\mathfrak{p}$. If $\mathfrak{p}$ is a class of archimedean absolute values we call $\mathfrak{p}$ ***archimedean*** and otherwise ***non-archimedean***. If K is an algebraic number field the infinite places are just the archimedean places. For a non-archimedean place $\mathfrak{p}$ we set $v_\mathfrak{p}$ as the unique normed valuation corresponding to $\mathfrak{p}$, write $\mathcal{O}_\mathfrak{p}$ for the valuation ring of $\mathfrak{p}$ in $K_\mathfrak{p}$ and $\kappa_\mathfrak{p}$ for its residue field and use the notation $\mathfrak{p}|p$ if $\mathrm{char}(\kappa_\mathfrak{p}) = p$. If $\mathfrak{p}$ is archimedean we set $\mathcal{O}_\mathfrak{p} := K_\mathfrak{p}$ and $v_\mathfrak{p}(x) = -\log(|x|_\mathfrak{p})$, where $|\ |_\mathfrak{p}$ is the unique representative of $\mathfrak{p}$ with $|\ |_{\mathfrak{p}|\mathbb{Q}} = |\ |_\infty$.

Let K be a global field, $\mathfrak{p}$ a place of K and $|\ |_\mathfrak{p}$ a representative of $\mathfrak{p}$. We denote the unique extension of $|\ |_\mathfrak{p}$ onto $K_\mathfrak{p}$ also by $|\ |_\mathfrak{p}$. By Theorem 1.3.1 there is a unique extension of $|\ |_\mathfrak{p}$ onto $\overline{K_\mathfrak{p}}$ which is denoted by $|\ |_{\overline{\mathfrak{p}}}$.
Let $L|K$ be an algebraic field extension and τ, τ' two K-linear embeddings from L into $\overline{K_\mathfrak{p}}$. Then τ and τ' are called ***conjugated*** over $\overline{K_\mathfrak{p}}$ if there is a $K_\mathfrak{p}$-linear automorphism $\sigma \in \mathrm{Aut}_{K_\mathfrak{p}}(\overline{K_\mathfrak{p}})$ of $\overline{K_\mathfrak{p}}$ such that $\tau = \sigma \circ \tau'$.

Theorem 1.3.4 *Let $L|K$ be an algebraic field extension of global fields, $\mathfrak{p}$ a place of K and $|\ |_\mathfrak{p}$ a representative of $\mathfrak{p}$.*

(i) Every extension $|\ |$ of $|\ |_\mathfrak{p}$ onto L is given by $|\ | = |\ |_{\overline{\mathfrak{p}}} \circ \tau$, where $\tau : L \to \overline{K_\mathfrak{p}}$ is a K-linear homomorphism of fields.

(ii) Two extensions $|\ |_{\overline{\mathfrak{p}}} \circ \tau$ and $|\ |_{\overline{\mathfrak{p}}} \circ \tau'$ are equal, iff τ and τ' are conjugated over $\overline{K_\mathfrak{p}}$.

Proof. See [6, p. 170]. □

Examples 1.3.5 (i) The absolute values on $\mathbb{Q}$ are by Theorem 1.1.5 equivalent to $|\ |_p$ or $|\ |_\infty$. We denote the corresponding places by p or ∞ respectively and obtain

$$V_{\mathbb{Q}} = \{p|p \textit{ is prime number}\} \cup \{\infty\}.$$

(ii) Let K be an algebraic number field and $\mathcal{O}_K$ its ring of integers. Every finite place $\mathfrak{p}$ of K corresponds to a prime ideal of $\mathcal{O}_K$ and vice versa: For every prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ we obtain a normed valuation $v_{\mathfrak{p}}$ given by $v_{\mathfrak{p}}(x) = \nu_{\mathfrak{p}}(x)$, where

$$(x) = \prod_{\mathfrak{p} \textit{ prime ideal}} \mathfrak{p}^{\nu_{\mathfrak{p}}(x)},$$

is the unique factorisation of (x) into prime ideals. Let on the other hand $\mathfrak{p}$ be a finite place of K and $\mathfrak{p}|p$. The Ideal $\mathfrak{m} = \mathcal{O}_K \cap \{x \in K | v_{\mathfrak{p}}(x) > 0\}$ is non empty, since $p \in \mathfrak{m}$. Further $\mathfrak{m}$ is obviously a prime ideal. The valuation $v_{\mathfrak{m}}$ as defined above coincides with the normed valuation $v_{\mathfrak{p}}$.

Let $\rho_1, \cdots, \rho_r$ be the embeddings of K into $\mathbb{R}$ and $\sigma_1, \overline{\sigma_1}, \cdots \sigma_s, \overline{\sigma_s}$ be the embeddings of K into $\mathbb{C}$. By Theorem 1.3.4 all archimedean absolute values of K are up to equivalence of the form $|x|_{\rho_i} = |\rho_i(x)|_\infty$ or $|x|_{\sigma_j} = |\sigma_j(x)|_\infty$. Further all the $|\ |_{\rho_i}$ and $|\ |_{\sigma_j}$ are pairwise non equivalent. Hence the set of places of K is given by

$$V_K = \{\mathfrak{p}|\mathfrak{p} \textit{ is prime ideal of } \mathcal{O}_K\} \cup \{\rho_1, \cdots, \rho_r, \sigma_1, \cdots, \sigma_s\}.$$

For every element $x \in K$ the factorisation of (x) into prime ideals is finite. Therefore holds $v_{\mathfrak{p}}(x) = 0$ for almost all finite places $\mathfrak{p}$ of K.

(iii) Let $\mathbb{F}_p(x)$ be the rational function field over the finite field $\mathbb{F}_p$. By Theorem 1.1.5 every absolute value on $\mathbb{F}_p(x)$ is equivalent to either $|\ |_q$ for a irreducible polynomial $q \in \mathbb{F}_p[x]$ or $|\ |_\infty$. The corresponding places are denoted by q or ∞ respectively and we obtain

$$V_{\mathbb{F}_p(x)} = \{q|q \in \mathbb{F}_p[x] \textit{ irreducible}\} \cup \{\infty\}.$$

(iv) Let K be a separable extension of $\mathbb{F}_p(x)$. Analogous to *(ii)* the finite places of K correspond to the prime ideals of the integral closure $\mathcal{O}_K$ of $\mathbb{F}_p[x]$ in K. The infinite places of K correspond by Theorem 1.3.4 to the embeddings $\sigma_1, \cdots, \sigma_n$ of K into $F_p((\frac{1}{x}))$ which are pairwise not conjugated over $\overline{\mathbb{F}_p((\frac{1}{x}))}$.

Lemma 1.3.6 *Set $K = \mathbb{Q}$ or $K = \mathbb{F}_p(x)$ and $\mathcal{O}_K = \mathbb{Z}$ or $\mathcal{O}_K = \mathbb{F}_p[x]$ respectively. Then we obtain for the intersection*

$$\bigcap_{\mathfrak{p} \nmid \infty} \mathcal{O}_{\mathfrak{p}} = \mathcal{O}_K.$$

Proof. One has obviously $\bigcap_{\mathfrak{p}\nmid\infty} \mathcal{O}_{\mathfrak{p}} \supseteq \mathcal{O}_K$. Now let $x \in \bigcap_{\mathfrak{p}\nmid\infty} \mathcal{O}_{\mathfrak{p}}$. For a place $\mathfrak{p}'$ we can choose for a prime element $\pi_{\mathfrak{p}'}$ in $\mathcal{O}_{\mathfrak{p}'}$ a prime number $p \in \mathbb{Z}$ if $K = \mathbb{Q}$ or a irreducible polynomial $q \in \mathbb{F}_p[x]$ if $K = \mathbb{F}_p(x)$. In both cases we obtain

$$v_{\mathfrak{p}}(\pi_{\mathfrak{p}'}) = \begin{cases} 1 & \mathfrak{p} = \mathfrak{p}', \\ 0 & \textit{otherwise}. \end{cases}$$

We can write $x = \pi_{\mathfrak{p}'}^m \sum_{i>0} a_i \pi_{\mathfrak{p}'}^i$ for $m \in \mathbb{Z}$ and $a_i \in R$, where $R \subseteq \mathcal{O}_K$ is a set of representatives of $\mathcal{O}_{\mathfrak{p}'}/\mathfrak{p}'$ with $0 \in R$. If the sum is not finite there exists a place $\mathfrak{p}$ such that $v_{\mathfrak{p}}(a_i) \neq 0$ for infinitely many i. This is a contradiction to $x \in \mathcal{O}_{\mathfrak{p}}$ since the sum does not converge in $\mathcal{O}_{\mathfrak{p}}$. Therefore $x \in \mathcal{O}_K$. $\square$

Definition 1.3.7 *Let K be a global field, $L|K$ an algebraic field extension and $\mathfrak{p}$, $\mathfrak{P}$ be places of K or L respectively. We say $\mathfrak{P}$* **lies over** $\mathfrak{p}$*, denoted by $\mathfrak{P}|\mathfrak{p}$, if the restriction $|\ |_{\mathfrak{P}|_K}$ is equivalent to $|\ |_{\mathfrak{p}}$ for representatives $|\ |_{\mathfrak{P}}$ and $|\ |_{\mathfrak{p}}$ of $\mathfrak{P}$ and $\mathfrak{p}$ respectively.*

The expression $\mathfrak{P}|\mathfrak{p}$ is well-defined since the restriction of two representatives $|\ |_{\mathfrak{P}}, |\ |'_{\mathfrak{P}}$ of $\mathfrak{P}$ onto K are equivalent.

Let $L|K$ be an algebraic field extension of a global field K and let $\mathfrak{P}, \mathfrak{p}$ be places of L or K respectively with $\mathfrak{P}|\mathfrak{p}$. Denote by $\mathfrak{m}_L, \mathfrak{m}_K$ the set of null sequences in L or K. We can write every element of $L_{\mathfrak{P}}$ or $K_{\mathfrak{p}}$ as $(x_n) + \mathfrak{m}_{\mathfrak{P}}$ or $(x_n) + \mathfrak{m}_{\mathfrak{p}}$ for a Cauchy-sequence (x_n) in L or K respectively. Since the restriction of the absolute value $|\ |_{\mathfrak{P}}$ onto K is equivalent to $|\ |_{\mathfrak{p}}$ a Cauchy-sequence (x_n) in K is also a Cauchy-sequence in L and hence $\mathfrak{m}_K \subseteq \mathfrak{m}_L$. Therefore the map

$$\iota : K_{\mathfrak{p}} \longrightarrow L_{\mathfrak{P}}, \quad (x_n) + \mathfrak{m}_K \mapsto (x_n) + \mathfrak{m}_L,$$

is well-defined. It is also an homomorphism of fields since the addition and multiplication in both $K_{\mathfrak{p}}$ and $L_{\mathfrak{P}}$ are component-wise. Finally ι preserves the absolute value:

$$|\iota((x_n) + \mathfrak{m}_K)|_{\mathfrak{P}} = |(x_n) + \mathfrak{m}_L|_{\mathfrak{P}} = \lim_{n\to\infty}(|(x_n)|_{\mathfrak{P}}) = \\ = \lim_{n\to\infty}(|(x_n)|_{\mathfrak{p}}) = |(x_n) + \mathfrak{m}_K|_{\mathfrak{p}}.$$

Therefore we can embed $K_{\mathfrak{p}}$ into $L_{\mathfrak{P}}$ such that the restriction of $|\ |_{\mathfrak{P}}$ onto $K_{\mathfrak{p}}$ is equivalent to $|\ |_{\mathfrak{p}}$.

The following Theorem is very useful to lift properties from $K_{\mathfrak{p}}$ onto $L_{\mathfrak{P}}$ if $\mathfrak{P}|\mathfrak{p}$.

Theorem 1.3.8 *Let $L|K$ be a separable field extension of global fields and $\mathfrak{p}$ a place of K. Then there exists an isomorphism of topological $K_{\mathfrak{p}}$-algebras*

$$K_{\mathfrak{p}} \otimes_K L \longrightarrow \prod_{\mathfrak{P}|\mathfrak{p}} L_{\mathfrak{P}},$$

if we choose the topology on $K_{\mathfrak{p}} \otimes_K L$ as in the following Remark.

Proof. See [2, p. 57]. □

Remark Let $\varphi : K_\mathfrak{p}^n \longrightarrow K_\mathfrak{p} \otimes_K L$ be an isomorphism of $K_\mathfrak{p}$-vector spaces and the topology on $K_\mathfrak{p}^n$ be the product topology. We equip $K_\mathfrak{p} \otimes_K L$ with the initial topology with respect to φ, i. e., a subset $O \subseteq K_\mathfrak{p} \otimes_K L$ is open iff O is the image of an open subset of $K_\mathfrak{p}^n$. Since every change of basis in $K_\mathfrak{p}^n$ is a homeomorphism the topology on $K_\mathfrak{p} \otimes_K L$ does not depend on the choice of the isomorphism φ. The addition and multiplication on $K_\mathfrak{p} \otimes_K L$ are continuous as one proves easily.

Proposition 1.3.9 *Let $L|K$ be a separable field extension of global fields and $\mathfrak{p}$ a place of K. Then the following holds:*
(i) $[L : K] = \sum_{\mathfrak{P}|\mathfrak{p}} [L_\mathfrak{P} : K_\mathfrak{p}]$.
(ii) $N_{L|K}(x) = \prod_{\mathfrak{P}|\mathfrak{p}} N_{L_\mathfrak{P}|K_\mathfrak{p}}(x)$.

Proof. See [6, p. 172]. □

Definition 1.3.10 *Let K be a global field, $L|K$ an algebraic field extension and $\mathfrak{p}$,$\mathfrak{P}$ be places of K or L respectively with $\mathfrak{P}|\mathfrak{p}$. We define the* **ramification index**

$$e(\mathfrak{P}|\mathfrak{p}) := \begin{cases} (v_\mathfrak{P}(L^*) : v_\mathfrak{p}(K^*)) & \mathfrak{p} \text{ not archimedean}, \\ 1 & \mathfrak{p} \text{ archimedean}, \end{cases}$$

and the **inertial degree**

$$f(\mathfrak{P}|\mathfrak{p}) := \begin{cases} [\kappa_\mathfrak{P} : \kappa_\mathfrak{p}] & \mathfrak{p} \text{ not archimedean}, \\ [L_\mathfrak{P} : K_\mathfrak{p}] & \mathfrak{p} \text{ archimedean}, \end{cases}$$

where $\kappa_\mathfrak{p}$ is the residue field of $K_\mathfrak{p}$ if $\mathfrak{p}$ is non-archimedean. Further we use the abbreviation $e_\mathfrak{p} = e(\mathfrak{p}|p)$ for a finite place $\mathfrak{p}$ and $e_\mathfrak{p} = e(\mathfrak{p}|\infty)$ for an infinite place $\mathfrak{p}$ and the analogue for $f_\mathfrak{p}$. We define

$$\mathfrak{N}(\mathfrak{p}) = \begin{cases} p^{f_\mathfrak{p}} & \mathfrak{p}|p, \\ e^{f_\mathfrak{p}} & \mathfrak{p} \text{ is archimedean}, \end{cases}$$

and the **normed absolute value** $|\ |_\mathfrak{p}$ *corresponding to $\mathfrak{p}$ as $|x|_\mathfrak{p} := \mathfrak{N}(\mathfrak{p})^{-v_\mathfrak{p}(x)}$ for all $x \in K$.*

Let K be a global field and $\mathfrak{p}$ a place of K. We denote for the rest of the thesis with $|\ |_\mathfrak{p}$ the normed absolute value corresponding to the place $\mathfrak{p}$.

Proposition 1.3.11 *Let K be a global field, $L|K$ an algebraic extension and $\mathfrak{p}$, $\mathfrak{P}$ be places of K or L respectively with $\mathfrak{P}|\mathfrak{p}$.*
(i) $\mathfrak{N}(\mathfrak{P}) = \mathfrak{N}(\mathfrak{p})^{f(\mathfrak{P}|\mathfrak{p})}$.
(ii) $v_\mathfrak{P}(x) = e(\mathfrak{P}|\mathfrak{p}) v_\mathfrak{p}(x)$ *for all $x \in K$.*
(iii) $\sum_{\mathfrak{P}|\mathfrak{p}} e(\mathfrak{P}|\mathfrak{p}) f(\mathfrak{P}|\mathfrak{p}) = [L : K]$ *where the sum is over all places $\mathfrak{P}$ of L with $\mathfrak{P}|\mathfrak{p}$.*

(iv) $|x|_{\mathfrak{P}} = |N_{L_{\mathfrak{P}}|K_{\mathfrak{p}}}(x)|_{\mathfrak{p}}$ *for all* $x \in L$ *if* $L|K$ *is separable.*

Proof. We will prove the Theorem only for separable field extensions $L|K$. For the inseparable case see [8, p. 71, 74].

(i) For an archimedean place $\mathfrak{p}$ holds

$$\mathfrak{N}(\mathfrak{P}) = e^{f_{\mathfrak{P}}} = e^{f(\mathfrak{P}|\infty)} = e^{f(\mathfrak{P}|\mathfrak{p})f(\mathfrak{p}|\infty)} = \mathfrak{N}(\mathfrak{p})^{f(\mathfrak{P}|\mathfrak{p})}.$$

One proves the claim analogously for a non-archimedean place $\mathfrak{p}$.

(ii) For infinite places $\mathfrak{p}, \mathfrak{P}$ this follows directly form the definition of $v_{\mathfrak{p}}, v_{\mathfrak{P}}$. Since $v_{\mathfrak{p}}$ and $v_{\mathfrak{P}}$ are for finite places $\mathfrak{p}, \mathfrak{P}$ the unique normed valuations corresponding to $\mathfrak{p}$ or $\mathfrak{P}$ respectively this is a direct consequence from the definition of $e(\mathfrak{P}|\mathfrak{p})$.

(iii) Proposition 1.3.9 states $[L : K] = \sum_{\mathfrak{P}|\mathfrak{p}}[L_{\mathfrak{P}} : K_{\mathfrak{p}}]$. The claim is therefore obviously true for an archimedean place $\mathfrak{p}$. Let $\mathfrak{p}$ be a non-archimedean place. Denote by $\mathcal{O}_{\mathfrak{p}}$ and $\mathcal{O}_{\mathfrak{P}}$ the valuation rings corresponding to $\mathfrak{p}$ or $\mathfrak{P}$ respectively and by $\mathfrak{p}$ and $\mathfrak{P}$ their only prime ideals - it is clear out of the context if the letters $\mathfrak{p}, \mathfrak{P}$ stand for the places or for the prime ideals. By Theorem 1.1.11 $\mathcal{O}_{\mathfrak{p}}$ and $\mathcal{O}_{\mathfrak{P}}$ are principal ideal domains. Since $v_{\mathfrak{P}}, v_{\mathfrak{p}}$ are the normed valuations corresponding to $\mathfrak{P}, \mathfrak{p}$ the ramification index $e(\mathfrak{P}|\mathfrak{p})$ of the places $\mathfrak{P}, \mathfrak{p}$ satisfies

$$\mathfrak{p}\mathcal{O}_{\mathfrak{P}} = \mathfrak{P}^{e(\mathfrak{P}|\mathfrak{p})}.$$

Since the initial degree $f(\mathfrak{P}|\mathfrak{p})$ is defined the same way for the places $\mathfrak{P}, \mathfrak{p}$ and for the prime ideals $\mathfrak{P}, \mathfrak{p}$ we obtain

$$[L_w : K_v] = e(\mathfrak{P}|\mathfrak{p})f(\mathfrak{P}|\mathfrak{p}),$$

which implies the claim.

(iv) By Theorem 1.3.1 we know for $x \in L$

$$w(x) = \frac{1}{[L_{\mathfrak{P}} : K_{\mathfrak{p}}]} v_{\mathfrak{p}}(N_{L_{\mathfrak{P}}|K_{\mathfrak{p}}}(x)),$$

is an extension of $v_{\mathfrak{p}}$ onto L which satisfies $w_{|K} = v_{\mathfrak{p}}$. The property (ii) implies $v_{\mathfrak{P}}(x) = e(\mathfrak{P}|\mathfrak{p})w(x)$. By (i) and the proof of (iii) we obtain

$$|x|_{\mathfrak{P}} = \mathfrak{N}(\mathfrak{P})^{-v_{\mathfrak{P}}(x)} = \mathfrak{N}(\mathfrak{p})^{-f(\mathfrak{P}|\mathfrak{p})e(\mathfrak{P}|\mathfrak{p})w(x)} = \\ = \mathfrak{N}(\mathfrak{p})^{-v_{\mathfrak{p}}(N_{L_{\mathfrak{P}}|K_{\mathfrak{p}}}(x))} = |N_{L_{\mathfrak{P}}|K_{\mathfrak{p}}}(x)|_{\mathfrak{p}}.$$ □

The following Theorem justifies the definition of a normed absolute value corresponding to a place.

Theorem 1.3.12 (Product Formula) *Let* K *be a global field and* $x \in K^*$. *Then* $|x|_{\mathfrak{p}} = 1$ *for almost all places* $\mathfrak{p}$ *and*

$$\prod_{\mathfrak{p} \in V_K} |x|_{\mathfrak{p}} = 1.$$

Proof. Set $K_0 = \mathbb{Q}$ if $\mathrm{char}(K) = 0$ and $K_0 = \mathbb{F}_p(s)$ if $\mathrm{char}(K) = p$, where $s \in K$ is a separating element for K, and $\mathcal{o}$ to be the ring $\mathbb{Z}$ or $\mathbb{F}_p[s]$ respectively. Theorem 1.3.3 states the existence of such a separating element s for $\mathrm{char}(K) = p$ and that $K|\mathbb{F}_p(s)$ is separable.
By Example 1.3.5 every global field K has only finitely many infinite places. Every finite place $\mathfrak{P}$ corresponds to a prime ideal $\mathfrak{P}$ of the integral closure $\mathcal{o}_K$ of $\mathcal{o}$ in K, which is a Dedekind domain. The normed valuations $v_\mathfrak{P}$ corresponding to a finite places $\mathfrak{P}$ are of the form $v_\mathfrak{P}(x) = \nu_\mathfrak{P}(x)$ where $x \in K$ and $(x) = \prod_\mathfrak{P} \mathfrak{P}^{\nu_\mathfrak{P}(x)}$ is the unique factorisation of the fractional ideal (x) into prime ideals $\mathfrak{P}$ in $\mathcal{o}_K$. This implies now the first claim.

As a consequence of Proposition 1.3.9 and Proposition 1.3.11 we obtain

$$\prod_{\mathfrak{P} \in V_K} |x|_\mathfrak{P} = \prod_{\mathfrak{p} \in V_{K_0}} \prod_{\mathfrak{P}|\mathfrak{p}} |x|_\mathfrak{P} = \prod_{\mathfrak{p} \in V_{K_0}} \prod_{\mathfrak{P}|\mathfrak{p}} |N_{K_\mathfrak{P}|K_{0\mathfrak{p}}}(x)|_\mathfrak{p} = \prod_{\mathfrak{p} \in V_{K_0}} |N_{K|K_0}(x)|_\mathfrak{p}.$$

Therefore it suffices to prove the product formula for $K = K_0$.
Let $K = \mathbb{Q}$. For every element $x \in \mathbb{Q}^*$ there is a unique prime factorisation $x = \mathrm{sign}(x) \prod_p p^{\nu_p}$, where $\nu_p \in \mathbb{Z}$ and almost all equal to zero. Hence we obtain

$$\prod_{\mathfrak{p} \in V_\mathbb{Q}} |x|_\mathfrak{p} = |x|_\infty \cdot \prod_{p \ prime} p^{-\nu_p} = \prod_{p \ prime} p^{\nu_p} \cdot \prod_{p \ prime} p^{-\nu_p} = 1.$$

Now let $K = \mathbb{F}_p(s)$. For $r \in \mathbb{F}_p(s)$ let $r = u \prod_q q^{\nu_p(x)}$ be the unique factorisation of r into irreducible polynomials $q \in \mathbb{F}_p[s]$ and $u \in \mathbb{F}_p^*$. The degree of r is given by $\deg(r) = \sum_q \nu_q(r) \deg(q)$ where the sum is over all irreducible polynomials q. Then we obtain

$$\prod_{\mathfrak{p} \in V_{\mathbb{F}_p(s)}} |r|_\mathfrak{p} = |r|_\infty \cdot \prod_{q \ irreducible} |r|_q = p^{\deg(r)} \cdot \prod_{q \ irreducible} p^{-\deg(q)\nu_q(r)} =$$

$$= p^{\sum_{q \ irreducible} \nu_q(r) \deg(q)} \cdot \prod_{q \ irreducible} p^{-\deg(q)\nu_q(r)} = 1. \qquad \square$$

1.4. Local fields

Definition 1.4.1 *A* **local field** *is a complete field K with respect to a discrete valuation v such that the residue field $\kappa_v = \mathcal{o}_v/\mathfrak{p}_v$ is finite.*

Proposition 1.4.2 *A local field K is locally compact and its valuation ring $\mathcal{o}_v$ is compact.*

Proof. As a consequence of Theorem 1.2.4 $\mathcal{o}_v/\mathfrak{p}_v^n$ is finite for every $n \geq 1$ and therefore compact. Theorem 1.2.7 states $\mathcal{o}_v \cong \varprojlim_n \mathcal{o}_v/\mathfrak{p}_v^n$ as topological rings. The topological space $\prod_{n \geq 1} \mathcal{o}_v/\mathfrak{p}_v^n$ is by Tychonoff's Theorem compact. Since $\varprojlim_n \mathcal{o}_v/\mathfrak{p}_v^n$ is a closed subset of $\prod_{n \geq 1} \mathcal{o}_v/\mathfrak{p}_v^n$ the valuation ring $\mathcal{o}_v$

is also compact. For any $x \in K$ the set $x + \mathcal{O}_v$ is a compact neighbourhood which proves that K is locally compact. □

Theorem 1.4.3 *Every finite field extensions of $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ is a local field and every local field is isomorphic to a finite field extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, where $\mathbb{F}_p((t))$ is the field of formal Laurent series in t over the finite field $\mathbb{F}_p$.*

Proof. See [6, p. 141]. □

Since every local field K is obviously Hausdorff, Theorem A.5 states that there exists a Haar measure μ on $(K, +)$. Let $\mathcal{O}_v$ be the residue field of K and $\mathfrak{p}_v$ the maximal ideal of $\mathcal{O}_v$. We normalize μ for the rest of the thesis by setting $\mu(\mathcal{O}_v) = 1$. With this normalization we obtain a unique measure μ on K given by

$$\mu(x + \mathfrak{p}_v^n) := q^{-n},$$

where $x \in K$, $n \in \mathbb{Z}$ and $q = |\mathcal{O}_v/\mathfrak{p}_v|$. For an element $x \in K$ we obtain

$$\mu(x\mathcal{O}_v) = |x|_v,$$

where $|\ |_v$ is the normalized absolute value corresponding to v given by $|x|_v = q^{-v(x)}$.

2. Ring of adeles

With the background knowledge of valuations provided in the first chapter we can now introduce the *adele ring* of a global field. We mainly follow [2], [6] and [7]. The notation follows mostly [6] and [7].

2.1. Restricted direct product

Before we are able to define the adele ring we need the concept of the restricted direct product.

Definition 2.1.1 *Let J be an index set and let $J_\infty \subseteq J$ be a finite subset. For $j \in J$ let G_j be a locally compact topological group and for $j \notin J_\infty$ let H_j be a compact open subgroup of G_j. The* **restricted direct product** *of the G_j with respect to the H_j is the topological group*

$$\prod_{j\in J}{}'(G_j : H_j) = \{(x_j)_{j\in J} \in \prod_{j\in J} G_j : x_j \in H_j \text{ for almost all } j \in J\},$$

where the group multiplication is component-wise and the topology is generated by the base of all sets of the form $\prod_{j\in J} O_j$ with O_j open in G_j and $O_j = H_j$ for almost all $j \in J$.

Proposition 2.1.2 *Let $G = \prod_{j\in J}{}'(G_j : H_j)$ be the restricted direct product of the G_j with respect to the H_j and G_j, H_j as above. Then G is a locally compact topological group.*

Proof. Let $(g_j)_{j\in J}$ be an arbitrary element of G. Let $S \subseteq J$ be the finite subset consisting of all indices $j \in J$ such that either $j \in J_\infty$ or $g_j \notin H_j$. We set $N_j \subseteq G_j$ to be a compact neighbourhood of g_j for $j \in S$ and $N_j = H_j$ for $j \notin S$. The set

$$G_S := \prod_{j\in J} N_j,$$

is then a compact neighbourhood of (g_j): G_S is by Tychonoff's Theorem compact in the product topology. Hence G_S is compact with respect to the subspace topology of G_S in G, since the subspace topology is coarser than the product topology. □

Proposition 2.1.3 *Let $G = \prod_{j\in J}{}'(G_j : H_j)$ be the restricted direct product of the G_j with respect to the H_j and let G_j be Hausdorff for all $j \in J$.Denote by μ_j the Haar measure on G_j with the normalization $\mu_j(H_j) = 1$ for all $j \notin J_\infty$. Then there exists a unique Haar measure μ on G given by*

$$\mu(\prod_j M_j) = \prod_j \mu_j(M_j),$$

where the $M_j \subseteq G_j$ are μ_j-measurable and $M_j = H_j$ for almost all $j \in J$.

Proof. See [2, p. 63] and [7, p. 185]. $\square$

2.2. Ring of adeles and idele group

Definition 2.2.1 *Let K be a global field, denote by $K_\mathfrak{p}$ the completion of K with respect to the place $\mathfrak{p}$ and by $\mathcal{O}_\mathfrak{p}$ the valuation ring of $K_\mathfrak{p}$ if $\mathfrak{p}$ is non-archimedean. The* **adele ring** *$\mathbb{A}_K$ over K is the restricted direct product*

$$\mathbb{A}_K := \prod_{\mathfrak{p} \in V_K}{}'(K_\mathfrak{p} : \mathcal{O}_\mathfrak{p})$$

of the additive groups $K_\mathfrak{p}$ with respect to the $\mathcal{O}_\mathfrak{p}$. It can be endowed with a multiplicative structure given by component-wise multiplication such that $\mathbb{A}_K$ is a topological ring. An element of the adele ring is called **adele***.*

Let K be a global field. We can see K as a subfield of its completion $K_\mathfrak{p}$ for every place $\mathfrak{p}$ of K. Theorem 1.3.12 implies then that every element $x \in K$ lies in $\mathcal{O}_\mathfrak{p}$ for almost all places $\mathfrak{p}$. Therefore we can embed K into its ring of adeles $\mathbb{A}_K$ via

$$x \mapsto (x)_{\mathfrak{p} \in V_K}.$$

We identify for the rest of the thesis the field K with its embedding into $\mathbb{A}_K$ and call an element $x \in K$ ***principal adele***.

We will need later the following

Lemma 2.2.2 *Let K be a global field. Then there exists an embedding of K-vector spaces $\iota : K^n \to \mathbb{A}_K^n$ such that every K-basis $\{b_1, \cdots, b_n\}$ of K^n is mapped onto an $\mathbb{A}_K$-basis $\{\iota(b_1), \cdots, \iota(b_n)\}$.*

Proof. Let $\varphi : K^n \to K^n \otimes_K \mathbb{A}_K$ be the homomorphism of K-vector spaces given by $\varphi : v \mapsto v \otimes 1_K$. Then φ maps K-bases onto $\mathbb{A}_K$-bases (cf. [10, p. 216]). Denote by $\psi : K^n \otimes \mathbb{A}_K \to \mathbb{A}_K^n$ the isomorphism of $\mathbb{A}_K$-modules which is induced by $\psi : (x_i)_{1 \le i \le n} \otimes a \mapsto (x_i \cdot a)_{1 \le i \le n}$. Therefore the map $\iota := \psi \circ \varphi$ is an embedding of K-vector spaces which maps K-bases of K^n onto $\mathbb{A}_K$-bases of $\mathbb{A}_K^n$. $\square$

Remark The above defined map $\iota : K^n \to \mathbb{A}_K^n$ is given in every component as the embedding of K into $\mathbb{A}_K$. Since we have identified K with its embedding in $\mathbb{A}_K$ we want to do the same with the embedding of K^n into $\mathbb{A}_K^n$ and write $v = \iota(v)$ for $v \in K^n$. Hence the above Lemma states that every K-basis of K^n is also an $\mathbb{A}_K$-basis of $\mathbb{A}_K^n$.

Definition 2.2.3 *Let K be a global field. The* **idele group** $\mathbb{I}_K$ *over K is defined as the restricted direct product*

$$\mathbb{I}_K := \prod_{\mathfrak{p} \in V_K}{}' (K_\mathfrak{p}^* : \mathcal{O}_\mathfrak{p}^*),$$

of the groups $K_\mathfrak{p}^*$ with respect to the $\mathcal{O}_\mathfrak{p}^*$. An element of the idele group is called **idele**.

We can obviously see the idele group $\mathbb{I}_K$ as a subset of the ring of adeles $\mathbb{A}_K$. The idele group is then exactly the group of units of the adele ring : Every idele is clearly a unit element of the adele ring. For an element $x = (x_\mathfrak{p}) \in \mathbb{A}_K^*$ we have $x_\mathfrak{p} \in K_\mathfrak{p}^*$ for every place $\mathfrak{p}$ since the multiplication is defined component-wise. Since $(x_\mathfrak{p}^{-1}) \in \mathbb{A}_K^*$ we have $x_\mathfrak{p}, x_\mathfrak{p}^{-1} \in \mathcal{O}_\mathfrak{p}$ for almost all $\mathfrak{p}$. Therefore $x_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}^*$ holds for almost all places $\mathfrak{p}$ and $x = (x_\mathfrak{p})$ is an idele. As a consequence the embedding $K \hookrightarrow \mathbb{A}_K$ induces an embedding of the group of units $K^* \hookrightarrow \mathbb{I}_K$. The elements in the image are called ***principal ideles***.
While the algebraic structure is the same for $\mathbb{I}_K$ and $\mathbb{A}_K^*$ they differ in their topology. Since the inversion $x \mapsto x^{-1}$ is not continuous on $\mathbb{A}_K$ the topology on $\mathbb{I}_K$ can not coincide with the subspace topology on $\mathbb{A}_K$. However the embedding

$$\mathbb{I}_K \to \mathbb{A}_K \times \mathbb{A}_K \quad x \mapsto (x, x^{-1}),$$

induces an isomorphism of topological groups from $\mathbb{I}_K$ onto the image of ι (cf. [7, p. 68]).

Definition 2.2.4 *Let K be a global field. Then we define the* **absolute value** $|\ |_{\mathbb{A}_K}$ *on $\mathbb{A}_K$ as*

$$|\ |_{\mathbb{A}_K} : \mathbb{A}_K \longrightarrow \mathbb{R}, \quad (x_\mathfrak{p}) \mapsto \prod_{\mathfrak{p} \in V_K} |x_\mathfrak{p}|_\mathfrak{p}.$$

The above map is not an absolute value in the sense of the definition given in the first chapter, nevertheless the name suits for this map and is commonly used in literature. Since the product in the definition of $|\ |_{\mathbb{A}_K}$ is infinite we have to think about its existence. For any adele $x = (x_\mathfrak{p}) \in \mathbb{A}_K$ almost all components $x_\mathfrak{p}$ are elements of $\mathcal{O}_\mathfrak{p}$, i.e., $|x_\mathfrak{p}|_\mathfrak{p} \le 1$. Hence the product is either converging or 0. One can easily prove $|x|_{\mathbb{A}_K} \neq 0$ iff $x \in \mathbb{I}_K$. For principal ideles $x \in K^*$ Theorem 1.3.12 states $|x|_{\mathbb{A}_K} = 1$.

The restriction of $|\ |_{\mathbb{A}_K}$ onto $\mathbb{I}_K$ is obviously a continuous map. It is also easy to see, that $|\ |_{\mathbb{A}_K}$ is not continuous on $\mathbb{A}_K$:
Let $(x_\mathfrak{p})$ be an adele with $|(x_\mathfrak{p})|_{A_K} \neq 0$. If $|\ |_{\mathbb{A}_K}$ is continuous on $\mathbb{A}_K$, then there exists a neighbourhood $U \subseteq \mathbb{A}_K$ of $(x_\mathfrak{p})$ such that

$$\frac{1}{2}|(x_\mathfrak{p})|_{A_K} \le |(y_\mathfrak{p})|_{A_K} \le 2|(x_\mathfrak{p})|_{A_K},$$

for every $(y_\mathfrak{p}) \in U$. But in every neighbourhood of $(x_\mathfrak{p})$ there is an element $(y_\mathfrak{p})$ with at least one component $y_\mathfrak{p} = 0$, which is a contradiction.

Theorem 2.2.5 *Let K be a global field and $L|K$ a separable field extension. The map*

$$\varphi : \mathbb{A}_K \otimes_K L \longrightarrow \mathbb{A}_L, \quad (x_\mathfrak{p})_{\mathfrak{p}\in V_K} \otimes a \mapsto (x_\mathfrak{p} \cdot \tau_\mathfrak{P}(a))_{\mathfrak{P}\in V_L, \mathfrak{P}|\mathfrak{p}},$$

is an isomorphism of topological rings, where the map $\tau_\mathfrak{P} : L \to L_\mathfrak{P}$ is the canonical embedding of L into $L_\mathfrak{P}$.

Proof. See [2, p. 64]. □

Remark Let K be a global field, $L|K$ a separable field extension and $\{b_1, \cdots, b_n\}$ be a K-basis of the field L. The map

$$\psi : \prod_{i=1}^{[L:K]} \mathbb{A}_K \to \mathbb{A}_K \otimes_K L,$$
$$((x_{\mathfrak{p},i})_{\mathfrak{p}\in V_K})_{1\le i\le [L:K]} \mapsto \sum_{1\le i\le [L:K]} (x_{\mathfrak{p},i})_{\mathfrak{p}\in V_K} \otimes b_i,$$

is an isomorphism of topological groups. By the above Theorem we obtain an isomorphism

$$\psi : \prod_{i=1}^{[L:K]} \mathbb{A}_K \to \mathbb{A}_L$$

of topological groups given by $(x_{\mathfrak{p},i}) \mapsto \sum_{i=1}^{[L:K]} b_i (x_{\mathfrak{p},i})_{\mathfrak{p}\in V_L, \mathfrak{P}|\mathfrak{p}}$.

2.3. Strong approximation

In this subchapter we prove two important Theorems which give some insight on the relation between a global field K and its adele ring $\mathbb{A}_K$. The first is Theorem 2.3.1 which states that K lies discrete and cocompact in its adele ring $\mathbb{A}_K$. The other is the so-called *strong approximation*, Theorem 2.3.5 which states that K is „almost dense“ in its adele ring $\mathbb{A}_K$.

Theorem 2.3.1 *Let K be a global field. We identify K with its image of the map*

$$K \longrightarrow \mathbb{A}_K, \quad x \mapsto (x)_{\mathfrak{p}\in V_K}.$$

Then K is a discrete and cocompact subring of the adele ring $\mathbb{A}_K$, i. e., the quotient $\mathbb{A}_K/K$ is compact.

Proof. We set $K_0 = \mathbb{Q}$ if $\mathrm{char}(K) = 0$ and $K_0 = \mathbb{F}_p(s)$ if $\mathrm{char}(K) = p$, where $s \in K$ is a separating element for K and set $n = [K : K_0]$. Theorem 1.3.3 states the separability of $K|K_0$ for $\mathrm{char}(K) \neq 0$. As a consequence of Theorem 2.2.5 we obtain a commutative diagram of topological K_0-vector spaces

$$\begin{array}{ccc} \prod_{i=1}^{n} \mathbb{A}_{K_0} & \xrightarrow{\cong} & \mathbb{A}_K \\ \uparrow & & \uparrow \\ \prod_{i=1}^{n} K_0 & \xrightarrow{\cong} & K \end{array}$$

Therefore $\mathbb{A}_K/K$ is compact iff $\prod_{j=1}^{n}(\mathbb{A}_{K_0}/K_0)$ is compact, which is true iff $\mathbb{A}_{K_0}/K_0$ is compact. Analogous it suffices to prove that K_0 is discrete in $\mathbb{A}_{K_0}$. Let $\mathcal{O}_\mathfrak{p}$ be the valuation ring corresponding to a place $\mathfrak{p}$ of K and C be the compact set

$$C := \{(x_\mathfrak{p}) \in \mathbb{A}_{K_0} : |x_\infty|_\infty \leq \frac{1}{2}, |x_\mathfrak{p}|_\mathfrak{p} \leq 1 \textit{ for } \mathfrak{p} \nmid \infty\}.$$

Then the following holds

(i) $C \cap K_0 = \{0\}$,
(ii) $\mathbb{A}_{K_0} = C + K_0 := \{x + y | x \in C, y \in K_0\}$.

Proof:

(i) Let $x \in K_0 \cap C$, then we obtain $x \in \mathcal{O}_\mathfrak{p}$ for all finite places $\mathfrak{p}$. As a consequence of Lemma 1.3.6 we get

$$x \in \bigcap_{\mathfrak{p}\nmid\infty} \mathcal{O}_\mathfrak{p} = \mathcal{O}_{K_0} := \begin{cases} \mathbb{Z} & \mathrm{char}(K) = 0, \\ \mathbb{F}_p[s] & \mathrm{char}(K) = p. \end{cases}$$

The condition $|x|_\infty \leq \frac{1}{2}$ implies $x = 0$.

(ii) Let $x = (x_\mathfrak{p}) \in \mathbb{A}_{K_0}$ be arbitrary and denote by S the finite set of all finite places $\mathfrak{p}$ with $x_\mathfrak{p} \notin \mathcal{O}_\mathfrak{p}$. For $\mathfrak{p} \in S$ we can write $x_\mathfrak{p} = \frac{a_\mathfrak{p}}{\pi_\mathfrak{p}^{\nu_\mathfrak{p}}} + b_\mathfrak{p}$ with $a_\mathfrak{p} \in K_0$, $b_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}$, $\nu_\mathfrak{p} \in \mathbb{N}$ and $\pi_\mathfrak{p} \in K_0$ a prime element of $\mathcal{O}_\mathfrak{p}$. We set

$$y = x - \sum_{\mathfrak{p}\in S} \frac{a_\mathfrak{p}}{\pi_\mathfrak{p}^{\nu_\mathfrak{p}}}.$$

For all finite places $\mathfrak{p}$ we have $y_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}$. We can choose an element $r \in \mathcal{O}_{K_0}$ with $|y_\infty - r|_\infty \leq \frac{1}{2}$. Hence we obtain

$$x = (y - r) + (r + \sum_{\mathfrak{p}\in S} \frac{a_\mathfrak{p}}{\pi_\mathfrak{p}^{\nu_\mathfrak{p}}}),$$

where $(y - r) \in C$ and $r + \sum_{p\in S} \frac{a_\mathfrak{p}}{\pi_\mathfrak{p}^{\nu_\mathfrak{p}}} \in K_0$, which implies the claim.

Since $\mathbb{A}_{K_0}$ is a topological group with respect to addition and C is a neighbourhood of 0, the subring K_0 is discrete in $\mathbb{A}_{K_0}$ by (i). The map $\pi : \mathbb{A}_{K_0} \to \mathbb{A}_{K_0}/K_0$ is continuous. By (ii) we have $\pi(\mathbb{A}_{K_0}) = \pi(C)$. The compactness of C implies therefore the compactness of $\mathbb{A}_{K_0}/K_0$. □

Corollary 2.3.2 *Let K be a global field. There exists a set $C \subseteq \mathbb{A}_K$ of the form $C = \{(x_\mathfrak{p}) \in \mathbb{A}_K : |x_\mathfrak{p}|_\mathfrak{p} \leq \delta_\mathfrak{p}\}$ with $\delta_\mathfrak{p} = 1$ for almost all places $\mathfrak{p}$ such that $C + K = \mathbb{A}_K$, i. e., every adele $z \in \mathbb{A}_K$ can be written as $z = x + y$ with $x \in C$ and $y \in K$.*

Proof. Let $K_0 = \mathbb{Q}$ if $\mathrm{char}(K) = 0$ and $K_0 = \mathbb{F}_p(s)$ if $\mathrm{char}(K) = p$, where s is a separating element for K. By the proof of the above Theorem there exists a compact set $C_0 \subseteq \mathbb{A}_{K_0}$ with $\mathbb{A}_{K_0} = K_0 + C_0$ and $C_0 = \{(x_\mathfrak{p}) \in A_{K_0} : |x_\mathfrak{p}|_\mathfrak{p} \leq \delta_\mathfrak{p}\}$ with $\delta_\mathfrak{p} = 1$ for almost all $\mathfrak{p} \in V_{K_0}$. Let $\{b_1, \cdots, b_n\}$ be a K_0-basis of K and define

$$\psi : \prod_{i=1}^{n} \mathbb{A}_{K_0} \to \mathbb{A}_K, \qquad ((x_{\mathfrak{p},i})_{\mathfrak{p}\in V_{K_0}})_{1\leq i\leq n} \mapsto \sum_{i=1}^{n} b_i (x_{\mathfrak{p},i})_{\mathfrak{P}\in V_K, \mathfrak{P}|\mathfrak{p}}.$$

Then ψ is an isomorphism of topological groups by the Remark after Theorem 2.2.5 with $\psi(K_0^n) = K$. This implies

$$\psi^{-1}(K) + \prod_{i=1}^{n} C_0 = \prod_{i=1}^{n} (K_0 + C_0) = \prod_{i=1}^{n} A_{K_0}.$$

We have $\psi(\prod_{i=1}^{n} C_0) = \iota(C_0)b_1 + \iota(C_0)b_2 + \cdots + \iota(C_0)b_n$, where $\iota : \mathbb{A}_{K_0} \to \mathbb{A}_K$ is the embedding of $\mathbb{A}_{K_0}$ into $\mathbb{A}_K$ given by $(x_\mathfrak{p})_{\mathfrak{p}\in V_{K_0}} \mapsto (x_\mathfrak{p})_{\mathfrak{P}\in V_K, \mathfrak{P}|\mathfrak{p}}$. Since we have $|b_i|_\mathfrak{P} \leq 1$ for $1 \leq i \leq n$ and for almost places $\mathfrak{P}$ of K there exists a compact subset $C' \subseteq \mathbb{A}_K$ of the form $C' = \{(x_\mathfrak{P}) \in \mathbb{A}_K : |x_\mathfrak{P}|_\mathfrak{P} \leq \delta'_\mathfrak{P}\}$ with $\delta'_\mathfrak{P} = 1$ for almost all $\mathfrak{P} \in V_K$ and $\psi(\prod_{i=1}^{n} C_0) \subseteq C'$. The set C' satisfies obviously $C' + K = \mathbb{A}_K$. □

Remark Let K be a global field. As a consequence of Proposition 2.1.3 there exists a Haar measure μ on the additive group $(\mathbb{A}_K, +)$ with $\mu(x \cdot B) = |x|_{\mathbb{A}_K} \mu(B)$ for every $x \in \mathbb{A}_K$ and every Borel subset B of $\mathbb{A}_K$. Let μ' be the Haar measure on $\mathbb{A}_K/K$ such that Proposition **??** in the Appendix holds without rescaling μ'.

Lemma 2.3.3 *Let K be a global field. Then there exists a constant c_K only depending on the field K with the following property:*
Let $y \in \mathbb{A}_K$ and $|y|_{\mathbb{A}_K} > c_K$ then there exists an element $x \in K^*$ with $|x|_\mathfrak{p} \leq |y_\mathfrak{p}|_\mathfrak{p}$ for every place $\mathfrak{p}$.

Proof. Let $c_0 = \mu'(\mathbb{A}_K/K)$ and

$$c_1 = \mu\left(\left\{(z_\mathfrak{p}) \in \mathbb{A}_K : |z_\mathfrak{p}|_\mathfrak{p} \leq \begin{cases} 1 & \mathfrak{p} \nmid \infty, \\ \frac{1}{2} & \mathfrak{p}|\infty, \end{cases}\right\}\right).$$

We obtain $0 < c_0, c_1 < \infty$ since both sets are compact. Set $c_K := \frac{c_0}{c_1}$ and let T be the set

$$T = \left\{(z_\mathfrak{p}) \in \mathbb{A}_K : |z_\mathfrak{p}|_\mathfrak{p} \leq \begin{cases} y_\mathfrak{p} & \mathfrak{p} \nmid \infty, \\ \frac{1}{2} y_\mathfrak{p} & \mathfrak{p}|\infty, \end{cases}\right\},$$

for an adele $(y_\mathfrak{p}) \in \mathbb{A}_K$ with $|(y_\mathfrak{p})|_{\mathbb{A}_K} > c_K$. The set T has Haar measure

$$\mu(T) = |(y_\mathfrak{p})|_{\mathbb{A}_K} c_1 > c_K c_1 = c_0 = \mu'(\mathbb{A}_K/K).$$

By Proposition **??** there exist two distinct elements $t, t' \in T$ with $t \equiv t' \mod K$. For the element $x = t - t' \in K$ holds $|x|_\mathfrak{p} = |t_\mathfrak{p} - t'_\mathfrak{p}|_\mathfrak{p} \leq |y_\mathfrak{p}|_\mathfrak{p}$ for all places $\mathfrak{p}$ of K. □

Corollary 2.3.4 *Let K be a global field, $\mathfrak{p}_0$ an arbitrary place of K and $\delta_\mathfrak{p} > 0$ real numbers for every place $\mathfrak{p} \neq \mathfrak{p}_0$ with $\delta_\mathfrak{p} = 1$ for almost all places $\mathfrak{p}$. Then there exists an element $x \in K^*$ with*

$$|x|_\mathfrak{p} \leq \delta_\mathfrak{p}, \quad \mathfrak{p} \neq \mathfrak{p}_0.$$

Proof. Choose $(y_\mathfrak{p}) \in \mathbb{A}_K$ such that $|y_\mathfrak{p}|_\mathfrak{p} \leq \delta_\mathfrak{p}$ for all places $\mathfrak{p} \neq \mathfrak{p}_0$ and $y_\mathfrak{p}$ such that $|y|_{\mathbb{A}_K} > c_K$, where c_K is as above. Then there exists by the previous Lemma an element $x \in K^*$ with

$$|x_\mathfrak{p}|_\mathfrak{p} \leq |y_\mathfrak{p}|_\mathfrak{p} \leq \delta_\mathfrak{p}, \quad \textit{for } \mathfrak{p} \neq \mathfrak{p}_0. \qquad \square$$

Theorem 2.3.5 *Let K be a global field $(y_\mathfrak{p}) \in \mathbb{A}_K$ and $\mathfrak{p}_0$ an arbitrary place of K. Then there exists for every $\epsilon > 0$ and for every finite set S of places containing all infinite places and $\mathfrak{p}_0 \notin S$ an element $x \in K$ with*

(i) $|x - y_\mathfrak{p}|_\mathfrak{p} < \epsilon$ for $\mathfrak{p} \in S$,
(ii) $x \in \mathcal{O}_\mathfrak{p}$ for $\mathfrak{p}_0 \neq \mathfrak{p} \notin S$.

Proof. According to Corollary **??** there is a set

$$C := \{(z_\mathfrak{p}) \in \mathbb{A}_K : |z_\mathfrak{p}|_\mathfrak{p} \leq \delta_\mathfrak{p}\},$$

with $\delta_\mathfrak{p} = 1$ for almost all places $\mathfrak{p}$, such that $\mathbb{A}_K = K + C$. By Corollary 2.3.4 there is an element $\lambda \in K^*$ with $|\lambda|_\mathfrak{p} \leq \epsilon\delta_\mathfrak{p}^{-1}$ for $\mathfrak{p} \in S$ and $|\lambda|_\mathfrak{p} \leq \delta_\mathfrak{p}^{-1}$ for $\mathfrak{p} \notin S$ and $\mathfrak{p} \neq \mathfrak{p}_0$. We set

$$\tilde{y}_\mathfrak{p} = \begin{cases} y_\mathfrak{p} & \mathfrak{p} \in S \\ 0 & \mathfrak{p} \notin S \end{cases}.$$

Then there exist $x \in K$ and $(z_\mathfrak{p}) \in C$ with $\lambda^{-1}(\tilde{y}_\mathfrak{p}) = x + (z_\mathfrak{p})$. The element $\lambda x \in K$ fulfils obviously the requested properties. $\square$

Remark Theorem 2.3.5 can also be formulated in the following way:
Let $\mathfrak{p}_0$ be an arbitrary place of a global field K. Then the embedding $x \mapsto (x)_{v \in V_K \setminus \{\mathfrak{p}_0\}}$ form K into $\prod'_{\mathfrak{p} \neq \mathfrak{p}_0}(K_\mathfrak{p} : \mathcal{O}_\mathfrak{p})$ is dense.

3. Height functions

We are now ready to deal with heights. In the first section we look at the *absolute multiplicative / logarithmic height function* on the projective space $\mathbb{P}^n(\overline{K})$ over the algebraic closure of a global field K. This will be the last preparation for our main goal: the treatment of height functions on $\mathbb{A}^n_K$ and Theorem 3.2.7 together with its proof.

3.1. Height functions on the projective space

For this section we followed mostly [4] and also [1].

Definition 3.1.1 *Let K be a global field and $\mathbb{P}^n(K)$ be the n-dimensional projective space over K. For a point $P \in \mathbb{P}^n(K)$ with homogeneous coordinates $P = [x_0 : \cdots : x_n]$ we define the* **multiplicative height** *relative to K as*

$$H_K(P) := \prod_{\mathfrak{p} \in V_K} \max\{|x_0|_{\mathfrak{p}}, \cdots, |x_n|_{\mathfrak{p}}\}.$$

The **logarithmic height** *relative to K is defined via*

$$h_K(P) := \log(H_K(P)) = \sum_{\mathfrak{p} \in V_K} -f_{\mathfrak{p}} \log(\mathfrak{N}(\mathfrak{p})) \min\{v_{\mathfrak{p}}(x_0), \cdots, v_{\mathfrak{p}}(x_n)\},$$

where $f_{\mathfrak{p}}$ and $\mathfrak{N}(\mathfrak{p})$ are defined as in Definition 1.3.10.

We have to show that these height functions exist and are well-defined. The latter will be done in Lemma 3.1.3. Let $[x_0 : \cdots : x_n]$ be homogeneous coordinates of a point $P \in \mathbb{P}^n(K)$. As a consequence of Theorem 1.3.12 for almost all $\mathfrak{p} \in V_K$ holds $|x_i|_{\mathfrak{p}} = 1$ for $0 \leq i \leq n$. Since $P \in \mathbb{P}^n(K)$ there exists an index i with $x_i \neq 0$ which implies $\max\{|x_0|_{\mathfrak{p}}, \cdots, |x_n|_{\mathfrak{p}}\} > 0$ for all places $\mathfrak{p} \in V_K$. The product $\prod_{\mathfrak{p} \in V_K} \max\{|x_0|_{\mathfrak{p}}, \cdots, |x_n|_{\mathfrak{p}}\}$ is therefore a finite product and greater than 0. Hence the height $H_K(P)$ and $h_k(P) = \log(H_K(P))$ exist.

Examples 3.1.2 (i) Let $P \in \mathbb{P}^n(\mathbb{Q})$ be a point in the projective space over $\mathbb{Q}$. Then there are up to multiplication with (-1) unique homogeneous coordinates $[x_0 : \cdots : x_n]$ for P with $x_0, \cdots, x_n \in \mathbb{Z}$ and $gcd(x_0, \cdots, x_n) = 1$. For every prime number p we have therefore $\max\{|x_0|_p, \cdots, |x_n|_p\} = 1$. Hence the multiplicative height of P is given by

$$H_{\mathbb{Q}}(P) = \max\{|x_0|_\infty, \cdots, |x_n|_\infty\}.$$

(ii) For $P \in \mathbb{P}^n(\mathbb{F}_p(x))$ we can choose homogeneous coordinates $P = [f_0 : \cdots : f_n]$ of P with $f_0, \cdots, f_n \in \mathbb{F}_p[x]$ and $(f_0, \cdots, f_n) = \mathbb{F}_p[x]$. Let $q \in \mathbb{F}_p[x]$ be an irreducible polynomial, then the choice of the homogeneous coordinates of P implies $\max\{|f_0|_q, \cdots, |f_n|_q\} = 1$. Hence we obtain for the height of $P = [f_0 : \cdots : f_n]$

$$H_{\mathbb{F}_p(x)}(P) = \max\{|f_0|_\infty, \cdots, |f_n|_\infty\}.$$

Conclusion: With the above considerations it is obvious that for every real number $B \geq 0$ there are only finitely many points $P \in \mathbb{P}^n(K_0)$ with $H_{K_0}(P) \leq B$ for $K_0 = \mathbb{Q}$ or $K_0 = \mathbb{F}_p(x)$.

Lemma 3.1.3 *Let K be a global field.*

(i) The height of a point $P \in \mathbb{P}^n(K)$ is well-defined, i. e., it is independent of the choice of the homogeneous coordinates of P.
(ii) $H_K(P) \geq 1$ for all points $P \in \mathbb{P}^n(K)$.
(iii) Let $L|K$ be a separable field extension. Then we have for all points $P \in \mathbb{P}^n(K)$

$$H_L(P) = H_K(P)^{[L:K]}.$$

Proof. (i) Let $[x_0 : \cdots : x_n]$ and $[\lambda x_0 : \cdots : \lambda x_n]$ be two homogeneous coordinates for a point $P \in \mathbb{P}^n(K)$ with $\lambda \in K^*$. Then we have

$$\begin{aligned} H_K([\lambda x_0 : \cdots : \lambda x_n]) &= \prod_{\mathfrak{p} \in V_K} \max\{|\lambda x_0|_\mathfrak{p}, \cdots, |\lambda x_n|_\mathfrak{p}\} = \\ &= \prod_{\mathfrak{p} \in V_K} |\lambda|_\mathfrak{p} \max\{|x_0|_\mathfrak{p}, \cdots, |x_n|_\mathfrak{p}\}. \end{aligned}$$

Since the product is finite and $|\lambda|_{\mathbb{A}_K} = 1$ by Theorem 1.3.12 we obtain

$$H_K([\lambda x_0 : \cdots : \lambda x_n]) = |\lambda|_{\mathbb{A}_K} H_K([x_0 : \cdots : x_n]) = H_K([x_0 : \cdots : x_n]).$$

(ii) Let $P \in \mathbb{P}^n(K)$. There exist homogeneous coordinates $[x_0 : \cdots : x_n]$ of P with $x_i = 1$ for an index i. This implies

$$H_K(P) = \prod_{\mathfrak{p} \in V_K} \max\{1, |x_0|_\mathfrak{p}, \cdots, |x_n|_\mathfrak{p}\} \geq 1.$$

(iii) Let $P \in \mathbb{P}^n(K)$ with homogeneous coordinates $P = [x_0 : \cdots, : x_n]$. For an element $x \in K$ and $\mathfrak{p}, \mathfrak{P}$ places of K and L respectively with $\mathfrak{P}|\mathfrak{p}$ Proposition 1.3.11 states

$$|x|_\mathfrak{P} = |N_{L_\mathfrak{P}|K_\mathfrak{p}}(x)|_\mathfrak{p} = |x|_\mathfrak{p}^{[L_\mathfrak{P}:K_\mathfrak{p}]}.$$

Using this and Proposition 1.3.9 we obtain

$$\begin{aligned} H_L(P) &= \prod_{\mathfrak{P} \in V_L} \max\{|x_0|_\mathfrak{P}, \cdots, |x_n|_\mathfrak{P}\} = \\ &= \prod_{\mathfrak{p} \in V_K} \prod_{\mathfrak{P}|\mathfrak{p}} \max\{|x_0|_\mathfrak{p}^{[L_\mathfrak{P}:K_\mathfrak{p}]}, \cdots, |x_n|_\mathfrak{p}^{[L_\mathfrak{P}:K_\mathfrak{p}]}\} = \\ &= \prod_{\mathfrak{p} \in V_K} \prod_{\mathfrak{P}|\mathfrak{p}} \max\{|x_0|_\mathfrak{p}, \cdots, |x_n|_\mathfrak{p}\}^{[L_\mathfrak{P}:K_\mathfrak{p}]} = \\ &= \prod_{\mathfrak{p} \in V_K} \max\{|x_0|_\mathfrak{p}, \cdots, |x_n|_\mathfrak{p}\}^{[L:K]} = H_K(P)^{[L:K]}. \end{aligned}$$

□

Definition 3.1.4 *Let* $K_0 = \mathbb{Q}$ *or* $K_0 = \mathbb{F}_p(x)$*. For a point* $P \in \mathbb{P}^n(\overline{K_0})$ *and an algebraic field extension* $K|K_0$ *we say* P **is defined over** K *if there are homogeneous coordinates* $P = [x_0 : \cdots : x_n]$ *with* $x_0, \cdots, x_n \in K$*. We define* $K_0(P)$ *as the smallest field such that* P *is defined over* $K_0(P)$*.*

The **absolute multiplicative height function** *is the map*

$$H : \mathbb{P}^n(\overline{K_0}) \longrightarrow [1, \infty), \quad P \mapsto H_{K_0(P)}(P)^{\frac{1}{[K_0(P):K_0']}},$$

where K_0' *is defined as* $K_0' := K_0$ *if* $\mathrm{char}(K_0) = 0$ *and* $K_0' = \mathbb{F}_p(s)$ *if* $\mathrm{char}(K_0) = p$*, where* s *is a separating element for* K*. The* **absolute logarithmic height function** *is the map*

$$h : \mathbb{P}^n(\overline{\mathbb{Q}}) \longrightarrow [0, \infty), \quad P \mapsto \log(H(P)) = \frac{1}{[K_0(P) : K_0']} h_{K_0(P)}(P),$$

where K_0' *is defined as above.*

Lemma 3.1.3 implies that the absolute multiplicative/logarithmic height function is well-defined. Let $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\overline{K_0})$ be homogeneous coordinates. One proves easily

$$K_0(P) = K_0(\frac{x_0}{x_i}, \cdots \frac{x_n}{x_i}),$$

for any $x_i \neq 0$.

Lemma 3.1.5 *Let* K_0 *be either* $\mathbb{Q}$ *or* $\mathbb{F}_p(x)$ *for a prime number* p*,* $P \in \mathbb{P}^n(\overline{K_0})$ *and* $\sigma \in \mathrm{Gal}(\overline{K_0}, K_0)$ *then* $H(P) = H(\sigma(P))$ *where* $\sigma(P) = \sigma([x_0 : \cdots : x_n]) := [\sigma(x_0) : \cdots : \sigma(x_n)]$*.*

Remark The field $\overline{\mathbb{F}_p(x)}|\mathbb{F}_p(x)$ for a prime number p is not separable and therefore not galois. The group $\mathrm{Gal}(\overline{\mathbb{F}_p(x)}, \mathbb{F}_p(x))$ stands in this case for the group $\mathrm{Aut}_{\mathbb{F}_p(x)}(\overline{\mathbb{F}_p(x)})$ of $\mathbb{F}_p(x)$-automorphisms of $\overline{F_p(x)}$.

Proof. Set $K = K_0(P)$ and $K_0' := K_0$ if $\mathrm{char}(K_0) = 0$ and $K_0' := \mathbb{F}_p(s)$ if $\mathrm{char}(K_0) = p$ where s is a separating element for K. The map σ induces an isomorphism $\sigma : K \to \sigma(K)$ and the restriction of σ onto K_0' is the identity $id_{K_0'}$. The map σ also induces a bijection

$$\sigma : V_K \to V_{\sigma(K)}, \quad \mathfrak{p} \mapsto \sigma(\mathfrak{p}),$$

where $\sigma(\mathfrak{p})$ is the place of $\sigma(K)$ containing the absolute value $|x|_{\sigma(\mathfrak{p})} := |\sigma^{-1}(x)|_{\mathfrak{p}}$ for $x \in \sigma(K)$. The absolute value $|\ |_{\sigma(\mathfrak{p})}$ is then clearly the normed absolute value to $\sigma(\mathfrak{p})$. Therefore we obtain:

$$H(P) = H_K(P)^{\frac{1}{[K:K_0']}} = \left(\prod_{\mathfrak{p}\in V_K} \max\{|x_0|_{\mathfrak{p}}, \cdots, |x_n|_{\mathfrak{p}}\}\right)^{\frac{1}{[K:K_0']}} =$$

$$= \left(\prod_{\sigma(\mathfrak{p})\in V_{\sigma(K)}} \max\{|\sigma(x_0)|_{\sigma(\mathfrak{p})}, \cdots, |\sigma(x_n)|_{\sigma(\mathfrak{p})}\}\right)^{\frac{1}{[\sigma(K):K_0']}} =$$

$$= H_{\sigma(K)}(\sigma(P))^{\frac{1}{[\sigma(K):K_0']}} = H(\sigma(P)). \qquad \square$$

For an element $x \in K$ of a global field K we define $H_K(x) := H_K([1 : x]) = \prod_{\mathfrak{p}\in V_K} \max\{1, |x|_{\mathfrak{p}}\}$ and $H(x) := H([1 : x])$.

Theorem 3.1.6 *Let K_0 be either $\mathbb{Q}$ or $\mathbb{F}_p(x)$, let $B, D > 0$ be real numbers and n an integer. Then the set*

$$P(K_0, B, D) := \{P \in \mathbb{P}^n(\overline{K_0}) : H(P) \le B, [K_0(P) : K_0] \le D, \\ K_0(P)|K_0 \textit{ is separable}\}$$

is finite.

Proof. Let $P \in \mathbb{P}^n(\overline{K_0})$ with $K_0(P)|K_0$ separable and choose homogeneous coordinates $P = [x_0 : \cdots : x_n]$ such that $x_j = 1$ for an index j. For every place $\mathfrak{p} \in V_{K_0(P)}$ we have

$$\max\{|x_0|_{\mathfrak{p}}, \cdots, |x_{j-1}|_{\mathfrak{p}}, 1, |x_{j+1}|_{\mathfrak{p}}, \cdots |x_n|_{\mathfrak{p}}\} \ge \max\{1, |x_i|_{\mathfrak{p}}\},$$

for $0 \le i \le n$. By multiplying over all places $\mathfrak{p} \in V_{K_0(P)}$ and taking the $[K_0(P) : K_0]$-th root we obtain

$$H(P) \ge H(x_i),$$

for all $0 \le i \le n$. It suffices to prove that the set

$$C(K_0, B, d) := \{y \in \overline{K_0} : H(y) \le B, [K_0(y) : K_0] = d, y \textit{ is separable over } K_0\},$$

is finite for $1 \le d \le D$. Indeed, if $H(P) \le B$ then its coordinates $x_0, \cdots, x_n$ lie in $C(K_0, B, d)$ for an appropriate d, if we choose homogeneous coordinates for P as above. If the sets $C(K_0, B, d)$ are finite for $1 \le d \le D$ there are only finitely many possibilities to choose homogeneous coordinates for a point $P \in \mathbb{P}^n(\overline{K_0})$ such that one coordinate is 1 and the other coordinates are in the above set.

Let $y \in \overline{K_0}$ be separable over K_0 and

$$m_y(T) = \prod_{1\le i\le d} (T - y_i) = \sum_{r=0}^{d} (-1)^r s_r(y) T^{d-r},$$

the minimal polynomial of y over K_0, where $y = y_1$ and $s_r(y)$ is the r-th symmetric polynomial in $y_1, \cdots, y_d$. We set $K = K_0(y)$. Let $\mathfrak{p}$ be a place of

K, then we obtain

$$|s_r(y)|_{\mathfrak{p}} = \left| \sum_{1 \le i_1 < \cdots < i_r \le d} y_{i_1} \cdots y_{i_r} \right|_{\mathfrak{p}} \le c(\mathfrak{p}, r, d) \max_{1 \le i_1 < \cdots < i_r \le d} \{ |y_{i_1} \cdots y_{i_r}|_{\mathfrak{p}} \} \le$$
$$\le c(\mathfrak{p}, r, d) \max_{1 \le i \le d} \{ |y_i|_{\mathfrak{p}}^r \},$$

where $c(\mathfrak{p}, r, d) = \binom{d}{r} \le 2^d$ for $\mathfrak{p}$ archimedean and $c(\mathfrak{p}, r, d) = 1$ for $\mathfrak{p}$ non-archimedean. We get

$$\max\{|s_0(y)|_{\mathfrak{p}}, \cdots, |s_d(y)|_{\mathfrak{p}}\} \le c(\mathfrak{p}, d) \prod_{i=1}^{d} \max\{1, |y_i|_{\mathfrak{p}}\}^d,$$

where $c(\mathfrak{p}, d) = 2^d$ for $\mathfrak{p}$ archimedean and $c(\mathfrak{p}, d) = 1$ for $\mathfrak{p}$ non-archimedean. As a consequence of Example 1.3.5 there are at most $[K : K_0] = d$ archimedean places of K. Multiplying over all places $\mathfrak{p} \in V_K$ we obtain

$$H_K([s_0(y) : \cdots : s_d(y)]) \le 2^{d^2} \prod_{i=1}^{d} H_K(y_i)^d.$$

Lemma 3.1.5 implies $H_K(y_1) = H_K(y_2) = \cdots = H_K(y_d)$. Hence we obtain

$$H([s_0(y) : \cdots : s_d(y)]) \le 2^d H(y)^{d^2}.$$

If y satisfies $H(y) \le B$ then the Point $P_y = [s_0(y) : \cdots : s_d(y)] \in \mathbb{P}^d(K_0)$ satisfies $H(P_y) \le 2^d B^{d^2}$. By the conclusion in Example 3.1.2 there are only finitely many Points $P \in \mathbb{P}^d(K_0)$ satisfying $H(P) \le 2^d B^{d^2}$ and therefore only finitely many separable $y \in \overline{K_0}$ satisfying $H(y) \le B$. □

Corollary 3.1.7 *Let K be a global field, $B \ge 0$ a real number and n a natural number. The set $\{P \in \mathbb{P}^n(K) : H_K(P) \le B\}$ is finite.*

Proof. Follows directly of Theorem 3.1.6. □

3.2. Adelic norms and height functions on the adele ring

We are interested in height functions on the $\mathbb{A}_K$-module $\mathbb{A}_K^n$ for a global field K and their properties which are stated in Theorem 3.2.7 (cf. [3]). For the concepts of *adelic norms* and *primitive elements* we followed [9] and [3].

For the rest of the thesis we identify $\mathbb{A}_K^n$ with $\prod'_{\mathfrak{p} \in V_K} (K_{\mathfrak{p}}^n : \mathcal{O}_{\mathfrak{p}}^n)$ as topological $\mathbb{A}_K$-modules by the map

$$\varphi : ((x_{\mathfrak{p},i})_{\mathfrak{p} \in V_K})_{1 \le i \le n} \mapsto ((x_{\mathfrak{p},i})_{1 \le i \le n})_{\mathfrak{p} \in V_K}.$$

Indeed this is a well-defined isomorphism of $\mathbb{A}_K$-modules, whereby the operation of $\mathbb{A}_K$ onto $\prod'_{\mathfrak{p} \in V_K} (K_{\mathfrak{p}}^n : \mathcal{O}_{\mathfrak{p}}^n)$ is given by

$$(\lambda_{\mathfrak{p}}) \cdot (x_{\mathfrak{p},i}) = (\lambda_{\mathfrak{p}} x_{\mathfrak{p},i}).$$

Since the topology on $\mathbb{A}_K^n$ is the product topology, a basis for the open subsets of $\mathbb{A}_K^n$ are all sets of the form $\prod_{1\le i\le n, \mathfrak{p}\in V_K} O_{i,\mathfrak{p}}$, where $O_{i,\mathfrak{p}} \subseteq K_\mathfrak{p}$ is open in $K_\mathfrak{p}$ and for every $1 \le i \le n$ we have $O_{i,\mathfrak{p}} = \mathcal{O}_\mathfrak{p}$ for almost all $\mathfrak{p} \in V_K$. A basis for the open subsets of $\prod'_{\mathfrak{p}\in V_K}(K_\mathfrak{p}^n : \mathcal{O}_\mathfrak{p}^n)$ are all sets of the form $\prod_{\mathfrak{p}\in V_K} U_\mathfrak{p}$ where $U_\mathfrak{p} \subseteq K_\mathfrak{p}^n$ is open and $U_\mathfrak{p} = \mathcal{O}_\mathfrak{p}^n$ for almost all $\mathfrak{p} \in V_K$. The morphism ϕ and its inverse ϕ^{-1} map a basis of the open sets of one space onto a basis of the open sets of the other space. Hence the map φ is a homeomorphism.

The above identification induces a map

$$\tilde{\varphi} : GL_n(\mathbb{A}_K) \longrightarrow \prod_{\mathfrak{p}\in V_K}{}' \left(GL_n(K_\mathfrak{p}) : GL_n(\mathcal{O}_\mathfrak{p})\right),$$

whereby $GL_n(R)$ stands for the group of R-module automorphisms of R^n for an commutative ring R. The map is given by $\tilde{\varphi} : g \mapsto (g|_{K_\mathfrak{p}^n})_{\mathfrak{p}\in V_K}$ and is an isomorphism of local-compact groups.

Remark The identification φ form above could be formulated more generally:
Let $\{b_1, \cdots, b_n\}$ be an $\mathbb{A}_K$-basis of $\mathbb{A}_K^n$. We define another identification $\varphi' : \mathbb{A}_K^n \to \prod'_{\mathfrak{p}\in V_K}(K_\mathfrak{p}^n : \mathcal{O}_\mathfrak{p}^n)$ by mapping each element $x \in \mathbb{A}_K^n$ onto $\varphi(\lambda_1, \cdots, \lambda_n)$, where $x = \sum_{i=1}^n \lambda_i b_i$ with $\lambda_i \in \mathbb{A}_K$ for $1 \le i \le n$. Let $g \in GL_n(\mathbb{A}_K)$ be the matrix with $b_1, \cdots, b_n$ as columns, then the map φ' is given by

$$\varphi' = (x \mapsto g^{-1} \cdot x) \circ \varphi.$$

The purpose of these identifications is to provide a simple way to formulate the definition of a height function H on $\mathbb{A}_K^n$. Lemma 3.2.6 implies that every height function on $\mathbb{A}_K^n$ for whose definition the identification φ is used can also be defined using the identification φ' and vice versa. Therefore the choice of this special identification means no loss of generality.

Definition 3.2.1 *Let K be a global field. A family $\mathcal{F} = \{\| \ \|_\mathfrak{p}\}_{\mathfrak{p}\in V_K}$ of norms $\| \ \|_\mathfrak{p} : K_\mathfrak{p}^n \to \mathbb{R}$ is called* **adelic norm** *on the $\mathbb{A}_K$-module $\mathbb{A}_K^n$ iff for almost all places $\mathfrak{p}$ holds $\|(x_1, \cdots, x_n)\|_\mathfrak{p} = \max\{|x_1|_\mathfrak{p}, \cdots, |x_n|_\mathfrak{p}\}$ for all $(x_1, \cdots, x_n) \in K_\mathfrak{p}^n$. We call the map*

$$H_\mathcal{F} : \mathbb{A}_K^n \longrightarrow \mathbb{R} : \quad (x_{\mathfrak{p},i}) \mapsto \prod_{\mathfrak{p}\in V_K} \|(x_{\mathfrak{p},1}, \cdots, x_{\mathfrak{p},n})\|_\mathfrak{p},$$

the **height function** *on $\mathbb{A}_K^n$ associated to $\mathcal{F}$.*

Since $x_{\mathfrak{p},1}, \cdots, x_{\mathfrak{p},n} \in \mathcal{O}_\mathfrak{p}$ for almost all $\mathfrak{p} \in V_K$ and $\|(x_{\mathfrak{p},1}, \cdots, x_{\mathfrak{p},n})\|_\mathfrak{p} = \max\{|x_{\mathfrak{p},1}|_\mathfrak{p}, \cdots, |x_{\mathfrak{p},n}|_\mathfrak{p}\}$ for almost all $\mathfrak{p} \in V_K$ the product

$$\prod_{\mathfrak{p}\in V_K} \|(x_{\mathfrak{p},1}, \cdots, x_{\mathfrak{p},n})\|_\mathfrak{p}$$

is finite. Hence the height function associated to an adelic norm is a well-defined map.

Remark (i) The concept of an *adelic norm* can be found e. g. in [9]. The above definition is however inspired by the definition of height functions in [3] and is hence different from the definition given in [9] which is the following:

An **adelic norm** *is a family* $\mathcal{F} = \{\| \ \|_{\mathfrak{p}}\}_{\mathfrak{p}\in V_K}$ *of norms* $\| \ \|_{\mathfrak{p}} : K_{\mathfrak{p}}^n \to \mathbb{R}$ *such that there exists a* K*-lattice* Λ *in* K^n *such that for almost all* $\mathfrak{p} \in V_K$ *and all* $x \in K_{\mathfrak{p}}^n$ *holds*

$$\|x\|_{\mathfrak{p}} = \inf_{\lambda\in K_{\mathfrak{p}}^*, \lambda x\in\Lambda_{\mathfrak{p}}} |\lambda|_{\mathfrak{p}}^{-1},$$

where $\Lambda_{\mathfrak{p}}$ *is the closure of* Λ *in* $K_{\mathfrak{p}}^n$*.*

We want to show that these two definitions are equivalent:

First Assume the now given definition. We can write $\Lambda = \bigoplus_{1\le i\le n} o_K \cdot b_i$, where $\{b_1, \cdots, b_n\}$ is an o_K-basis of Λ with $b_i = (b_{i,j}) \in K^n$. Let S be the finite set of places $\mathfrak{p} \in V_K$ such that $V_K \setminus S$ is the set of all non-archimedean places $\mathfrak{p}$ with $|b_{i,j}|_{\mathfrak{p}} = 1$ for $1 \le i, j \le n$ and the norm $\| \ \|_{\mathfrak{p}}$ is given as above. Let $K_{\mathfrak{p}}^n \setminus \{0\} \ni x = (x_i) = \sum_{1\le i\le n} \alpha_i b_i$. One has the following equivalences

$$\begin{aligned}\lambda x \in \Lambda_{\mathfrak{p}} \Leftrightarrow \lambda\alpha_i \in o_{\mathfrak{p}} \textit{ for all } i \Leftrightarrow |\lambda\alpha_i|_{\mathfrak{p}} \le 1 \textit{ for all } i \Leftrightarrow \\ \Leftrightarrow |\alpha_i|_{\mathfrak{p}} \le |\lambda|_{\mathfrak{p}}^{-1} \textit{ for all } i,\end{aligned}$$

where $\lambda \in K_{\mathfrak{p}}^*$. Hence we obtain for the norm of x

$$\begin{aligned}\|x\|_{\mathfrak{p}} &= \inf_{\lambda\in K_{\mathfrak{p}}^*, \lambda x\in\Lambda_{\mathfrak{p}}} |\lambda|_{\mathfrak{p}}^{-1} = \max_{1\le i\le n} \{|\alpha_i|_{\mathfrak{p}}\} = \\ &= \max_{1\le i\le n} \{|\sum_{1\le j\le n} \alpha_j b_{i,j}|_{\mathfrak{p}}\} = \max_{1\le i\le n} \{|x_i|_{\mathfrak{p}}\}.\end{aligned}$$

Now assume our definition of an adelic norm. Let $S \subseteq V_K$ be the finite set such that $V_K \setminus S$ is the set of all places $\mathfrak{p}$ of K with $\|(x_i)\|_{\mathfrak{p}} = \max_{1\le i\le n} |x_i|_{\mathfrak{p}}$ for all $(x_i) \in K_{\mathfrak{p}}^n$. Set $\Lambda = \bigoplus_{1\le i\le n} o_K \cdot b_i$ where the $b_i = (b_{i,j})$ form a K-basis of K^n with $|b_{i,j}|_{\mathfrak{p}} = 1$ for all $1 \le i, j \le n$ and $\mathfrak{p} \notin S$.

(ii) Let $\mathcal{F} = \{\| \ \|_{\mathfrak{p}}\}_{\mathfrak{p}\in V_K}$ be an adelic norm on $\mathbb{A}_K^n$ for a global field K. The requirement $\|(x_1, \cdots, x_n)\|_{\mathfrak{p}} = \max\{|x_1|_{\mathfrak{p}}, \cdots, |x_n|_{\mathfrak{p}}\}$ for $(x_i) \in K_{\mathfrak{p}}^n$ and for almost all places $\mathfrak{p}$ of K is very restrictive. Since this requirement is not necessary for a well-defined height function, as we will see in Example 3.2.9, one can ask oneself if it is possible to weaken it in a way such that one would obtain more general height functions which enjoy the same properties as stated in Theorem 3.2.7. We will state a possible generalisation as closure for this thesis and proof in Theorem 3.2.10 that Theorem 3.2.7 holds also in this more general setting.

Examples 3.2.2 (i) Let K be a global field and $\mathcal{F} = \{\| \ \|_{\mathfrak{p}}\}_{\mathfrak{p}\in V_K}$ be the adelic norm defined by $\|(x_1, \cdots, x_n)\|_{\mathfrak{p}} = \max\{|x_1|_{\mathfrak{p}}, \cdots, |x_n|_{\mathfrak{p}}\}$ for $(x_1, \cdots, x_n) \in K_{\mathfrak{p}}^n$ and all $\mathfrak{p} \in V_K$. The restriction of the height function $H_{\mathcal{F}}$ onto K^n induces the multiplicative height H_K on the projective space $\mathbb{P}^{n-1}(K)$. Therefore we denote the height function

$H_{\mathcal{F}}$ associated to this special adelic norm $\mathcal{F}$ also by H_K and call it ***standard height function*** on $\mathbb{A}_K^n$.

(ii) Let K be a global field and $\| \|_{\mathfrak{p}}$ be the norm on $K_{\mathfrak{p}}^n$ defined by

$$\|(x_1, \cdots, x_n)\|_{\mathfrak{p}} := \begin{cases} \left(\sum_{i=1}^n |x_i|_{\mathfrak{p}}^2\right)^{\frac{1}{2}} & \mathfrak{p}|\infty, \\ \max\{|x_1|_{\mathfrak{p}}, \cdots, |x_n|_{\mathfrak{p}}\} & \mathfrak{p} \nmid \infty. \end{cases}$$

The height $H_{\mathcal{F}}$ for $\mathcal{F} = \{\| \|_{\mathfrak{p}}\}_{\mathfrak{p} \in V_K}$ is called l^2 ***Northcott-Weil height*** (cf. [9, p. 94]).

Definition 3.2.3 *Let K be a global field. An element $x \in \mathbb{A}_K^n$ is called* **primitive** *iff there exists an $\mathbb{A}_K$-automorphism $g \in GL_n(\mathbb{A}_K)$ of $\mathbb{A}_K^n$ with $g \cdot x \in K^n \setminus \{0\}$.*

Proposition 3.2.4 *Let K be a global field and $x = ((x_{\mathfrak{p},i})_{1 \le i \le n})_{\mathfrak{p} \in V_K} \in \mathbb{A}_K^n$. Then the following are equivalent:*

(i) The element $x \in \mathbb{A}_K^n$ is primitive.

(ii) The set $\{x\}$ can be extended to an $\mathbb{A}_K$-basis of $\mathbb{A}_K^n$.

(iii) For almost all places $\mathfrak{p} \in V_K$ there is a component $x_{\mathfrak{p},i}$ of $(x_{\mathfrak{p},1}, \cdots, x_{\mathfrak{p},n})$ with $x_{\mathfrak{p},i} \in \mathcal{O}_{\mathfrak{p}}^*$.

Proof. *(i)* $\Rightarrow$ *(ii)*:
Extend $b_1 = g \cdot x$ to a K-basis $\{b_1, \cdots, b_n\}$ of K^n. By Lemma 2.2.2 the set $\{b_1, \cdots, b_n\}$ is an $\mathbb{A}_K$-basis of $\mathbb{A}_K^n$. Therefore the set $\{x, g^{-1} \cdot b_2, \cdots, g^{-1} \cdot b_n\}$ is an $\mathbb{A}_K$-basis of $\mathbb{A}_K^n$.

(ii) $\Rightarrow$ *(iii)*:
Assume the converse. Choose for every finite place $\mathfrak{p}$ an element $\lambda_{\mathfrak{p}} \in K_{\mathfrak{p}}^*$, such that the vector $\lambda_{\mathfrak{p}} \cdot (x_{\mathfrak{p},1}, \cdots, x_{\mathfrak{p},n}) \in \mathcal{O}_{\mathfrak{p}}^n$ and there is at least one index i with $\lambda_{\mathfrak{p}} x_{\mathfrak{p},i} \in \mathcal{O}_{\mathfrak{p}}^*$. By the assumption we have $\lambda_{\mathfrak{p}} \notin \mathcal{O}_{\mathfrak{p}}$ for infinitely many finite places $\mathfrak{p}$. We set $\lambda_{\mathfrak{p}} = 1$ for every infinite place $\mathfrak{p}$ and set $\tilde{x} = (\lambda_{\mathfrak{p}} x_{\mathfrak{p},i})$ which is obviously an element of $\mathbb{A}_K^n$. By *(ii)* we can extend $\{x\}$ to an $\mathbb{A}_K$-basis $\{x, b_2, \cdots b_n\}$ of $\mathbb{A}_K^n$ with $b_j = (b_{j,\mathfrak{p}})$ where $b_{j,\mathfrak{p}} \in K_{\mathfrak{p}}^n$ for $2 \le j \le n$ and all places $\mathfrak{p}$. Thus there exist $\alpha_1, \cdots, \alpha_n \in \mathbb{A}_K$ with $\tilde{x} = \alpha_1 x + \alpha_2 b_2 + \cdots + \alpha_n b_n$. Denote $\alpha_i = (\alpha_{i,\mathfrak{p}})$ for every i. We obtain for every place $\mathfrak{p} \in V_K$

$$\begin{aligned} \lambda_{\mathfrak{p}}(x_{\mathfrak{p},i})_{1 \le i \le n} &= (\tilde{x}_{\mathfrak{p},i})_{1 \le i \le n} = \\ &= \alpha_{1,\mathfrak{p}}(x_{\mathfrak{p},i})_{1 \le i \le n} + \alpha_{2,\mathfrak{p}} b_{2,\mathfrak{p}} + \cdots + \alpha_{n,\mathfrak{p}} b_{n,\mathfrak{p}}. \end{aligned}$$

This implies $\alpha_{1,\mathfrak{p}} = \lambda_{\mathfrak{p}}$ and $\alpha_{2,\mathfrak{p}} = \cdots = \alpha_{n,\mathfrak{p}} = 0$ for all places $\mathfrak{p}$, which is a contradiction since $(\lambda_{\mathfrak{p}}) \notin \mathbb{A}_K$.

(iii) $\Rightarrow$ *(i)*:
Let S be the finite set of places $\mathfrak{p}$ for which $(x_{\mathfrak{p},i})_{1 \le i \le n}$ has no component lying in $\mathcal{O}_{\mathfrak{p}}^*$. For every place $\mathfrak{p} \in S$ we extend $(x_{\mathfrak{p},i})_{1 \le i \le n}$ to a $K_{\mathfrak{p}}$-basis $\{(x_{\mathfrak{p},i})_{1 \le i \le n}, b_{2,\mathfrak{p}}, \cdots, b_{n,\mathfrak{p}}\}$ of $K_{\mathfrak{p}}^n$. For every place $\mathfrak{p} \notin S$ let $i_{\mathfrak{p}}$ denote an index such that $x_{\mathfrak{p},i_{\mathfrak{p}}} \in \mathcal{O}_{\mathfrak{p}}^*$. We choose $b_{2,\mathfrak{p}}, \cdots, b_{n,\mathfrak{p}}$ such that $\{b_{2,\mathfrak{p}}, \cdots b_{n,\mathfrak{p}}\}$ is the standard basis of $K_{\mathfrak{p}}^n$ without $e_{i_{\mathfrak{p}}}$.
Let $g_{\mathfrak{p}} \in GL_n(K_{\mathfrak{p}})$ be the matrix with the vectors $(x_{\mathfrak{p},i})_{1 \le i \le n}, b_{2,\mathfrak{p}}, \cdots, b_{n,\mathfrak{p}}$ as

columns for every place $\mathfrak{p}$. For $\mathfrak{p} \notin S$ we have $g_\mathfrak{p} \in GL_n(\mathcal{O}_\mathfrak{p})$ since the columns of $g_\mathfrak{p}$ are obvious an $\mathcal{O}_\mathfrak{p}$-basis of $\mathcal{O}_\mathfrak{p}^n$. If we set $g = (g_\mathfrak{p})_{\mathfrak{p}\in V_K} \in GL_n(\mathbb{A}_K)$ then we obtain

$$g^{-1} \cdot x = e_1 \in K^n. \qquad \square$$

Remark In the article [3] the height is only defined for primitive elements. The previous Proposition implies that the height of an element $x \in \mathbb{A}_K^n$ is non-zero iff the element is primitive.

In order to shorten the proof of Godement's Theorem and make it clearer, we need the following two Lemmas.

Lemma 3.2.5 *Let K be a global field, $\mathfrak{p}$ an arbitrary place, $\|x\|_\mathfrak{p} = \max\{|x_1|_\mathfrak{p}, \cdots |x_n|_\mathfrak{p}\}$ for $x \in K_\mathfrak{p}^n$ and let $g = (g_{i,j}) \in GL_n(K_\mathfrak{p})$. We define the maps $\| \|_{s,\mathfrak{p}}, \| \|_{i,\mathfrak{p}} : GL_n(K_\mathfrak{p}) \to \mathbb{R}^+$ by*

$$\|g\|_{s,\mathfrak{p}} := \sup_{\|x\|_\mathfrak{p}=1} \|g\cdot x\|_\mathfrak{p} = \sup_{\|x\|_\mathfrak{p}\le 1} \|g\cdot x\|_\mathfrak{p} = \sup_{x\neq 0} \frac{\|g\cdot x\|_\mathfrak{p}}{\|x\|_\mathfrak{p}},$$

$$\|g\|_{i,\mathfrak{p}} := \inf_{\|x\|_\mathfrak{p}=1} \|g\cdot x\|_\mathfrak{p} = \inf_{x\neq 0} \frac{\|g\cdot x\|_\mathfrak{p}}{\|x\|_\mathfrak{p}}.$$

Then the following holds:

(i) $\|g\|_{s,\mathfrak{p}} = \max_{1\le i\le n}\{\sum_{j=1}^n |g_{i,j}|_\mathfrak{p}\} < \infty$ *for all* $g \in GL_n(K_\mathfrak{p})$ *and* $\mathfrak{p}|\infty$.

(ii) $\|g\|_{s,\mathfrak{p}} = \max_{1\le i,j\le n}\{|g_{i,j}|_\mathfrak{p}\} < \infty$ *for all* $g \in GL_n(K_\mathfrak{p})$ *and* $\mathfrak{p} \nmid \infty$.

(iii) $\|g\|_{s,\mathfrak{p}} = \dfrac{1}{\|g^{-1}\|_{i,\mathfrak{p}}}$ *for all* $g \in GL_n(K_\mathfrak{p})$.

(iv) For a finite place $\mathfrak{p}$ and $g \in GL_n(\mathcal{O}_\mathfrak{p})$ one has

$$\|g\|_{s,\mathfrak{p}} = \|g\|_{i,\mathfrak{p}} = 1.$$

(v) The maps $\| \|_{s,\mathfrak{p}}, \| \|_{i,\mathfrak{p}}$ are continuous.

Proof. (i) $\|g\|_{s,\mathfrak{p}} = \sup_{\|x\|_\mathfrak{p}\le 1; 1\le i\le n} \left| \sum_{1\le j\le n} g_{i,j}x_j \right|_\mathfrak{p} \le$

$$\le \sup_{1\le i\le n; |x_i|_\mathfrak{p}\le 1} \sum_{1\le j\le n} |g_{i,j}|_\mathfrak{p}|x_j|_\mathfrak{p} = \max_{1\le i\le n}\{\sum_{j=1}^n |g_{i,j}|_\mathfrak{p}\}.$$

Let k be an index such that $\sum_{j=1}^n |g_{k,j}|_\mathfrak{p}$ is maximal within the set $\{\sum_{j=1}^n |g_{k,j}|_\mathfrak{p} : 1 \le i \le n\}$. If we set $y_j = \mathrm{sign}(g_{k,j})$ we obtain

$$\left| \sum_{1\le j\le n} g_{i,j}y_j \right|_\mathfrak{p} = \sum_{1\le j\le n} |g_{k,j}|_\mathfrak{p}.$$

Together with the previous inequality this implies (i).

(ii) We have

$$\|g\|_{s,\mathfrak{p}} = \sup_{1\leq i,j\leq n, |x_j|_\mathfrak{p}\leq 1} \{|g_{i,j}|_\mathfrak{p}|x_j|_\mathfrak{p}\} \leq \max_{1\leq i,j\leq n}\{|g_{i,j}|_\mathfrak{p}\}.$$

Let l, k be indices such that $|g_{l,k}|_\mathfrak{p}$ is maximal within the set $\{|g_{i,j}|_\mathfrak{p} : 1 \leq i, j \leq n\}$. If we set $x_i = 1$ for $i = k$ and otherwise $x_i = 0$ we get $\|g\|_{s,\mathfrak{p}} = |g_{l,k}|_\mathfrak{p} = \max_{1\leq i,j\leq n}\{|g_{i,j}|_\mathfrak{p}\}$ which proves the claim.

(iii) $$\|g\|_{s,\mathfrak{p}} = \sup_{x\neq 0} \frac{\|g\cdot x\|_\mathfrak{p}}{\|x\|_\mathfrak{p}} = \sup_{x\neq 0} \frac{\|x\|_\mathfrak{p}}{\|g^{-1}\cdot x\|_\mathfrak{p}} = \\ = \sup_{\|x\|_\mathfrak{p}=1} \frac{1}{\|g^{-1}\cdot x\|_\mathfrak{p}} = \frac{1}{\inf_{\|x\|_\mathfrak{p}=1}\|g^{-1}\cdot x\|_\mathfrak{p}} = \frac{1}{\|g^{-1}\|_{i,\mathfrak{p}}}.$$

(iv) As a consequence of (ii) we have $\|g\|_{s,\mathfrak{p}}, \|g^{-1}\|_{s,\mathfrak{p}} \leq 1$. Assume we have $\|g\|_{s,\mathfrak{p}} < 1$. Then (iii) implies $\|g^{-1}\|_{i,\mathfrak{p}} > 1$, but $\|g^{-1}\|_{i,\mathfrak{p}} \leq \|g^{-1}\|_{s,\mathfrak{p}} \leq 1$. Hence we obtain $\|g\|_{s,\mathfrak{p}} = \|g^{-1}\|_{s,\mathfrak{p}} = 1$ and (iii) implies $\|g\|_{i,\mathfrak{p}} = 1$.

(v) With the explicit formulas for $\|\ \|_{s,\mathfrak{p}}$ given in (i) and (ii) the map $\|\ \|_{s,\mathfrak{p}}$ is obviously continuous for arbitrary places $\mathfrak{p}$. Since the map $g \mapsto g^{-1}$ is continuous on $GL_n(K_\mathfrak{p})$ and the map $x \mapsto \frac{1}{x}$ is continuous on $\mathbb{R}^+$ the map

$$\|\ \|_{i,\mathfrak{p}} = (x \mapsto \frac{1}{x}) \circ \|\ \|_{s,\mathfrak{p}} \circ (g \mapsto g^{-1}),$$

is also continuous. □

Lemma 3.2.6 *Let K be a global field, $\mathcal{F}$ an adelic norm on $\mathbb{A}_K^n$ and $g = (g_\mathfrak{p}) \in GL_n(\mathbb{A}_K)$. Set $\|x\|'_\mathfrak{p} = \|g_\mathfrak{p} \cdot x\|_\mathfrak{p}$ for $x \in K_\mathfrak{p}^n$. Then the family $\mathcal{F}' = \{\|\ \|'_\mathfrak{p}\}_{\mathfrak{p}\in V_K}$ is also an adelic norm.*

Proof. Let $S \subseteq V_K$ be the finite set consisting of all infinite places, all places $\mathfrak{p}$, such that $\|(x_1, \cdots, x_n)\|_\mathfrak{p} \neq \max\{|x_1|_\mathfrak{p}, \cdots, |x_n|_\mathfrak{p}\}$ for all $(x_1, \cdots, x_n) \in K_\mathfrak{p}^n$ and all places $\mathfrak{p}$ such that $g_\mathfrak{p} \notin GL_n(\mathcal{O}_\mathfrak{p})$. By the previous Lemma we obtain for $\mathfrak{p} \notin S$

$$\|x\|'_\mathfrak{p} = \|g_\mathfrak{p} x\|_\mathfrak{p} = \|x\|_\mathfrak{p} = \max\{|x_1|_\mathfrak{p}, \cdots, |x_n|_\mathfrak{p}\},$$

for all $x = (x_1, \cdots, x_n) \in K_\mathfrak{p}^n$. Since S is finite $\mathcal{F}'$ is an adelic norm. □

Theorem 3.2.7 *Let K be a global field and $\mathcal{F} = \{\|\ \|_\mathfrak{p}\}_{\mathfrak{p}\in V_K}$ an adelic norm on the module $\mathbb{A}_K^n$ over the adele ring $\mathbb{A}_K$ of K.*

(i) Let $\mathcal{F}' = \{\|\ \|'_\mathfrak{p}\}_{\mathfrak{p}\in V_K}$ be another adelic norm on $\mathbb{A}_K^n$. Then there exist real numbers $c_1, c_2 > 0$ with

$$c_1 H_{\mathcal{F}'}(x) \leq H_\mathcal{F}(x) \leq c_2 H_{\mathcal{F}'}(x),$$

for all $x \in \mathbb{A}_K^n$.

(ii) $H_\mathcal{F}(t\cdot x) = |t|_{\mathbb{A}_K} \cdot H_\mathcal{F}(x)$ for all $t \in \mathbb{A}_K^*$ and $x \in \mathbb{A}_K^n$.

(iii) Let $(x_k)_{k\in\mathbb{N}} = ((x_{k,\mathfrak{p}})_{\mathfrak{p}\in V_K})_{k\in\mathbb{N}}$ *be a sequence of primitive elements* $x_k \in \mathbb{A}_K^n$ *with* $x_{k,\mathfrak{p}} \in K_{\mathfrak{p}}^n$ *and* $\lim_{k\to\infty} x_k = 0$*, then*

$$\lim_{k\to\infty} H_{\mathcal{F}}(x_k) = 0.$$

(iv) Let $(x_k)_{k\in\mathbb{N}}$ *be a sequence of primitive elements* $x_k \in \mathbb{A}_K^n$ *with* $\lim_{k\to\infty} H_{\mathcal{F}}(x_k) = 0$*. Then there exists a sequence* (λ_k) *of principal ideles* $\lambda_k \in K^*$ *such that*

$$\lim_{k\to\infty} (\lambda_k \cdot x_k) = 0.$$

(v) For given $g \in GL_n(\mathbb{A}_K)$*,* $c > 0$ *the set*

$$Z_{g,c} := \{x \in K^n : H_{\mathcal{F}}(g\cdot x) < c\}/K^*,$$

is finite.

(vi) Let $M \subseteq GL_n(\mathbb{A}_K)$ *be a compact subset. Then there exist real numbers* $c_1, c_2 > 0$ *with*

$$c_1 H_{\mathcal{F}}(x) \le H_{\mathcal{F}}(m\cdot x) \le c_2 H_{\mathcal{F}}(x),$$

for all $m \in M$ *and* $x \in \mathbb{A}_K^n$*.*

Proof. (i) Let S be the finite set of all places $\mathfrak{p}$ with $\| \ \|_{\mathfrak{p}} \neq \| \ \|'_{\mathfrak{p}}$. By Theorem 1.2.9 there exist for every $\mathfrak{p} \in S$ real numbers $c_{\mathfrak{p},1}, c_{\mathfrak{p},2} > 0$ with

$$c_{\mathfrak{p},1}\|y\|'_{\mathfrak{p}} \le \|y\|_{\mathfrak{p}} \le c_{\mathfrak{p},2}\|y\|'_{\mathfrak{p}},$$

for $y \in K_{\mathfrak{p}}^n$. By multiplying over all places we obtain

$$\prod_{\mathfrak{p}\in S} c_{\mathfrak{p},1} \cdot H_{\mathcal{F}'}(x) \le H_{\mathcal{F}}(x) \le \prod_{\mathfrak{p}\in S} c_{\mathfrak{p},2} \cdot H_{\mathcal{F}'}(x),$$

for all $x \in \mathbb{A}_K^n$.

(ii) We have for all $t \in \mathbb{A}_K^*$ and $x \in \mathbb{A}_K^n$

$$\begin{aligned} H_{\mathcal{F}}(t\cdot x) &= \prod_{\mathfrak{p}\in V_K} \|t_{\mathfrak{p}} \cdot (x_{\mathfrak{p},1}, \cdots, x_{\mathfrak{p},n})\|_{\mathfrak{p}} = \\ &= \prod_{\mathfrak{p}\in V_K} |t_{\mathfrak{p}}|_{\mathfrak{p}} \|(x_{\mathfrak{p},1}, \cdots, x_{\mathfrak{p},n})\|_{\mathfrak{p}} = \\ &= \prod_{\mathfrak{p}\in V_K} |t_{\mathfrak{p}}|_{\mathfrak{p}} \prod_{\mathfrak{p}\in V_K} \|(x_{\mathfrak{p},1}, \cdots, x_{\mathfrak{p},n})\|_{\mathfrak{p}} = |t|_{\mathbb{A}_K} H_{\mathcal{F}}(x). \end{aligned}$$

Since the product in the second last line is finite the splitting up into two products which is done in the second last step is allowed.

(iii) Let S be the finite set of all places $\mathfrak{p}$ with $\mathfrak{p}|\infty$ or $\|x\|_{\mathfrak{p}} \neq \max\{|x_i|_{\mathfrak{p}}, \cdots, |x_n|_{\mathfrak{p}}\}$ for $x \in K_{\mathfrak{p}}^n$. Since the sequence (x_k) converges to 0 there exists an integer N_0 with

$$x_{k,\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}^n,$$

for $k \geq N_0$ and $\mathfrak{p} \notin S$. Let $\varepsilon > 0$ be arbitrary. Then there exists an integer N_ε such that we have

$$\|x_{k,\mathfrak{p}}\|_\mathfrak{p} < \varepsilon,$$

for every place $\mathfrak{p} \in S$ and $k \geq N_\varepsilon$. Hence we obtain for $k \geq \max\{N_0, N_\varepsilon\}$

$$H_\mathcal{F}(x_k) = \prod_{\mathfrak{p}\in S} \|x_{k,\mathfrak{p}}\|_\mathfrak{p} \prod_{\mathfrak{p}\in V_K\setminus S} \|x_{k,\mathfrak{p}}\|_\mathfrak{p} < \prod_{\mathfrak{p}\in S} \varepsilon \prod_{\mathfrak{p}\in V_K\setminus S} 1 = \varepsilon^{|S|},$$

which proves the claim since $S \neq \{\}$.

(iv) Let $S \subseteq V_K$ be a finite set of places containing all non-archimedean places and let $\varepsilon > 0$ be an arbitrary real number. As a consequence of (i) we have

$$\lim_{k\to\infty} H_\mathcal{F}(x_k) = 0 \quad \Leftrightarrow \quad \lim_{k\to\infty} H_K(x_k) = 0,$$

where H_K is the standard height function. Hence we set without loss of generality $H_\mathcal{F} = H_K$. Since the set V_K of all places of K is countable we can choose an enumeration $V_K = \{\mathfrak{p}_1, \mathfrak{p}_2, \cdots\}$ of all places. We denote by $(x_k) = (x_{k,\mathfrak{p}_i})$ with $x_{k,\mathfrak{p}_i} \in K^n_{\mathfrak{p}_i}$ the given sequence, set $d_{k;i} := \|x_{k,\mathfrak{p}_i}\|_{\mathfrak{p}_i}$ and $h_k := H_K(x_k) = \prod_{i\geq 1} d_{k;i}$. Since x_k is primitive the height $H_K(x_k) = h_k \neq 0$. We have to show the existence of an integer N such that $|x_{k,\mathfrak{p}_i}|_{\mathfrak{p}_i} < \varepsilon$ for $\mathfrak{p}_i \in S$ and $x_{k,\mathfrak{p}_i} \in \mathcal{O}_{\mathfrak{p}_i}$ for $\mathfrak{p}_i \notin S$ and for all $k \geq N$.

Let $c(K)$ be the constant only depending on K as in Lemma 2.3.3. For every integer l we set N_l as the minimal integer such that

$$\frac{1}{h_k 2^{l^2}} > c(K),$$

holds for all $k \geq N_l$. Since $\lim_{k\to\infty} h_k = 0$ the integer N_l exists for every l. We set

$$\delta_{k;i} = \begin{cases} 1 & k < N_0, \\ d_{k;i}^{-1} & N_0 \leq k < N_i, \\ d_{k;i}^{-1} 2^{-l} & N_i \leq N_l \leq k < N_{l+1}. \end{cases}$$

By the definition of N_l we have $\prod_{i\geq 1} \delta_{k;i} > c(K)$ for all $k > N_0$. Hence Lemma 2.3.3 implies that there exists an element $\lambda_k \in K^*$ with

$$|\lambda_k|_{\mathfrak{p}_i} \leq \delta_{k;i},$$

for all places $\mathfrak{p}_i$ and $k \geq N_0$. This implies $\|\lambda_k(x_{k;\mathfrak{p}_i})\|_{\mathfrak{p}_i} \leq 1$ for all $k \geq N_0$ which is equivalent to $(x_{k;\mathfrak{p}_i}) \in \mathcal{O}^n_{\mathfrak{p}_i}$ for every archimedean place $\mathfrak{p}_i$. For every $i \geq 1$ and $k \geq \max\{N_i, N_l\}$ we obtain

$$\|\lambda_k(x_{k;\mathfrak{p}_i})\|_{\mathfrak{p}_i} \leq 2^{-l}.$$

Now set $L = \max\left\{0, \left\lceil \frac{-\log(\varepsilon)+1}{\log(2)} \right\rceil\right\}$ and $N = \max_{\mathfrak{p}_i\in S}\{N_0, N_L, N_i\}$. Then for all $k \geq N$ holds

$$|x_{k,\mathfrak{p}_i}|_{\mathfrak{p}_i} \leq 2^{-L} < \varepsilon,$$

for $\mathfrak{p}_i \in S$ and

$$|x_{k,\mathfrak{p}_i}|_{\mathfrak{p}_i} \in \mathcal{O}_\mathfrak{p},$$

for $\mathfrak{p}_i \notin S$, which proves the claim.

(v) By Lemma 3.2.6 there is an adelic norm $\mathcal{F}'$ on $\mathbb{A}_K^n$ such that $H_{\mathcal{F}}(g \cdot x) = H_{\mathcal{F}'}(x)$ for all $x \in \mathbb{A}_K^n$. Therefore without loss of generality let $g = id_{\mathbb{A}_K^n}$. By (i) there are real numbers $c_1, c_2 > 0$ with

$$c_1 H_K(x) \leq H_{\mathcal{F}}(x) \leq c_2 H_K(x),$$

for all $x \in \mathbb{A}_K^n$, where H_K is the standard height function on $\mathbb{A}_K^n$. The restriction of H_K onto K^n induces on the projective space $\mathbb{P}^{n-1}(K)$ the multiplicative height function H_K. By Corollary 3.1.7 the set

$$\{x \in K^n : H_{\mathcal{F}}(x) < c\}/K^* \subseteq \{x \in K^n : H_K(x) < \frac{c}{c_2}\}/K^* = \\ = \{x \in \mathbb{P}^{n-1}(K) : H_K(x) \leq \frac{c}{c_2}\},$$

is finite.

(vi) With (i) already proven it suffices to show the claim for $H_{\mathcal{F}} = H_K$. We define two maps $H_s, H_i : GL_n(\mathbb{A}_K) \to \mathbb{R}$ via

$$H_s((g_{\mathfrak{p}})) := \prod_{\mathfrak{p} \in V_K} \|g_{\mathfrak{p}}\|_{s,\mathfrak{p}},$$
$$H_i((g_{\mathfrak{p}})) := \prod_{\mathfrak{p} \in V_K} \|g_{\mathfrak{p}}\|_{i,\mathfrak{p}}.$$

By Lemma 3.2.5 the above products are both finite. We show that H_s and H_i are continuous:

Let $1 > \varepsilon > 0$ be arbitrary, $(g_{\mathfrak{p}}) \in GL_n(\mathbb{A}_K)$ and S be the set of places $\mathfrak{p}$ such that $g_{\mathfrak{p}} \notin \mathcal{O}_{\mathfrak{p}}$. Since the maps $\| \ \|_{s,\mathfrak{p}}, \| \ \|_{i,\mathfrak{p}}$ are for all places continuous we can choose for all $\mathfrak{p} \in S$ neighbourhoods $U_{\mathfrak{p}}$ of $g_{\mathfrak{p}}$ with

$$(1-\varepsilon)^{\frac{1}{|S|}} \|g_{\mathfrak{p}}\|_{s,\mathfrak{p}} \leq \|h_{\mathfrak{p}}\|_{s,\mathfrak{p}} \leq (1+\varepsilon)^{\frac{1}{|S|}} \|g_{\mathfrak{p}}\|_{s,\mathfrak{p}},$$
$$(1-\varepsilon)^{\frac{1}{|S|}} \|g_{\mathfrak{p}}\|_{i,\mathfrak{p}} \leq \|h_{\mathfrak{p}}\|_{i,\mathfrak{p}} \leq (1+\varepsilon)^{\frac{1}{|S|}} \|g_{\mathfrak{p}}\|_{i,\mathfrak{p}},$$

for all $\mathfrak{p} \in S$ and $h_{\mathfrak{p}} \in U_{\mathfrak{p}}$. For $\mathfrak{p} \notin S$ and $h_{\mathfrak{p}} \in GL_n(\mathcal{O}_{\mathfrak{p}})$ Lemma 3.2.5 states $\|h_{\mathfrak{p}}\|_{i,\mathfrak{p}} = \|h_{\mathfrak{p}}\|_{s,\mathfrak{p}} = \|g_{\mathfrak{p}}\|_{i,\mathfrak{p}} = \|g_{\mathfrak{p}}\|_{s,\mathfrak{p}} = 1$. Hence we have for an element $(h_{\mathfrak{p}})$ of the neighbourhood $U = \prod_{\mathfrak{p} \in S} U_{\mathfrak{p}} \times \prod_{\mathfrak{p} \in V_K \setminus S} GL_n(\mathcal{O}_{\mathfrak{p}})$ of $(g_{\mathfrak{p}})$:

$$(1-\varepsilon) H_s((g_{\mathfrak{p}})) \leq H_s((h_{\mathfrak{p}})) \leq (1+\varepsilon) H_s((g_{\mathfrak{p}})),$$
$$(1-\varepsilon) H_i((g_{\mathfrak{p}})) \leq H_i((h_{\mathfrak{p}})) \leq (1+\varepsilon) H_i((g_{\mathfrak{p}})).$$

This implies that H_s, H_i are continuous in $(g_{\mathfrak{p}})$ and since $(g_{\mathfrak{p}})$ was arbitrary, both maps are continuous on $GL_n(\mathbb{A}_K)$.

Hence there are constants $c_1, c_2 > 0$ such that

$$H_s(M), H_i(M) \subseteq (c_1, c_2).$$

Together with the inequality

$$H_i((g_{\mathfrak{p}})) H_K(x) \leq H_K((g_{\mathfrak{p}}) \cdot x) \leq H_s((g_{\mathfrak{p}})) H_K(x),$$

this proves the the claim. □

Remark Let $\mathcal{H}_n$ be the set of all height functions h on $\mathbb{A}_K^n$ for a global field K. Indeed $\mathcal{H}_n$ is a set, since it is a subset of $\prod_{\mathfrak{p}\in V_K} \mathcal{H}_{n,\mathfrak{p}}$ where $\mathcal{H}_{n,\mathfrak{p}}$ is the set of all norms on $K_\mathfrak{p}^n$ which is a subset of the set of all functions mapping $K_\mathfrak{p}^n$ into $\mathbb{R}$. We can define a topology on $\mathcal{H}_n$ by taking all sets of the form

$$B_{h,\epsilon} := \{h' \in \mathcal{H}_n : \epsilon h(x) \leq h'(x) \leq \frac{1}{\epsilon} h(x) \text{ for all } x \in \mathbb{A}_K^n\},$$

with $h \in \mathcal{H}_n$ and $0 < \epsilon < 1$ as a basis for the open sets. Lemma 3.2.6 implies that

$$\varphi : GL_n(\mathbb{A}_K) \times \mathcal{H}_n \longrightarrow \mathcal{H}_n, \quad (g, h) \mapsto h \circ g^{-1}$$

is a left group action. The proof of (vi) in the above Theorem shows basically that $\varphi(-, h)$ is continuous for all $h \in \mathcal{H}_n$. As a consequence the group action φ is continuous, as one proves easily.

The definition of an adelic norm $\mathcal{F}$ on $\mathbb{A}_K^n$ for a global field K is very restrictive. As mentioned in the remark after Definition 3.2.1 we want to consider a possible generalisation of height functions on $\mathbb{A}_K^n$.
This generalisation is on the one hand motivated by the question why we have chosen this special definition for an adelic norm $\mathcal{F} = \{\| \ \|_\mathfrak{p}\}$ - is the property that almost all norms $\| \ \|_\mathfrak{p}$ are given by $\|(x_i)\|_\mathfrak{p} = \max_i\{|x_i|_\mathfrak{p}\}$ only a necessary condition for the proof of Theorem 3.2.7 or can we weaken it? On the other hand we want to know if there exist other functions than the height functions on $\mathbb{A}_K^n$ which enjoy the same properties. Hence the motivation for this generalisation is more of philosophical nature than of mathematical utility. The wanted general height functions should enjoy the following properties:

Firstly $H(x) \neq 0$ for every general height function H on $\mathbb{A}_K^n$ iff $x \in \mathbb{A}_K^n$ is primitive. Secondly $H \circ (x \mapsto g \cdot x)$ should be a general height function for every $g \in GL_n(\mathbb{A}_K)$. Thirdly every general height function H should satisfy Theorem 3.2.7. These three properties are satisfied by the general height functions defined in the following

Definition 3.2.8 *Let K be a global field and let H_K be the standard height on K. A* **general adelic norm** *$\mathcal{F}$ on the $\mathbb{A}_K$-module $\mathbb{A}_K^n$ is a family $\{\| \ \|_\mathfrak{p}\}_{\mathfrak{p}\in V_K}$ of norms $\| \ \|_\mathfrak{p}$ on $K_\mathfrak{p}^n$ such that there exist two real numbers $c_1, c_2 > 0$ with*

$$c_1 H_K((x_\mathfrak{p})) \leq H_\mathcal{F}((x_\mathfrak{p})) \leq c_2 H_K((x_\mathfrak{p})),$$

for all $(x_\mathfrak{p}) \in \mathbb{A}_K^n$ and such that $H_\mathcal{F}((x_\mathfrak{p})) := \prod_{\mathfrak{p}\in V_K} \|x_\mathfrak{p}\|_\mathfrak{p}$ is well-defined for all $(x_\mathfrak{p}) \in \mathbb{A}_K^n$. The map $H_\mathcal{F}$ is called the **general height function** *associated to $\mathcal{F}$.*

The definition of the general adelic norm implies that $H_\mathcal{F}(x) \neq 0$ iff $x \in \mathbb{A}_K^n$ is primitive since this is true for the standard height function H_K. Let $g \in GL_n(\mathbb{A}_K)$ be arbitrary. Then there exists an adelic norm $\mathcal{F}'$ such that $H_{\mathcal{F}'}(x) = H_K(g \cdot x)$ for all $x \in \mathbb{A}_K^n$. By Theorem 3.2.7 there exist constants $c_1', c_2' > 0$ such that

$$c_1' H_K(x) \leq H_{\mathcal{F}'}(x) = H_K(g \cdot x) \leq c_2' H_K(x),$$

holds for all $x \in \mathbb{A}_K^n$. Therefore we obtain

$$c_1 c_1' H_K(x) \leq c_1 H_K(g \cdot x) \leq H_{\mathcal{F}}(g \cdot x) \leq c_2 H_K(g \cdot x) \leq c_2 c_2' H_K(x),$$

for all $x \in \mathbb{A}_K^n$. Finally general height functions enjoy Theorem 3.2.7 which is stated in Theorem 3.2.10.

Example 3.2.9 We want to describe in this example a general adelic norm, which is not an adelic norm.

Let $\| \ \|$ be the norm on $\mathbb{R}^2$ given by

$$\|(x,y)\| := \sqrt{x^2 - xy + y^2}, \quad (x,y) \in \mathbb{R}^2.$$

For every prime number p we obtain a norm $\| \ \|_p'$ on $\mathbb{Q}_p^2$ via

$$\|(x,y)\|_p' := \|(|x|_p, |y|_p)\|,$$

for all $(x,y) \in \mathbb{Q}_p^2$. It is easy to see that $\|(x,y)\|_p' \leq \max\{|x|_p, |y|_p\}$ for all $(x,y) \in \mathbb{Q}_p^2$. We want to find a maximal real number $c_p > 0$ which satisfies

$$c_p \max\{|x|_p, |y|_p\} \leq \|(x,y)\|_p',$$

for all $(x,y) \in \mathbb{Q}_p^2$. Let $(x,y) \in \mathbb{Q}_p^2$ with $\max\{|x|_p, |y|_p\} = 1$. Without loss of generalisation let $|x|_p = 1$ and therefore $y \in \mathbb{Z}_p$. Then the constant c_p satisfies

$$c_p \leq \|(x,y)\|_p' = \sqrt{1 - |y|_p + |y|_p^2}.$$

Since $|y|_p = \frac{1}{p^n}$ with suitable $n \in \mathbb{N}$ and the function $f : \mathbb{R} \to \mathbb{R}, x \mapsto \sqrt{1 - x + x^2}$ is convex with a minima at $\frac{1}{2}$ and satisfies $f(\frac{1}{2} + x) = f(\frac{1}{2} - x)$ we obtain for the constant $c_p = \sqrt{1 - \frac{1}{p} + \frac{1}{p^2}} = \frac{\sqrt{p^2 - p + 1}}{p}$.

Let $\{q_n, n \in \mathbb{N}\}$ be an infinite set of prime numbers satisfying $2^n \leq q_n < q_{n+1}$ for all $n \in \mathbb{N}$. We set $\mathcal{F} = \{\| \ \|_p\}_{p \in V_{\mathbb{Q}}}$ the family of norms $\| \ \|_p : \mathbb{Q}_p^2 \to \mathbb{R}$ with

$$\|(x,y)\|_p = \begin{cases} \|(x,y)\|_p', & p \in \{q_n, n \in \mathbb{N}\}, \\ \max\{|x|_p, |y|_p\} & \textit{otherwise}. \end{cases}$$

We set $c = \prod_{n \in \mathbb{N}} c_{q_n}$ where c_{q_n} is defined as above. Then c exists and is greater than zero:

The product $\prod_{n \in \mathbb{N}} c_{q_n}$ converges iff the sum $\sum_{n \in \mathbb{N}} \log(c_{q_n})$ converges. We obtain for the infinite sum even absolute convergence:

$$\sum_{n \in \mathbb{N}} |\log(c_{q_n})| = \sum_{n \in \mathbb{N}} \left| \frac{1}{2} \log \left(\frac{q_n^2 - q_n + 1}{q_n^2} \right) \right| = \sum_{n \in \mathbb{N}} \frac{1}{2} \log \left(1 + \frac{q_n - 1}{q_n^2 - q_n + 1} \right) \leq$$

$$\leq \sum_{n \in \mathbb{N}} \frac{1}{2} \log \left(1 + \frac{1}{q_n} \right) \leq \sum_{n \in \mathbb{N}} \frac{1}{2} \log \left(1 + \frac{1}{2^n} \right)$$

The power series expansion of the logarithm implies $\log(1+\frac{1}{2^n}) < \frac{1}{2^n}$ for $n \in \mathbb{N}$. This yields

$$\sum_{n\in\mathbb{N}} |\log(c_{q_n})| \leq \sum_{n\in\mathbb{N}} \frac{1}{2}\log\left(1+\frac{1}{2^n}\right) \leq \sum_{n\in\mathbb{N}} \frac{1}{2^{n+1}} = \frac{1}{2}.$$

Hence the product $c = \prod_{n\in\mathbb{N}} c_{q_n}$ exists for every rearrangement. If we set $\delta_p = c_p$ for $p \in \{q_n, n \in \mathbb{N}\}$ and $\delta_p = 1$ otherwise we obtain

$$\prod_{p\in V_{\mathbb{Q}}} \|(x,y)\|_p \leq \prod_{p\in V_{\mathbb{Q}}} \max\{|x|_p, |y|_p\} = H_{\mathbb{Q}}((x,y)),$$

and

$$\prod_{p\in V_{\mathbb{Q}}} \|(x,y)\|_p \geq \prod_{p\in V_{\mathbb{Q}}} \delta_p \max\{|x|_p, |y|_p\} \geq$$
$$\geq \prod_{p\in V_{\mathbb{Q}}} \delta_p \prod_{p\in V_{\mathbb{Q}}} \max\{|x|_p, |y|_p\} = c \cdot H_{\mathbb{Q}}((x,y)),$$

for all $(x,y) \in \mathbb{A}^2_{\mathbb{Q}}$. Hence $\mathcal{F}$ is an general adelic norm.

Theorem 3.2.10 *Let K be a global field, $\mathcal{F}$ a general adelic norm and $H_{\mathcal{F}}$ the general height function corresponding to $\mathcal{F}$. Then Theorem 3.2.7 is also true for the general height function $H_{\mathcal{F}}$.*

Proof. The first property is a part of the definition of a general adelic norm.

The proof of the second property is analogue to the proof for height functions expect one may split the product in the second last step since the product converges for every rearrangement of the factors.

Using the first property it suffices to prove the properties $(iii) - (vi)$ only for the case of $H_{\mathcal{F}} = H_K$, which is true by Theorem 3.2.7. □

A. Haar measure

We need for the understanding of this thesis some techniques form measure theory. See for more details and the proofs [7] and [11].

Definition A.1 *Let X be a set. A* **σ-algebra** *Σ over X is a subset of the power set 2^X of X with the following properties:*

(i) $X \in \Sigma$,
(ii) $X \setminus A \in \Sigma$ *for all* $A \in \Sigma$,
(iii) $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma$ *for all families* $\{A_n \in \Sigma, n \in \mathbb{N}\}$.

The pair (X, Σ) is called **measurable space**. *If X is a topological space the* **Borel algebra** *$\mathcal{B}$ of X is the smallest σ-algebra containing all open subsets of X. An element $B \in \mathcal{B}$ is called* **Borel subset** *of X.*

Definition A.2 *A* **measure** *μ on a measurable space (X, Σ) is a map $\mu : \Sigma \to [0, \infty)$ such that*

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n),$$

for all families $\{A_n\}$ of pairwise disjoint sets in Σ. For a topological space X with Borel algebra $\mathcal{B}$ a measure μ on $(X, \mathcal{B})$ is called **Borel measure** *on X.*

Definition A.3 *Let X be a locally compact Hausdorff space. A* **Radon measure** *on X is a Borel measure on X such that*

(i) $\mu(K)$ *is finite for all compact* $K \subseteq X$.
(ii) $\mu(B) = \inf\{\mu(O) : B \subseteq O, O \subseteq X \text{ open}\}$ *for all* $B \in \mathcal{B}$.
(iii) $\mu(O) = \sup\{\mu(K) : K \subseteq O, K \subseteq X \text{ compact}\}$ *for all* $O \subseteq X$ *open.*

Definition A.4 *Let G be a locally compact Hausdorff group. A* **left-Haar measure** *μ on G is a nonzero Radon measure on G such that*

$$\mu(g \cdot B) = \mu(B),$$

for all $g \in G$ and all Borel sets B of G.

Theorem A.5 *Let G be a locally compact Hausdorff group. Then there exists a left-Haar measure μ on G, which is up to a positive multiplicative constant unique.*

Proof. See [7, p. 12]. □

Proposition A.6 *Let μ be a left-Haar measure on a locally compact Hausdorff group G. Then $\mu(G)$ is finite iff G is compact*

Proof. See [7, p. 10]. □

Proposition A.7 *Let μ be a left-Haar measure on a locally compact Hausdorff group G, $\Gamma \subset G$ be a discrete subgroup such that G/Γ is compact. We can scale the left-Haar measure μ' on G/Γ such that for all Borel subsets B of G with $\mu(B) > \mu'(G/\Gamma)$ there exist two distinct elements $x, y \in B$ with $xy^{-1} \in \Gamma$.*

Proof. See [11, p. 36]. □

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Abstract

This thesis deals with height functions on free modules over the adele ring $\mathbb{A}_K$ over a global field K and is aimed at proofing six fundamental properties of these height functions. It starts with an introduction into valuation theory to give the needed background knowledge. By using the direct restricted product of locally compact groups we can define the adele ring $\mathbb{A}_K$ and the idele group $\mathbb{I}_K$ over a global field K and prove their properties pertinent to this thesis. The first example for height functions given in this thesis are height functions on the projective space over global fields. Finally we deal with height functions on free modules over the adele ring over a global field K. The six properties of these height functions, as mentioned above, can be summarised as following:

All height functions are equivalent, where the equivalence is defined analogously as for norms on vector spaces. The image of a null sequence under a height function is a null sequence. Conversely if the image of a sequence of primitive elements under a height function is a null sequence, we can multiply every element of the sequence by a non zero scalar from K and obtain thereby a null sequence. For every real number B and every height function h there exist up to multiplication by scalars from K only finitely many points P with components in K and $h(P) \leq B$. The last two properties deal with the behaviour of height functions when the argument is multiplied by an scalar from $\mathbb{A}_K^*$ or when a change of $\mathbb{A}_K$-basis is applied.

Zusammenfassung

Diese Arbeit beschäftigt sich mit Höhenfunktionen auf freien Moduln über dem Adelring $\mathbb{A}_K$ über einem globalen Körper K mit dem Ziel, sechs grundlegende Eigenschaften von diesen zu beweisen. Die Arbeit beginnt mit einer Einführung in die Bewertungstheorie, um das nötigte Hintergrundwissen zu vermitteln. Mithilfe des direkten eingeschränkten Produktes von lokal kompakten Gruppen wird der Adelring $\mathbb{A}_K$ und die Idelgruppe $\mathbb{I}_K$ über einem globalen Körper K definiert und deren für diese Arbeit relevanten grundlegenden Eigenschaften präsentiert und bewiesen. Mit den Höhenfunktionen auf dem projektiven Raum über einem globalen Körper werden wir erste Beispiele von Höhenfunktionen sehen. Abschließend werden wir Höhenfunktionen auf freien Moduln über dem Adelring über einem globalen Körper K betrachten. Die sechs Eigenschaften von diesen Höhenfunktionen die wir beweisen werden lassen sich wie folgt zusammenfassen:
Alle Höhenfunktionen sind äquivalent zueinander, wobei Äquivalenz hier analog wie bei Normen auf Vektorräumen definiert ist. Das Bild einer Nullfolge unter einer Höhenfunktion ist wieder eine Nullfolge. Ist umgekehrt das Bild einer Folge von primitiven Elementen unter einer Höhenfunktion eine Nullfolge, so kann man jedes Folgenglied mit einem Skalar aus K ungleich Null multiplizieren sodass man eine Nullfolge erhält. Für jede reelle Zahl B und jeder Höhenfunktion h gibt es bis auf Multiplikation mit Skalaren aus K nur endlich viele Punkte P mit Komponenten aus K und $h(P) \leq B$. Die letzten beiden Eigenschaften behandeln das Verhalten von Höhenfunktionen unter Multiplikation der Argumente mit Skalaren aus $\mathbb{A}_K^*$ und unter $\mathbb{A}_K$-Basiswechsel.

Curriculum Vitae

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