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„Refined counting of alternating sign arrays“

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Contents

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Summary

In the early 80s, Mills, Robbins and Rumsey introduced *alternating sign matrices*, or short ASMs. ASMs are very remarkable since they have many connections to further combinatorial objects such as plane partitions or lozenge tilings, to different areas of mathematics such as representation theory or symmetric functions as well as to statistical physics. Further, working on problems related to ASMs constantly pushes the limits of our counting tools. The most intriguing and mysterious fact about ASMs is the existence of three further families of combinatorial objects which are equinumerous with ASMs but there is no explicit bijection between any of these family known. Beside ASMs, these are *alternating sign triangles* (ASTs), *descending plane partitions* (DPPs) and *totally symmetric self-complementary plane partitions* (TSSCPPs).

In this thesis, I study three refined enumerations of ASMs or AS-trapezoids (the latter generalise ASTs), respectively. First, we consider *fully packed loops* (FPLs), which are in bijection to ASMs, with respect to their link pattern. We use the theory of *wheel polynomials* to prove a conjecture from Zuber which states that the number of fully packed loop configurations whose link pattern consists of two noncrossing matchings, which are separated by m nested arches, is a polynomial function in m of certain degree and with certain leading coefficient.

The second part of this thesis is devoted to a refinement of AS-trapezoids, with respect to Catalan objects and Motzkin paths. We show that the number of AS-trapezoids associated to a Catalan object (resp. a Motzkin path) is a polynomial function in the length of the shorter base of the trapezoid. Further we study the rational roots of these polynomials and deduce a constant term identity for the refined counting of AS-trapezoids.

In the last part of this thesis, we provide a determinant formula for the Q-enumeration of ASMs, which is a weighted enumeration of ASMs with respect to the number of -1 entries. By evaluating a generalisation of this determinant we are able to present new proofs for the 1-,2- and 3-enumeration of alternating sign matrices and a factorisation in the 4-enumeration case. Finally we relate the 1-enumeration of our generalised determinant to the weighted enumeration of cyclically symmetric lozenge tilings of a hexagon with a triangular hole and also AS-trapezoids.

Zusammenfassung

In den frühen 80ern präsentierten Mills, Robbins und Rumsey *alternierende Vorzeichenmatrizen*, kurz ASMs (aus dem Englischen *"alternating sign matrices"*). ASMs zeichnen sich einerseits durch ihre Verbindungen zu weiteren kombinatorischen Objekten wie zweidimensionalen Partitionen oder Parkettierung mit Rhomben, zu verschiedenen Gebieten der Mathematik wie Darstellungstheorie oder symmetrischen Funktionen sowie zur statistischen Physik aus. Des Weiteren führen uns die Arbeiten an diesen Problemstellungen regelmäßig dazu, unser Repertoire an Methoden weiterzuentwickeln. Das wohl größte und faszinierenste Mysterium rund um ASMs ist die Tatsache, dass es drei weitere Familien von kombinatorischen Objekten gibt, die allesamt gleichmächtig zu ASMs sind, aber keine einzige explizite Bijektion zwischen diesen Familien bekannt ist. Neben ASMs sind dies *alternierende Vorzeichen Dreiecke* (ASTs), *"Descending plane partitions"* (DPPs) und *total symmetrische selbst-komplementäre zweidimensionale Partitionen* (TSSCPPs).

In dieser Dissertation werden drei verfeinerte Abzählungen von ASMs bzw. von AS-Trapezen, welche eine Verallgemeinerung von ASTs sind, untersucht. Als erstes betrachten wir *"Fully packed loops"* (*FPLs*) in Bezug auf deren *"link pattern"*. Wir benützen die Theorie der *"Wheel Polynome"* um eine Vermutung von Zuber zu beweisen, welche aussagt, dass die Anzahl der FPLs, deren "link pattern" aus zwei nichtkreuzenden "Matchings" besteht welche durch m verschachtelte Bögen getrennt sind, ein Polynom in m von bestimmten Grad und mit einem bestimmten Leitkoeffizienten ist.

Im zweiten Teil dieser Arbeit wird die verfeinerte Abzählung von *AS-Trapezen* bezüglich Catalan Objekten und Motzkin Pfaden betrachtet. Wir beweisen, dass die Anzahl der AS-Trapeze mit gegebenem Catalan Objekt bzw. Motzkin Pfad ein Polynom in der Länge der kürzeren Grundseite ist. Weiters untersuchen wir die rationalen Nullstellen dieser Polynome und leiten eine konstante Term-Identität für die verfeinerte Abzählung her.

Im letzten Teil dieser Dissertation präsentieren wir eine Determinantenformel für die Q-Abzählung von ASMs, das ist eine gewichtete Abzählung von ASMs bezüglich der Anzahl der -1 Einträge. Die Berechnung einer Verallgemeinerung dieser Determinante erlaubt es uns neue Beweise für die 1-, 2- und 3-Abzählung von ASMs zu beweisen und eine Faktorisierung der 4-Abzählung zu präsentieren. Abschließend bringen wir die 1-Abzählung der verallgemeinerten Determinante mit der gewichteten Abzählung von Zyklisch symmetrischen Parkettierungen eines Sechseckes mit einem dreieckigen Loch mit Rhomben und AS-Trapezen in Verbindung.

Chapter 1

Introduction

1.1 A brief historic overview

This section is meant as a historic overview. The following book [17] and articles [18, 64, 68, 78] give a very detailed overview of the developments concerning ASMs prior to 2001.

A story starting with 1, 2, 7, 42, 429, ...

In 1866 Dodgson, who is better known in general public as Lewis Carroll, the author of *Alice's Adventures in Wonderland*, proposed a new method for calculating determinants which he was calling *Condensation method* [28]. Roughly speaking, the essential part of this algorithm is iteratively calculating the determinant of certain 2×2 matrices

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

By using the condensation method and generalising the determinant of a 2×2 matrix to

$$\det{}_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad + \lambda bc,$$

Robbins and Rumsey [69] defined in 1986 the λ-determinant; for $\lambda = -1$ this is just the usual determinant. Generalising the Leibniz formula, which states that a determinant can be expressed as a sum over permutations, or permutation matrices, the λ-determinant of an $n \times n$ matrix $M = (m_{i,j})$ can be expressed as the sum

$$\det{}_\lambda(M) = \sum_{A=(a_{i,j}) \in \mathrm{ASM}_n} \lambda^{\mathrm{inv}(A)} (1 + \lambda^{-1})^{\mathcal{N}(A)} \prod_{1 \leq i,j \leq n} m_{i,j}^{a_{i,j}}, \tag{1.1}$$

where ASM_n is the set of $n \times n$ *alternating sign matrices* (ASMs), inv denotes the number of inversions and $\mathcal{N}$ the number of -1's of an ASM. Although this was the mathematical origin of ASMs, they were already introduced to the mathematical community in 1982 in a paper of Robbins and Rumsey together with Mills [59] in which they solved the *Macdonald Conjecture*. This paper also includes the *alternating* *sign matrix Conjecture* [59, Conjecture 1], which states that the number of ASMs of size n is given by the product formula

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}. \tag{1.2}$$

ASMs have been studied even earlier by physicists as *configurations of the six-vertex model* (with certain boundary conditions). The six-vertex model is one of the so called *ice-type models* which have been studied since the first half of the twentieth century, see for example [12].

Four classes of combinatorial objects

When Mills, Robbins and Rumsey introduced their conjecture, the sequence of numbers defined by the above formula was not unfamiliar to the mathematical community. As Stanley pointed out to Mills, Robbins and Rumsey, see [59, p. 79], this formula also counts *descending plane partitions* (DPPs). DPPs were introduced by Andrews [6] in 1979, where he proved beside the unweighted case of the Macdonald conjecture that DPPs are enumerated by the formula in (1.2).

A third class of combinatorial objects enumerated by the above formula was introduced in 1986. In [58], Mills, Robbins and Rumsey defined *totally symmetric self complementary plane partitions* (TSSCPPs) and Andrews [7] proved in 1994 their enumeration formula. As a "byproduct" this led to a new operation on plane partitions, namely the complement, which was used by Stanley to define five more symmetry classes of plane partitions, see [71].

Thirty years later in 2016, Ayyer, Behrend and Fischer [9] introduced a fourth class of combinatorial objects which are enumerated by the formula (1.2): *alternating sign triangles* (ASTs). These objects are connected to one instance of *extreme diagonally and antidiagonally symmetric ASMs.*

This completes the quartet (ASMs, ASTs, DPPs, TSSCPPs) of classes of combinatorial objects which are equinumerous but there is no explicit bijection between any pair of classes known.

Proofs of the ASM Conjecture

The first proof for the ASM Conjecture was due to Zeilberger [76] in 1996. While the first version of his proof was submitted already in 1992, it took two further years to fix errors and gaps and two more before it was published (a detailed elaboration on this process can be found in [78]) and resulted in very formal, structured and rigorous proof of the conjecture. His proof is based on manipulating constant term identities and shows in particular a more general statement, namely that *Gog-trapezoids* (which generalise ASMs) and *Magog-trapezoids* (which generalise TSSCPPs) are equinumerous; this is the only known proof of this fact until now.

While twelve years passed between the announcement of the ASM conjecture and the publication of its first proof, already in 1996, a second proof was published by Kuperberg [51]. Contrary to the proof of Zeilberger, Kuperberg used methods from the *six-vertex model*. The six-vertex model was originally studied from statistical physicists in the twentieth century allowing various boundary conditions, see for example [12]. Using these methods was made possible by a bijection between ASMs and six-vertex configurations with the *domain boundary wall condition* (DBWC). It seems that Robbins and Rumsey have already described this very bijection (perhaps unwittingly) in the construction of the matrix B^* in [69, p. 179f]. In order to derive the enumeration formula of ASMs, Kuperberg used a determinantal expression for the partition function, a weighted generating function for configurations of the six-vertex model, which is due to Korepin [47] and Izergin [45]. Following the method of Kuperberg, Zeilberger gave also in 1996 a proof for the refined ASM conjecture [77].

Starting in 2006, Fischer developed in a series of papers [29, 31, 32, 33, 37] the so called *operator formula* for monotone triangles and various generalisations which expresses the number of monotone triangles as a polynomial in shift operators applied to another polynomial. Using this formula, she could prove in 2007 [30] the refined ASM Theorem and hence the ASM Theorem. A very short and self contained version of this proof was published by her [35] in 2016. In a recent paper [3], the author found another proof for the ASM Theorem which is also based on the operator formula.

The Razumov-Stroganov Conjecture

We want to enclose our historic overview with another relation of ASMs to statistical physics which was discovered by Razumov and Stroganov.

One of the many faces of ASMs are *fully packed loop configurations* (FPLs), which were introduced in 1996 by Batchelor, Blöte, Nienhuis and Yung [53]. To every FPL we can associate in a very intriguing way a noncrossing matching, called its *link pattern*. In 2000, Wieland proved in [75] combinatorially, which is very impressive since the proofs in this area of combinatorics are usually of computational nature, that the number of FPLs with a given link pattern is invariant under rotating the link pattern, which was observed by Cohn and Propp. Only one year later, Razumov and Stroganov [65] connected conjecturally the number of FPLs with given link patterns to the ground state vector of the homogeneous $O(1)$ loop model, a model in statistical physics, and hence generalised the observations of Cohn and Propp. Variations of their conjecture were later found by Batchelor, de Gier, Mitra, Nienhuis, Pearce, Razumov, Rittenberg and Stroganov [11, 25, 62, 63, 66, 67] for symmetry classes of FPLs. In 2011 Cantini and Sportiello proved the Razumov-Stroganov Conjecture using Wieland gyration and a generalised version of the conjecture in 2014 [20].

An important contribution to this topic is the theory of wheel polynomials which was developed by Di-Francesco, Fonseca and Zinn-Justin among others in the following papers [26, 43, 80]. These wheel polynomials are solutions of the so-called *quantum Knizhnik-Zamolodchiknov equation* and are connected to the ground state vector of the homogeneous $O(1)$ loop model, when specialised.

1.2 Organisation of this thesis

Following this introduction, the thesis consists of four further chapters which I describe briefly.

Chapter 2 contains preliminary concepts, definitions and notations which lay the basis for this thesis and generally for the understanding of this research field.

In Chapter 3 we present the theory of wheel polynomials, introduce a new basis for the vector space of wheel polynomials and finally prove Conjecture 7 from [81] which states that the number of fully packed loop configurations whose link pattern consists of two noncrossing matchings, which are separated by m nested arches, is a polynomial function in m of certain degree and with certain leading coefficient. The content of this chapter can be found in [4] and its extended abstract [1] within FPSAC 2016.

Chapter 4 is devoted to a refinement of AS-trapezoids by means of Catalan objects and Motzkin paths. We show that the number of AS-trapezoids associated to a Catalan object (resp. a Motzkin path) is a polynomial function in the length of the shorter base of the trapezoid. Further we study the rational roots of these polynomials and deduce a constant term identity for the refined counting of AS-trapezoids. This chapter follows the paper [5]; an extended abstract [2] of it was presented at FPSAC 2017.

In Chapter 5 we provide a new type of binomial determinant for the Q-enumeration of ASMs, this is a weighted enumeration of ASMs. By evaluating a generalisation of this determinant we are able to reprove Conjecture 4 from [60], present new proofs for the 1-,2- and 3-enumeration of alternating sign matrices and a factorisation in the 4-enumeration case. Finally we relate the 1-enumeration of our generalised determinant to the weighted enumeration of cyclically symmetric lozenge tilings of a hexagon with a triangular hole and also to AS-trapezoids. The content of this chapter is presented in [3].

Chapter 2

Notations and Definitions

The following chapter is a summary of the main definitions and notations of this thesis, complemented by the most common notions of the research field of *alternating sign matrices*. Section 2.1 contains a small introduction to the world of Catalan objects; we define five objects enumerated by the *Catalan numbers* and present their relations to each other. Further it includes the definition of *Motzkin paths* and the description of a surjection from *Dyck paths* to them. In Section 2.2 we introduce the four classes of combinatorial objects enumerated by the formula in (1.2); these are the *ASM class*, the *AST class*, the *DPP class* and the *TSSCPP class*. Since two of these classes are connected to *plane partitions*, we give a small introduction to plane partitions in Subsection 2.2.3.

2.1 Catalan numbers and a selection of objects counted by them

One of the most famous numbers of enumerative and algebraic combinatorics are the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. They enumerate a variety of combinatorial objects, a list of more than 200 of such objects can for example be found in [72]. In the following section, we define five *Catalan objects*, i.e. combinatorial objects enumerated by the Catalan numbers, which are important in the context of this thesis and present their relations among themselves.

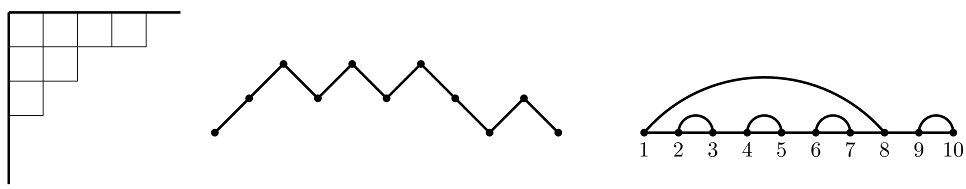


Figure 2.1: The objects from left to right: a Young diagram in the staircase shape, a Dyck path and a noncrossing matching. All of them are mapped to each other by the described bijections. The centred Catalan set associated to the Dyck path is $\{-3,-2,-1,0,4\}$.

Definition 2.1.1. *A Dyck path of length $2n$ is a path in the plane starting at $(0,0)$, ending at $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ such that it lies weakly above the x-axis.*

Dyck paths of length $2n$ are enumerated by the n-th Catalan number C_n. For an example of a Dyck path of length 10 see Figure 2.1.

Definition 2.1.2. *A* Young diagram *is a finite collection of boxes, arranged in left-justified rows and weakly decreasing row-length from top to bottom.*

Let λ be a Young diagram, we denote by $|\lambda|$ the number of boxes of λ. Young diagrams for which the i-th row from top has at most $n-i$ boxes for all $1 \leq i \leq n$ are in bijection to Dyck paths of length $2n$ in the following way. We associate to every Dyck path D the arrangement of unit boxes which lie between the Dyck path D and the path consisting of n north-east steps followed by n south-east steps. Rotated by 45 degrees this yields a Young diagram with at most $n-i$ boxes in the i-th row from top, see Figure 2.1 for an example. Let D be a Dyck path of length $2n$ and D' the Dyck path of length $2n+2$ which is obtained by adding a $(1,1)$ step in front of D and a $(1,-1)$ step at the end. Then both D and D' are mapped to the same Young diagram λ by the above described map, i.e. we can regard λ as a Catalan object of size n or larger. If we want to emphasise that we regard λ as a Catalan object of size n, we depict it together with the path consisting of n north steps followed by n south steps, for an example see Figure 2.1 left.

Definition 2.1.3. *A* partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ *of n is a weakly decreasing sequence of positive integers such that $\sum_{i=1}^{l} \lambda_i = n$. We use the notation $\lambda \vdash n$ to state that λ is a partition of n.*

Every partition can be presented as a Young diagram by putting λ_i boxes in the i-th row from top. Figure 2.1 shows a Young diagram associated to the partition $(4,2,1)$.

Definition 2.1.4. *A* noncrossing matching *of size n consists of $2n$ points on a line labelled from left to right with the numbers $1, \ldots, 2n$ together with n pairwise noncrossing arches above the line such that every point is endpoint of exactly one arch. We denote by NC_n the set of noncrossing matchings of size n.*

Given a noncrossing matching π of size n we associate a Dyck path of length $2n$ to it, where the i-th step of the Dyck path is $(1,1)$ if i is connected in π to a number larger than i and $(1,-1)$ otherwise. It is easy to see that this yields a bijection between noncrossing matchings of size n and Dyck paths of length $2n$. We define $\lambda(\pi)$ as the Young diagram obtained by applying the composition of the bijection from noncrossing matchings to Dyck path with the bijection from Dyck paths to Young diagrams to π.

A new family of objects enumerated by the Catalan numbers are *centred Catalan sets*, which were first introduced in [2].

Definition 2.1.5. *A* centred Catalan set S *of size n is an n-subset of $\{-n+1, -n+2, \ldots, n-1\}$ such that $|S \cap \{-i, -i+1, \ldots, i\}| \geq i+1$ for all $0 \leq i \leq n-1$, in particular $0 \in S$.*

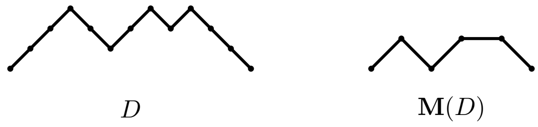


Figure 2.2: A Dyck path of length 12 and its associated Motzkin path of length 5.

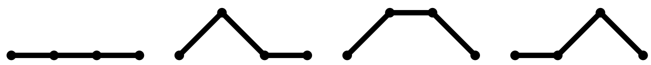
Centred Catalan sets of size n are in bijection with Dyck paths of length $2n$. For a given centred Catalan set S, we construct a Dyck path $\mathbf{D}(S)$ by reading the integers $-n+1, \ldots, n-1, n$ in the order $0, -1, 1, -2, 2, \ldots, -n+1, n-1, n$ and drawing a north-east step if the number is in S and a south-east step otherwise, for an example see Figure 2.1. In fact, there are 2^{n-1} different bijections of the above kind between centred Catalan sets of size n and Dyck paths of length $2n$. For every $1 \leq i \leq n-1$ we can switch the order of reading $-i, i$ in the above algorithm and obtain a new bijection.

Beside the above defined Catalan objects, we will also need *Motzkin paths.*

Definition 2.1.6. *A* Motzkin path *of length n is a path starting at $(0,0)$ and ending at $(n,0)$ with step-set $\{(1,1),(1,0),(1,-1)\}$ which is weakly above the x-axis.*

In the following, we encode a Motzkin path M of length n with a sequence $M = (m_1, \ldots, m_n)$ with $m_i \in \{1, 0, -1\}$ for $1 \leq i \leq n$, where m_i corresponds to the step $(1, m_i)$.

Example 2.1.7. *The following are all Motzkin paths of length 3, represented graphically and as sequences.*



$(0,0,0)$ $(1,-1,0)$ $(1,0,-1)$ $(0,1,-1)$

We define $\mathbf{M}(D)$ to be the Motzkin path of size $n-1$ obtained by "averaging" the steps in the Dyck path D of size $2n$: the i-th step of $\mathbf{M}(D)$ is the average of the $(2i)$-th and $(2i+1)$-st step of D; the first and last step of the Dyck path are omitted in the averaging process. By averaging we mean that two north-east steps result in a north-east step, a north-east step and a south-east step in an east step and two south-east steps in a south-east step. This map is a surjection from Dyck paths of length $2n$ to Motzkin paths of length $n-1$, for an example see Figure 2.2. For a centred Catalan set S of size n, the Motzkin path $\mathbf{M}(S) := \mathbf{M}(\mathbf{D}(S)) = (m_1(S), \ldots, m_{n-1}(S))$ is given by

$$m_i(S) := |\{-i, i\} \cap S| - 1.$$

2.2 Classes counted by the sequence 1, 2, 7, 42, 429, ...

As already described in Chapter 1, the four families of combinatorial objects ASMs, ASTs, DPPs and TSSCPPs are all equinumerous, however there is no explicit bijection between any pair of these families known. As we will see in the following section,

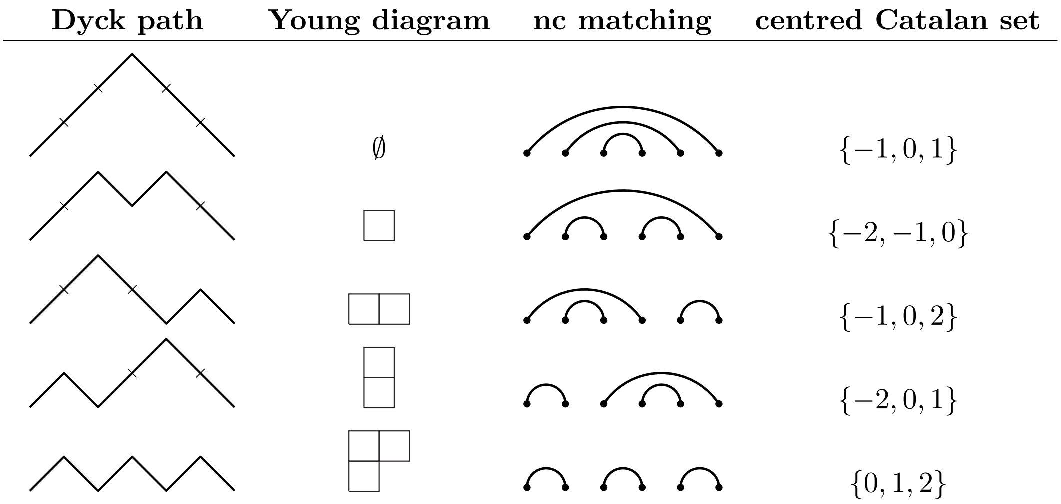


Figure 2.3: This table shows the above described Catalan objects enumerated by the third Catalan number $C_3 = 5$. In each row, the objects are mapped to each other by the previously described bijections.

there are many families of combinatorial objects for which we know an explicit bijection to one of the above four families. We group these families into four classes which are associated to one of the above four families in the following way. Every class consists of combinatorial objects for which we know an explicit bijection to the family associated to this class, i.e. ASMs, ASTs, DPPs or TSSCPPs respectively. In the following we describe these classes in separate subsections. A further subsection is dedicated to *plane partitions*.

2.2.1 The ASM class

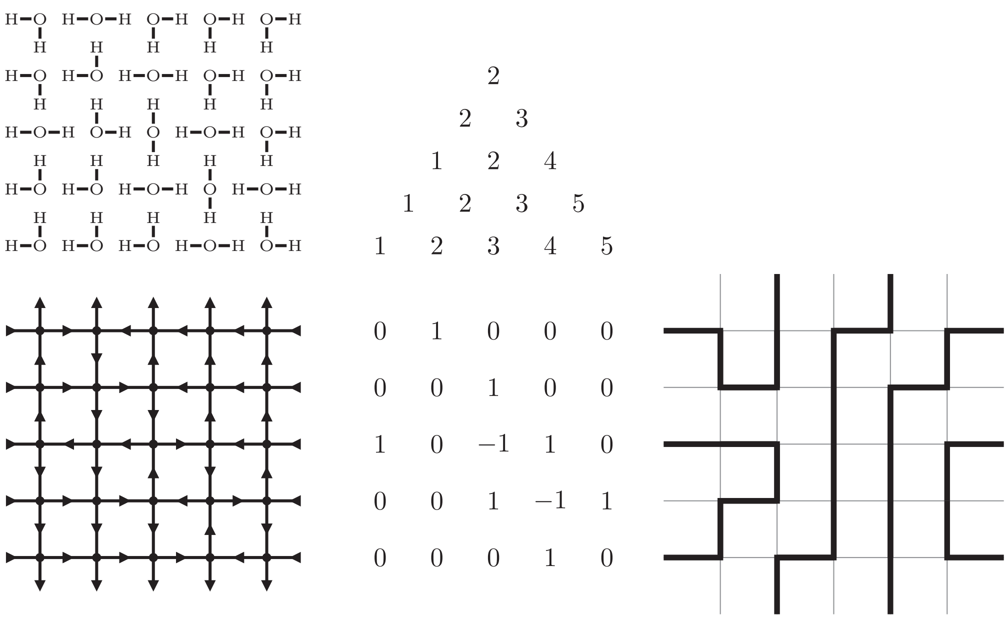
We are describing in this subsection some of the most prominent members of the ASM class which are *ASMs*, *FPLs*, *six-vertex configurations*, *square ice configurations* and *monotone triangles*. Further members which did not find their way into this thesis are *corner-sum matrices*, *height-function matrices*, certain *tuples of osculatig paths* as well as order ideals of a certain family of posets; for a detailed description see among others [64, 73].

Alternating sign matrices

Definition 2.2.1. *An* alternating sign matrix*, or short* ASM*, of size n is an $n \times n$ matrix with entries $-1, 0, 1$ such that*

- *all row- and column-sums are equal to* 1,
- *in all rows and columns the non-zero entries alternate.*

We denote by A_n the number of ASMs of size n. Every ASM has a unique 1 entry in its top row. Since the row-sum is equal to 1 there has to be at least one positive entry in the first row. Assume there are more than one 1 entry in the top row, then



$$\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 2 & & 3 & & & \\ & & 1 & & 2 & & 4 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}$$

Figure 2.4: Five different objects of the ASM class which are mapped to each other by the described bijections. The objects are from top to bottom and within rows from left to right: square ice configuration, monotone triangle, six-vertex configuration, ASM and FPL.

there has to be a -1 in the top row between two 1 entries. Hence the column-sum of the column with the -1 in the top row can not be positive which is a contradiction to the definition. We denote by $A_{n,i}$ the number of ASMs of size n whose unique 1 entry in the top row is in the i-th column. The following was conjectured by Mills, Robbins and Rumsey [59, 60] and was first proven by Zeilberger [77].

Theorem 2.2.2 (Refined alternating sign matrix Theorem)**.** *The number $A_{n,i}$ of ASMs of size n whose unique 1 in the top row is in the i-th column is*

$$\frac{\binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}}{\binom{3n-2}{n-1}} \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}. \tag{2.1}$$

Further important statistics are the positions of the unique 1 entry in the bottom row, the leftmost or rightmost column and the following three statistics. Let $\mathcal{N}(A)$ denote the number of -1 entries in an ASM A. Following the convention which was used among others in [37], we define the *inversion number* $\mathrm{inv}(A)$ and the *complementary inversion number* $\mathrm{inv}'(A)$ of an ASM $A = (a_{i,j})_{1 \le i,j \le n}$ of size n as

$$\mathrm{inv}(A) := \sum_{\substack{1 \le i' < i \le n \\ 1 \le j' \le j \le n}} a_{i',j} a_{i,j'},$$

$$\mathrm{inv}'(A) := \sum_{\substack{1 \le i' < i \le n \\ 1 \le j \le j' \le n}} a_{i',j} a_{i,j'}.$$

For an ASM A of size n, the inversion number, complementary inversion number and the number of -1 entries of A are connected in the following way

$$\mathrm{inv}(A) + \mathrm{inv}'(A) + \mathcal{N}(A) = \binom{n}{2}. \tag{2.2}$$

The refined enumeration of ASMs with respect to some or all of these statistics was elaborated among others in the following papers [10, 14, 34, 40, 42, 46, 74]. In the following we want to discuss the refined enumeration of ASMs with respect to their number of -1 entries in more detail. The *Q-enumeration of ASMs* $A_n(Q)$ is defined as the weighted sum

$$A_n(Q) = \sum_A Q^{\mathcal{N}(A)},$$

where the sum is over all ASMs A of size n. Analogously there exists a refined version $A_{n,i}(Q) = \sum_A Q^{\mathcal{N}(A)}$ where the sum is over all ASMs of size n with their unique 1 in the top row in the i-th column. For $Q = 0$ the Q-enumeration is counting ASMs with no -1 entries, i.e., permutation matrices, hence we have $A_n(0) = n!$. The 1-enumeration is the straight enumeration of ASMs which is given by the product formula in (1.2). The only further values of Q for which we have an explicit formula for the Q-enumerations are 2 and 3.

Theorem 2.2.3 ([60])**.** *The* 2*-enumeration of ASMs of size n is given by*

$$A_n(2) = 2^{\binom{n}{2}}. \tag{2.3}$$

The refined 2*-enumeration of ASMs is given by* $A_{n,k}(x) = 2^{\binom{n-1}{2}}\binom{n-1}{k-1}$.

The following result was also conjectured by Mills, Robbins and Rumsey in [60, Conjecture 6] and first proven by Kuperberg [51].

Theorem 2.2.4. *For all positive integers n holds*

$$\begin{aligned} A_{2n+1}(3) &= 3^{n(n+1)} \prod_{i=1}^{n} \frac{(3i-1)!^2}{(n+i)!^2}, \\ A_{2n+2}(3) &= 3^{n(n+2)} \frac{n!}{(3n+2)!} \prod_{i=1}^{n+1} \frac{(3i-1)!^2}{(n+i)!^2}. \end{aligned} \tag{2.4}$$

Fully packed loop configurations

Definition 2.2.5. *A* fully packed loop configuration*, or short* FPL*, F of size n is a subgraph of the $n \times n$ grid with n external edges, these are "edges" with only one vertex incident to them, on every side such that*

- *all vertices of the $n \times n$ grid have degree 2 in F,*
- *F contains every other external edge beginning with the topmost at the left side.*

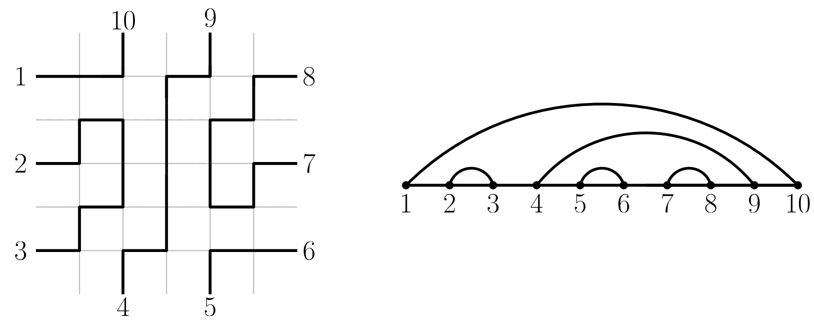


Figure 2.5: An example of an FPL of size 5 and its link pattern.

For an example see Figure 2.4 or Figure 2.5. The following yields a bijection between FPLs and ASMs of the same size. First, we colour the vertices of an FPL in a checker board manner white or black starting with the top leftmost vertex to be white. If the two edges at a vertex form a corner we assign 0 to the vertex, if the two edges form a horizontal line at a white (resp. black) vertex we assign 1 (resp. -1) to the vertex and if the edges form a vertical line at a white (resp. black) vertex we assign -1 (resp. 1) to the vertex; see Figure 2.4 for an example.

An FPL consists of pairwise disjoint paths and loops. Every path connects two external edges. We number the external edges in an FPL counter-clockwise with 1 up to $2n$, see Figure 2.5. This allows us to assign to every FPL F a noncrossing matching $\pi(F)$, where i and j are connected by an arch in $\pi(F)$ if they are connected in F. We call $\pi(F)$ the *link pattern* of F and write A_π for the number of FPLs F with link pattern $\pi(F) = \pi$. In [75] Wieland proved that A_π is invariant under the rotation of the link pattern π, i.e., $A_\pi = A_{\rho(\pi)}$ where $\rho(\pi)$ is the noncrossing matching connecting i and j iff $i-1$ and $j-1$ have been connected by π. This was conjecturally generalised by Razumov and Stroganov [65] where they related the vector $(A_\pi)_{\pi \in \mathrm{NC}_n}$ with the stationary distribution of a Markov process connected to the $O(\tau)$ loop model, see Subsection 3.2.3 for more details. This conjecture was proven by Cantini and Sportiello [19] in 2011.

The six-vertex model

Definition 2.2.6. *A* six-vertex configuration *of size n is an $n \times n$ grid with n external edges on every side together with an edge orientation such that*

- *every vertex has exactly two edges pointing towards it and two edges pointing away from it,*
- *the external edges at the top and bottom point outward and on the left and right point inward; this is called the* domain boundary wall condition[1] *(DBWC).*

[1]While this condition is in general not part of the definition of the six-vertex configuration - other boundary conditions yield also interesting and important results, see for example [12] - it is natural to include this condition in the setting of ASMs.

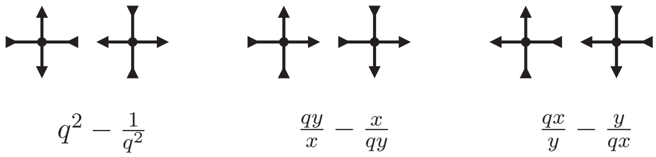


Figure 2.6: The six possible configurations at a vertex of a six-vertex configuration and their weights.

The name of this model originates from the fact that there are six possibilities to arrange the orientations at each vertex, see Figure 2.6. Six-vertex configurations are in bijection with ASMs by replacing the first vertex configuration in Figure 2.6 by 1, the second by -1 and the remaining vertex configurations by 0. A very important tool for the six-vertex model is the *partition function*

$$Z_n(q; \mathbf{x}, \mathbf{y}) := \sum_C \omega(C),$$

which is the weighted sum over all six-vertex configurations C of size n, where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. The weight $\omega(C)$ of a six-vertex configuration C is defined as

$$\omega(C) = \prod_{1 \leq i,j \leq n} \omega_{v(i,j)}(x_i, y_j),$$

where $v_{i,j}$ is the vertex in the i-th row from top and j-th column from left and the weight $\omega_v(x, y)$ is depending on the orientation of the edges incident to the vertex v, see Figure 2.6. It is possible to use different vertex weights, see for example [26, 51], which are equivalent to the weights described in Figure 2.6. Korepin found in [47] a set of properties which uniquely determine the partition function and Izergin [45] found a determinantal expression satisfying these properties. For the above choice of weights, which are for example used in [52], the partition function can be written as

$$Z_n(q; \mathbf{x}, \mathbf{y}) = \frac{\left(q^2 - \frac{1}{q^2}\right)^n \prod_{1 \leq i,j \leq n} \left(\frac{qx_i}{y_j} - \frac{y_j}{qx_i}\right)\left(\frac{qy_j}{x_i} - \frac{x_i}{qy_j}\right)}{\prod_{1 \leq i \neq j \leq n} \left(\frac{qx_i}{x_j} - \frac{x_j}{qx_i}\right)\left(\frac{qy_j}{y_i} - \frac{y_i}{qy_j}\right)} \times \det_{1 \leq i,j \leq n} \left(\frac{1}{\left(\frac{qx_i}{y_j} - \frac{y_j}{qx_i}\right)\left(\frac{qy_j}{x_i} - \frac{x_i}{qy_j}\right)} \right). \quad (2.5)$$

Square ice

Definition 2.2.7. *A* square ice configuration *of size n is an arrangement of n^2 water molecules where the oxygen atoms O are placed on an $n \times n$ square lattice, the hydrogen atoms H are placed between two oxygen atoms as well as to the left (resp. right) of the left (resp. right) boundary oxygen atoms, such that every oxygen atom forms together with two neighbouring hydrogen atoms a water molecule and every hydrogen atom is part of exactly one water molecule (for an example see Figure 2.4).*

For an example of a square ice configuration of size 5 see Figure 2.4. Square ice configurations are mapped bijectively to six-vertex configurations of the same size. Thereby every O atom is replaced by a vertex and every H atom induces an edge orientation towards the vertex corresponding to the O atom it is connected to. Since six-vertex configurations are in bijection to ASMs of the same size, this yields a bijection from square ice configurations to ASMs which is obtained by replacing horizontal aligned molecules by a 1, vertical aligned molecules by a -1 and all other molecules by 0.

Monotone triangles and Gogs

Definition 2.2.8. *Let m, n, k be non-negative integers with $k \leq n$. An (m, n, k)-*Gog trapezoid *is a trapezoidal array of positive integers of the form*

$$\begin{array}{cccc}
a_{1,1} & a_{1,2} & \cdots & a_{1,k} \\
\vdots & \vdots & \vdots & \vdots \\
a_{n-k,1} & a_{n-k,2} & \cdots & a_{n-k,k} \\
a_{n-k+1,1} & a_{n-k+1,2} & \cdots & a_{n-k+1,k} \\
\vdots & \vdots & \iddots & \\
a_{n-1,1} & a_{n-1,2} & & \\
a_{n,1} & & &
\end{array}$$

such that

- *the entries are weakly increasing along rows, weakly decreasing along columns, and strictly increasing along north-east diagonals,*
- *the entries of the i-th row from top are less than or equal to $m + n + 1 - i$ for $1 \leq i \leq n$.*

Gog trapezoids are sometimes defined as rotated versions of the above definition. They were first introduced by Mills, Robbins and Rumsey as $(0, n, k)$-Gog trapezoids in [58], however their name is due to Zeilberger [76]. The above presented generalisation is by Krattenthaler [48] (to be more precise its a variation of his definition). In [16], Biane and Cheballah presented a generalisation of the above, called *Gog pentagon*. There are two important statistics on Gog trapezoids, namely the number of *minima* and *maxima* of a Gog trapezoid.

- A *minimum* of a Gog trapezoid is a 1 entry.
- An entry $a_{i,k}$ of an (m, n, k)-Gog trapezoid is called a maximum iff $a_{i,k} = m + n + 1 - i$.

Definition 2.2.9. *A* monotone triangle *of size n is a $(0, n, n)$-Gog-trapezoid which is rotated by 45 degrees clockwise.*

We assign to a monotone triangle a weight Q^{Σ} where Σ is the number of entries $a_{i,j}$ of the triangle such that $a_{i+1,j} < a_{i,j} < a_{i+1,j+1}$.

We associate to an ASM of size n a monotone triangle of the same size, where the i-th row of the monotone triangle consists of the column-labels of the ASM for which the partial column-sum of the top i rows is positive. It is easy to see that this is a bijection between ASMs and monotone triangles of the same size where the weight of a monotone triangle corresponds to the number of -1 entries of the associated ASM.

2.2.2 The AST class

In the following we present *alternating sign triangles*, their generalisation to *AS-trapezoids* as well as their connection to centred Catalan sets and Motzkin paths. AS-trapezoids can further be represented as certain generalisations of *Gelfand-Tsetlin patterns* which are called $(\mathbf{s},\mathbf{t})$*-trees*, see Section 4.4 for more details.

Alternating sign triangles

Definition 2.2.10. *An* alternating sign triangle*, or short* AST*, of order n is a configuration of n centred rows where the i-th row, counted from bottom, has $2i-1$ entries in $\{0,\pm1\}$ such that*

- *all row-sums are* 1*,*
- *in all rows and columns the non-zero elements alternate,*
- *in every column the first non-zero entry from top is positive.*

The following is an example of an AST of order 6.

$$
\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & & 1 & -1 & 0 & 0 & 0 & 0 & 1 & & \\
 & & & 0 & 0 & 1 & 0 & 0 & & & \\
 & & & & 1 & -1 & 1 & & & & \\
 & & & & & 1 & & & & &
\end{array}
$$

We label the columns of an AST A of order n from left to right with $-n+1,\dots,n-1$ and the rows from bottom to top with $1,\dots,n$. In Proposition 4.2.2 we show that the set $\mathbf{S}(A)$ of columns with column-sum equal to 1 is a centred Catalan set of size n. In the above example the corresponding centred Catalan set is $\{-3,-1,0,1,3,4\}$. The refinement $\mathbf{M}(A) := \mathbf{M}(\mathbf{S}(A))$ of ASTs by Motzkin paths is due to Ayyer [8]. In the above example the associated Motzkin path is $\mathbf{M}(A) = (1,-1,1,0,-1)$. We are interested in the *weight function* $w(S)$ (resp. $w(M)$) of a centred Catalan set S (resp. Motzkin path M), which is defined as the number of ASTs A with $\mathbf{S}(A) = S$ (resp. $\mathbf{M}(A) = M$).

Alternating sign trapezoids

Following the definition of ASTs, we introduce a generalisation of ASTs to a trapezoidal shape.

Definition 2.2.11. *Let n, l be positive integers. An (n,l)-*AS-trapezoid *is an array of n centred rows where the i-th row from bottom has $l + 2i - 1$ entries, filled with $-1, 0$ or 1 such that*

- *all row-sums are 1,*
- *the column-sums are 0 for the central $l-1$ columns,*
- *in all rows and columns the non-zero elements alternate,*
- *in every column the first non-zero entry from top is positive.*

Alternating sign trapezoids were first introduced in [2] with bases of odd length. The above definition is more general as it allows bases of even length. The term (n,l)-AS-trapezoid in [2] corresponds to the term $(n,2l)$-AS-trapezoid in this thesis. ASTs of order $n+1$ and $(n,2)$-AS-trapezoids are in bijection by deleting the bottom 1 of an AST. The following enumeration formula for AS-trapezoids was independently conjectured by Behrend [13] and myself in [5, Theorem 2.8]. Behrend and Fischer [13] found a proof using the six-vertex model and Fischer [36] additionally found a proof based on her operator formula for monotone triangles.

Theorem 2.2.12 ([13, 36])**.** *The number of (n,l)-AS-trapezoids is*

$$2^{\lfloor \frac{n+1}{2} \rfloor \lfloor \frac{n+2}{2} \rfloor - \lfloor \frac{n}{2} \rfloor} \prod_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(i-1)!}{(n-i)!}$$
$$\times \prod_{i \geq 0} \left(\frac{l}{2} + 3i + 2 \right)_{\lfloor \frac{n-4i-1}{2} \rfloor} \left(\frac{l}{2} + 3i + 2 \right)_{\lfloor \frac{n-4i-2}{2} \rfloor}$$
$$\times \prod_{i \geq 0} \left(\frac{l}{2} + 2 \left\lfloor \frac{n}{2} \right\rfloor - i + \frac{1}{2} \right)_{\lfloor \frac{n-4i}{2} \rfloor} \left(\frac{l}{2} + 2 \left\lfloor \frac{n-1}{2} \right\rfloor - i + \frac{3}{2} \right)_{\lfloor \frac{n-4i-3}{2} \rfloor}, \quad (2.6)$$

where $(x)_n$ denotes the Pochhammer symbol $(x)_n := x(x+1)\cdots(x+n-1)$ for positive integers n and $(x)_n := 1$ if n is less than zero.

Remarkably, the above formula is related to two determinants; one is linked to the weighted enumeration of d-descending plane partitions (a generalisation of DPPs) and one is linked to the Q-enumeration of ASMs. For more details see Section 5.5.

We label the rows of an (n,l)-AS-trapezoid from bottom to top with $1, \dots, n$ and the columns from left to right by $-n+1, \dots, l+n-1$. Let A be an (n,l)-AS-trapezoid, we define $\mathbf{S}(A)$ to be the set obtained as follows: First, take the set of columns of A with positive column-sum, then subtract 1 from the columns with non-positive label. Now, subtract $l-1$ from the columns with positive labels, and finally add 0 to the set. In Proposition 4.2.2 we prove that $\mathbf{S}(A)$ is a centred Catalan set of size $n+1$.

The following is an example of a $(4,6)$-AS-trapezoid A with $\mathbf{S}(A) = \{-2,-1,0,1,3\}$.

$$\begin{array}{rcccccccccccccc}
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & \\
 & & & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & & \\
 & & & & 1 & 0 & 0 & 0 & -1 & 0 & 1 & & & \\
\textit{labels:} & -3 & -2 & \color{red}{-1} & \color{red}{0} & 1 & 2 & 3 & 4 & 5 & \color{red}{6} & 7 & \color{red}{8} & 9
\end{array}$$

Analogously to ASTs, we define $\mathbf{M}(A) := \mathbf{M}(\mathbf{S}(A))$ for an (n,l)-AS-trapezoid A. For a centred Catalan set S of size $n+1$ (resp. M a Motzkin path of length n), we define $w_l(S)$ (resp. $w_l(M)$) to be the number of (n,l)-trapezoids A with $\mathbf{S}(A) = S$ (resp. $\mathbf{M}(A) = M$). It is easy to see that the bijection between ASTs of order $n+1$ and $(n,2)$-AS-trapezoids commutes with the maps $\mathbf{S}$ and $\mathbf{M}$, which map ASTs and AS-trapezoids to centred Catalan sets or Motzkin paths, respectively. This implies

$$\begin{aligned} w_2(S) &= w(S), \\ w_2(M) &= w(M). \end{aligned}$$

The properties of $w_l(S)$ and $w_l(M)$ are the content of Chapter 4.

2.2.3 Plane Partitions and lozenge tilings

In the following subsection we are defining *plane partitions* and their connections to *lozenge tilings*. They can also be connected to *non-intersecting lattice paths* or to *perfect matchings*, a description of this can be found for example in [50].

Definition 2.2.13. *A* plane partition $\pi = (\pi_{i,j})$ *is an array of positive integers*

$$\begin{array}{ccccc}
\pi_{1,1} & \cdots & \cdots & \cdots & \pi_{1,\lambda_1} \\
\pi_{2,1} & \cdots & \cdots & \pi_{2,\lambda_2} & \\
\vdots & & \iddots & & \\
\pi_{n,1} & \cdots & \pi_{n,\lambda_n} & &
\end{array}$$

with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and such that the rows and columns are weakly decreasing, i.e., $\pi_{i,j} \geq \pi_{i+1,j}$ and $\pi_{i,j} \geq \pi_{i,j+1}$.

Plane partitions were introduced by MacMahon at the end of the 19th century [55]. One can visualise a plane partition π as stacks of unit cubes by putting a stack of $\pi_{i,j}$ many unit cubes at the position (i,j), see Figure 2.7 for an example. This point of view allows a second definition of plane partitions. A *plane partition* π is a finite subset of $(\mathbb{N} \setminus \{0\})^3$ such that if (i,j,k) is an element of π, then all triples (i',j',k') with $i' \leq i, j' \leq j, k' \leq k$ are elements of π. In the following we do not distinguish between the two definitions for plane partitions. We say that a plane partition fits inside an (r,s,t)-box if there are at most r rows, s columns and every entry is at most t; we call such a plane partition an (r,s,t)-boxed plane partition.

We can relate (r,s,t)-boxed plane partitions to *lozenge tilings* of a hexagon with side lengths r,s,t,r,s,t.

4	3	3	1
4	2	1	
2			

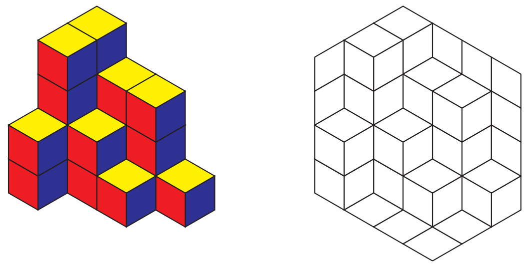


Figure 2.7: A plane partition which fits into a $(3,4,4)$-box, its graphical representation as stacks of cubes and its corresponding lozenge tiling of a hexagon with side lengths $4,4,3,4,4,3$.

Definition 2.2.14. *A* lozenge tiling*, or also* rhombus tiling*, of a hexagon with side lengths r,s,t,r,s,t (in clockwise direction) and all angles are $120°$, is a tiling by unit rhombi, i.e., they have side length 1 and the angles are $60°$ or $120°$, such that they do not overlap and cover the interior.*

We can associate to a lozenge tiling of a hexagon with side lengths r,s,t,r,s,t an (r,s,t)-boxed plane partition by shading the three types of rhombi differently and deleting the area of rhombi at the bottom, the area of rhombi at the right and the area of rhombi at the left. Then (i,j,k) is an element of the associated plane partition iff there is a box at position (i,j,k), see Figure 2.7 for an example. It is obvious that this yields a bijection between (r,s,t)-boxed plane partitions and lozenge tilings of a hexagon with side lengths r,t,s,r,t,s. The number of plane partitions inside a box is given by the following famous formula by MacMahon.

Theorem 2.2.15 ([57])**.** *The number of plane partitions fitting inside an (r,s,t)-box is given by*

$$\prod_{i=1}^{r}\prod_{j=1}^{s}\prod_{k=1}^{t}\frac{i+j+k-1}{i+j+k-2}.$$

There are two classes of plane partitions which are connected to ASMs. We will discuss them and their generalisations in the next two subsections.

2.2.4 The DPP class

This subsection contains the definitions of *column strict shifted plane partitions*, *d-descending plane partitions* and *lozenge tilings of a cored hexagon*. As a special case we present *cyclically symmetric plane partitions* and *descending plane partitions*.

Cyclically symmetric plane partitions and cored hexagons

Definition 2.2.16. *A* cyclically symmetric plane partition*, or short* CSPP*, is a plane partition π such that a triple (i,j,k) is an element of π iff (j,k,i) is also an element of π. In its graphical representation this means that the piles of boxes are*

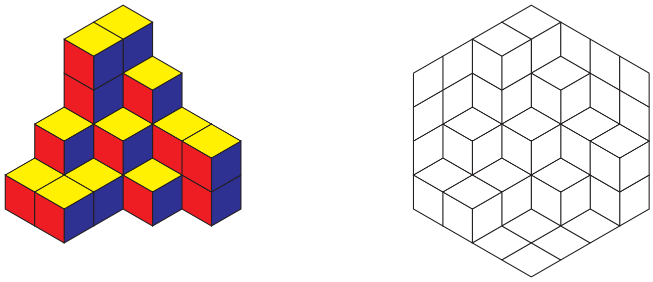


Figure 2.8: A cyclically symmetric plane partition of size 4 and its corresponding cyclically symmetric lozenge tiling.

invariant under the rotation of $120°$*, i.e., a cyclical change of the* x*-,* y*- and* z*-axis. We say that a CSPP is of size* n *if it fits inside an* (n, n, n)*-box.*

Cyclically symmetric plane partitions were first studied by MacMahon [56]. The following enumeration formula was conjectured by Macdonald [54] and proven by Mills, Robbins and Rumsey [59]. In [6], Andrews proved the special case of $q = 1$.

Theorem 2.2.17 ([54, 59])**.** *The weighted enumeration of cyclically symmetric plane partitions is given by*

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{n} \frac{1-q^{3i-1}}{1-q^{3i-2}} \prod_{j=1}^{n} \frac{1-q^{3(n+i+j-1)}}{1-q^{3(2i+j-1)}},$$

where the sum is over all cyclically symmetric plane partitions inside an (n, n, n)*-box and* $|\pi| = \sum_{1\le i,j\le n} \pi_{i,j}$*.*

Since (n, n, n)-boxed plane partitions are in bijection with lozenge tilings, it follows that CSPPs of size n are in bijection with cyclically symmetric (under rotation of $120°$) lozenge tilings of a regular hexagon with side length n. These lozenge tilings can be generalised in the following way.

Definition 2.2.18. *A* cyclically symmetric lozenge tiling of a cored hexagon *of side length* $a, a+x, a, a+x, a, a+x$ *is a lozenge tiling of a hexagon with side length* $a, a+x, a, a+x, a, a+x$ *where an equilateral triangle with side length* x *is removed from the centre, such that it vertices point towards the shorter edges of the hexagon (see Figure 2.9 for an example), and which is invariant under rotation of* $120°$*. We denote by* $CS(a, x)$ *the number of cyclically symmetric lozenge tilings of a cored hexagon with the above side lengths.*

These objects were first introduced by Ciucu and Krattenthaler [24]. They proved the following result for the number $CS(a, x)$, which was first proven by Andrews [6]

Theorem 2.2.19 ([24, Lemma 3.1])**.** *The number* $CS(a, x)$ *is given by*

$$CS(a, x) = \det_{0\le i,j\le a-1} \left(\delta_{ij} + \binom{x+i+j}{i} \right).$$

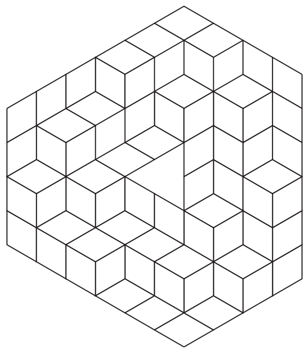


Figure 2.9: A cyclically symmetric lozenge tiling of a cored hexagon.

For an evaluation of this determinant see Theorem 8 of [6]. The above theorem was generalised in [23] to the weighted enumeration of cyclically symmetric lozenge tilings of cored hexagons. Further, this determinant is connected to the enumeration of AS-trapezoids and the Q-enumeration of ASMs, see Section 5.5.

Column strict shifted plane partitions and descending plane partitions

Definition 2.2.20. *Let k be a non-negative integer. A* column strict shifted plane partition π *of class k, or short CSSPP of class k, is an array of positive integers of the form*

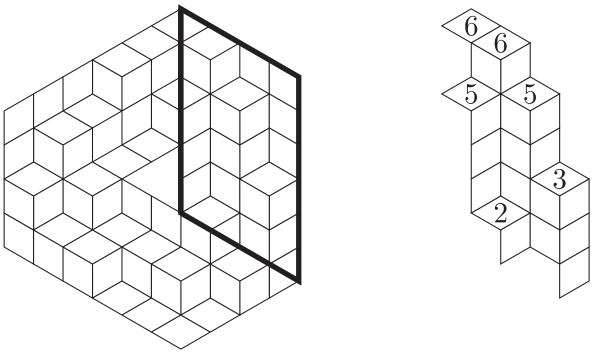
$$\begin{array}{ccccccccc}
\pi_{1,1} & & & \cdots & \cdots & \cdots & & & \pi_{1,\lambda_1} \\
& \pi_{2,2} & & \cdots & \cdots & \cdots & & \pi_{2,\lambda_2} & \\
& & \ddots & & & & \iddots & & \\
& & & \pi_{l,l} & \cdots & \pi_{l,\lambda_l} & & &
\end{array}$$

such that

- $\lambda_1 \geq \ldots \geq \lambda_l$,
- *the rows are weakly decreasing and the columns are strictly decreasing,*
- *the first entry in each row exceeds the number of entries in its row by k.*

We say that π has size n if all entries are less than or equal to $n+k-1$.

Column strict shifted plane partitions were first defined in [61]; the above definition was taken from [36]. CSPPs of class k and size n are in bijection to cyclically symmetric lozenge tilings of a cored hexagon of side length $n, n+k, n, n+k, n, n+k$. The bijection is obtained in the following way. Given a cyclically symmetric lozenge tiling, we first restrict the tiling to a fundamental region, see the region bounded by the thick lines in Figure 2.10 left. We regard the tiling of the fundamental region as a detail image of a three dimensional object, namely stacks of unit cubes; we label the ◇ lozenges by their height, where we interpret the bottom and rightmost ▱ lozenge as the face of a cube at height 1, see Figure 2.10 in the middle. It is an easy proof for the reader, that if there is no ▱ lozenge, then there is also no ◇ and the



6	6	5	3
	4	1	

Figure 2.10: A cyclically symmetric lozenge tiling of a cored hexagon (left), its fundamental region (middle) and its corresponding column strict shifted plane partition (right).

tiling will correspond to the empty CSSPP. Finally we rotate the array of numbers counter-clockwise by 30 degrees and subtract $i-1$ from the i-th row from top.

Definition 2.2.21. *Let $d \leq 1$. A* d-descending plane partition *π, or short d-DPP, is an array of positive integers of the same form as a CSSPP of class k such that*

- $\lambda_1 \geq \ldots \geq \lambda_l$,
- *the rows are weakly decreasing and the columns are strictly decreasing,*
- *the first entry in each row does not exceed the number of entries in the previous row less d and is larger than the number of entries in its own row less d.*

We say that π has size n if all entries of π are less than or equal to $n-d$.

These objects were first defined by Andrews in [6], however without giving them a name; the above definition was taken from [49]. An extension of the above definition to arbitrary d can be found in [49, p. 1144]. We can map CSSPPs of class k and size n bijectively to $(2-k)$-DPPs of size n by subtracting 1 of every entry of a CSSPP and deleting all 0 entries if $k>0$.

Beside CSPPs, the following is the most intriguing special case of d-DPPs.

Definition 2.2.22. *A* descending plane partition*, or short* DPP*, of size n is a 0-DPP of size n.*

As already mentioned in the introduction, DPPs of size n are equinumerous to ASMs of size n.

2.2.5 The TSSCPP class

We present in this subsection *TSSCPPs*, *Magog trapezoids* and the bijection from TSSCPPs to *Magog triangles*.

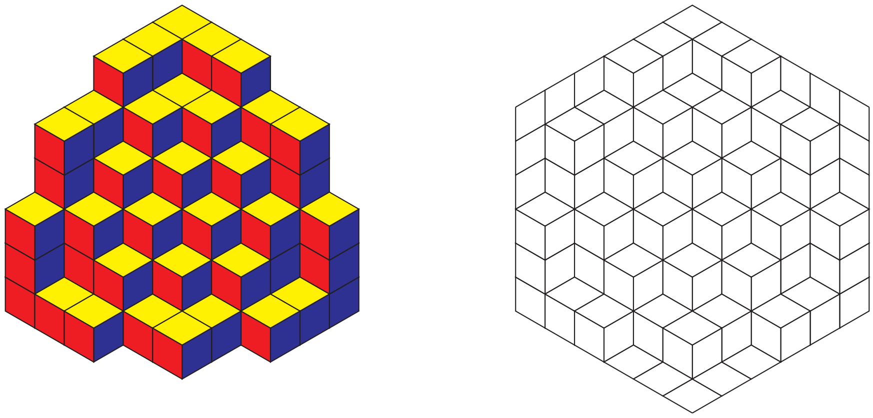


Figure 2.11: A TSSCPP of size 3 and its corresponding lozenge tiling.

Definition 2.2.23. *A* totally symmetric self complementary plane partition*, or short* TSSCPP*, π of size n is a $(2n, 2n, 2n)$-boxed plane partition which is totally symmetric, i.e., if $(i, j, k) \in \pi$ then all permutations of the triple (i, j, k) are elements of π and is self-complementary, i.e., the triple (i, j, k) is an element of π if and only if $(2n+1-i, 2n+1-j, 2n+1-k) \notin \pi$.*

TSSCPPs were first introduced by Mills, Robbins and Rumsey [58] and proven to have the same enumeration formula as DPPs (and hence also ASMs and ASTs) by Andrews in [7]. By looking at TSSCPP from a different point of view (by considering them as Magog triangles, see below) one can generalise them in the following way.

Definition 2.2.24. *An (m, n, k)-*Magog trapezoid *is an array of positive integers of the form*

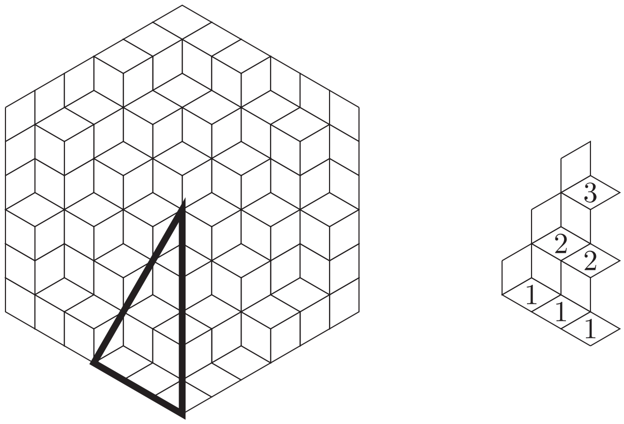
$$\begin{array}{cccccc}
a_{1,1} & a_{1,2} & \cdots & \cdots & \cdots & a_{1,n} \\
a_{2,1} & a_{2,2} & \cdots & \cdots & a_{2,n-1} & \\
\vdots & \vdots & & \iddots & & \\
a_{k,1} & a_{k,2} & \cdots & a_{k,n-k+1} & &
\end{array}$$

such that

- *the rows are weakly increasing and the columns are weakly decreasing,*
- *the i-th column from left is bound by $m+i$.*

We call a $(0, n, n)$-Magog trapezoid a Magog triangle *of size n.*

Magog trapezoids are sometimes defined as rotated versions of the above definition (which is due to Krattenthaler [48]). Magog-triangles and $(0, n, k)$-Magog trapezoids were first introduced by Mills, Robbins and Rumsey [58]; the name however is due to Zeilberger [76]. The above generalisation of $(0, n, k)$-Magogs to (m, n, k)-Magog trapezoids is due to Krattenthaler [48]. In [22] Biane and Cheballah introduced *GOGAm-trapezoids* which are in bijection to Magog trapezoids and were generalised to *GOGAm-pentagons* in [16].



1	2	3
1	2	
1		

Figure 2.12: A TSSCPP (left), its fundamental region (middle) and its corresponding Gog triangle (right).

TSSCPPs of size n are in bijection to Magog triangles of size n as follows. We start with a TSSCPP and restrict ourself to a twelfth (see the region within the thick triangle in Figure 2.12) of the corresponding lozenge tiling (the rest is forced by symmetry). We regard the twelfth piece of the TSSCPP as a part of a three dimensional picture and label the $\diamond$ lozenges by their height, where the bottom level of these lozenges has height 1. Finally we rotate the obtained array clockwise by 60 degree and obtain a Magog triangle of size n, see Figure 2.12 for an example.

There are two important statistics on Magog trapezoids, called *minima* and *maxima* which are defined as follows.

- A *minimum* of an (m, n, k)-Magog trapezoid is a 1 entry.
- A *maximum* of an (m, n, k)-Magog trapezoid is an entry $a_{i,j}$ with $a_{i,j} = i + m$.

The following conjecture by Krattenthaler is a generalisation of [58, Conjecture 7′].

Conjecture 2.2.25 ([48])**.** *The number of (m, n, k)-Gog trapezoids with s maxima in the rightmost north-east diagonal and t minima in the leftmost north-east diagonal is the same as the number of (m, n, k)-Magog trapezoids with t maxima in the first row and s minima in the last row.*

If we forget about the minima and maxima and set $m = 0$ the above was proven by Zeilberger [76, Lemma 1].

Chapter 3

Fully packed loop configurations: polynomiality and nested arches

The content of the following chapter has been published in [4] and appeared in the Proceedings of FPSAC 2016 [1].

3.1 Introduction

Zuber [81] formulated nine conjectures about the refined enumeration of fully packed loops (FPLs) with respect to their link pattern. In this chapter we prove one of these conjectures.

Theorem 3.1.1 ([81, Conjecture 7])**.** *For noncrossing matchings $\pi_1 \in \mathrm{NC}_{n_1}$, $\pi_2 \in \mathrm{NC}_{n_2}$ and an integer m, the number of FPLs with link pattern $(\pi_1)_m\pi_2$ is a polynomial in m of degree $|\lambda(\pi_1)| + |\lambda(\pi_2)|$ with leading coefficient $\frac{f^{\lambda(\pi_1)} f^{\lambda(\pi_2)}}{|\lambda(\pi_1)|!|\lambda(\pi_2)|!}$, where f^λ denotes the number of standard Young tableaux of shape λ.*

Caselli, Krattenthaler, Lass and Nadeau [21] proved this for empty π_2 and showed that $A_{(\pi_1)_m\pi_2}$ is a polynomial for large values of m with correct degree and leading coefficient. In this chapter we prove that the number $A_{(\pi_1)_m\pi_2}$ is a polynomial function in m, which is achieved without relying on the work of [21]. Together with the results of [21] this proves Theorem 3.1.1.

We conclude the introduction by sketching the theory on which the proof of Theorem 3.1.1 relies and giving an overview of this chapter. In the next section we supplement the definitions of Chapter 2 by further ones and present the Razumov-Stroganov-Cantini-Sportiello Theorem 3.2.4 which states that $(A_\pi)_{\pi\in\mathrm{NC}_n}$ is up to multiplication by a constant the unique eigenvector to the eigenvalue 1 of the Hamiltonian of the homogeneous $O(1)$ loop model. In Section 3.3 we show that solutions of the level 1 quantum Knizhnik-Zamolodchikov (qKZ) equations lie in the 1-dimensional eigenspace to the eigenvalue 1 of the Hamiltonian of the inhomogeneous $O(1)$ loop model, for the parameter q set equal to $e^{\frac{2\pi i}{3}}$. Di Francesco and

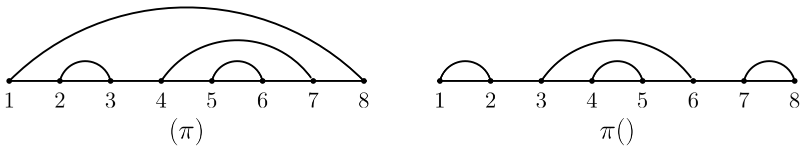


Figure 3.1: The noncrossing matchings (π) and $\pi()$ where $\pi = ()(())$.

Zinn-Justin [26] could characterise these solutions in a different way, namely as wheel polynomials. The specialisation of the inhomogeneous to the homogeneous $O(1)$ loop model means for wheel polynomials performing the evaluation $z_1 = \ldots = z_{2n} = 1$. Summarising, for every $\pi \in \mathrm{NC}_n$ there exists an element Ψ_π of the vector space $W_n[z]$ of wheel polynomials such that $A_\pi = \Psi_\pi(1, \ldots, 1)$.

$$
\begin{array}{lcccc}
\mathrm{FPLs} \overset{\mathrm{RSCS-Thm}}{\longleftrightarrow} \text{hom } O(1) & \overset{\text{specialisation}}{\longleftarrow} & \text{inhom } O(1) \overset{\text{Di F.}-\text{Z. J.}}{\longleftrightarrow} & & W_n[z] \\
A_\pi = \Psi_\pi(1, \ldots, 1) & \overset{\text{evaluation}}{\longleftarrow} & & & \Psi_\pi
\end{array}
$$

We introduce a new family of wheel polynomials D_{π_1,π_2} such that every $\Psi_{\rho^{n_2}(\pi_1\pi_2)}$ is a linear combination of D_{σ_1,σ_2}'s where n_2 is the size of π_2, $\pi_1\pi_2$ denotes the concatenation of π_1 and π_2, ρ is the rotation acting on noncrossing matchings and for $i = 1, 2$ the Young diagram $\lambda(\sigma_i)$ is included in the Young diagram $\lambda(\pi_i)$. The advantage of the wheel polynomials D_{π_1,π_2} over $\Psi_{\pi_1\pi_2}$ becomes clear in Section 3.4. We prove in Lemma 3.4.3 in a more general setting that $D_{\pi_1,\pi_2}(z_1, \ldots, z_n)|_{z_1=\ldots=z_n=1}$ is a polynomial function in n with degree at most $|\lambda(\pi_1)| + |\lambda(\pi_2)|$. This lemma applied in our situation and using the rotational invariance $A_\pi = A_{\rho(\pi)}$ imply the polynomiality in Theorem 3.1.1.

3.2 Definitions

The main objects have already been defined in Chapter 2. For the rest of this chapter we also need the following definitions.

3.2.1 Noncrossing matchings and Young diagrams

Denote for two noncrossing matchings σ, π by $\sigma\pi$ their concatenation. For an integer n we define $(\pi)_n$ as the noncrossing matching π surrounded by n *nested arches*, see Figure 3.1. We want to remind the reader that we associated bijectively to a noncrossing matching π of size n a Young diagram $\lambda(\pi)$, whose i-th row from top has at most $n - i$ boxes; for more details see Section 2.1. Further we depict $\lambda(\pi)$ together with a path consisting of n consecutive north steps followed by n east steps, see Figure 3.2. We define a partial order on the set NC_n of noncrossing matchings via $\sigma < \pi$ iff the Young diagram $\lambda(\sigma)$ is contained in the Young diagram $\lambda(\pi)$. For $2 \leq j \leq 2n - 2$ we write $\sigma \nearrow_j \pi$ if $\lambda(\pi)$ is obtained by adding a box to $\lambda(\sigma)$ on the j-th diagonal, where the diagonals are labelled as in Figure 3.2.

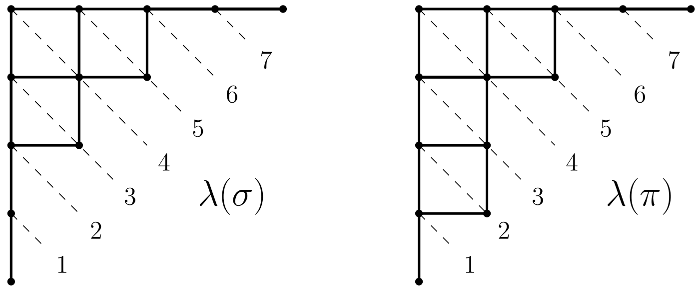


Figure 3.2: The matchings σ, π satisfy $\sigma \nearrow_2 \pi$.

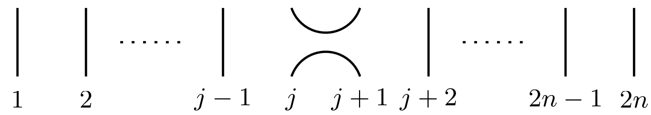


Figure 3.3: The graphical representation of e_j

3.2.2 The Temperley-Lieb Operators

The *rotation* ρ: $\mathrm{NC}_n \to \mathrm{NC}_n$. Two numbers i and j are connected in $\rho(\pi)$ for $\pi \in \mathrm{NC}_n$ iff $i-1$ and $j-1$ are connected in π, where we identify $2n+1$ with 1. The *Temperley-Lieb operator* e_j for $1 \leq j \leq 2n$ is a map from noncrossing matchings of size n to themselves. For a given $\pi \in \mathrm{NC}_n$ the noncrossing matching $e_j(\pi)$ is obtained by deleting the arches which are incident to the points $j, j+1$ and adding an arch between $j, j+1$ and an arch between the points which were connected to j and $j+1$, where we identify $2n+1$ with 1. There exists also a graphical representation of the Temperley-Lieb operators . Applying e_j on a noncrossing matching π is done by attaching the diagram of e_j, depicted in Figure 3.3, at the bottom of the diagram of π and simplifying the paths to arches. An example for this is given in Figure 3.4.

Since noncrossing matchings of size n are in bijection with Young diagrams whose i-th row from the top has at most $n-i$ boxes, we can define e_j also for such Young diagrams via $e_j(\lambda(\pi)) := \lambda(e_j(\pi))$. For $1 \leq j \leq 2n-1$ the action of e_j on Young diagrams is depicted in Figure 3.5. The operator e_{2n} maps a Young diagram to itself iff the i-th row has less than $n-i$ boxes for all $1 \leq i \leq n-1$. Otherwise the Young diagram corresponds to a noncrossing matching of the form $(\alpha)\beta(\gamma)$, where α, β, γ are noncrossing matchings of smaller size. In this case, e_{2n} maps this Young diagram to the one corresponding to the noncrossing matching $(\alpha(\beta)\gamma)$, as depicted in Figure 3.6. The next lemma is an easy consequence of the above observations.

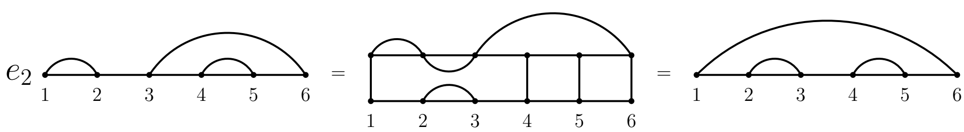


Figure 3.4: Calculating $e_j(\pi)$ graphically where π is from the previous example.

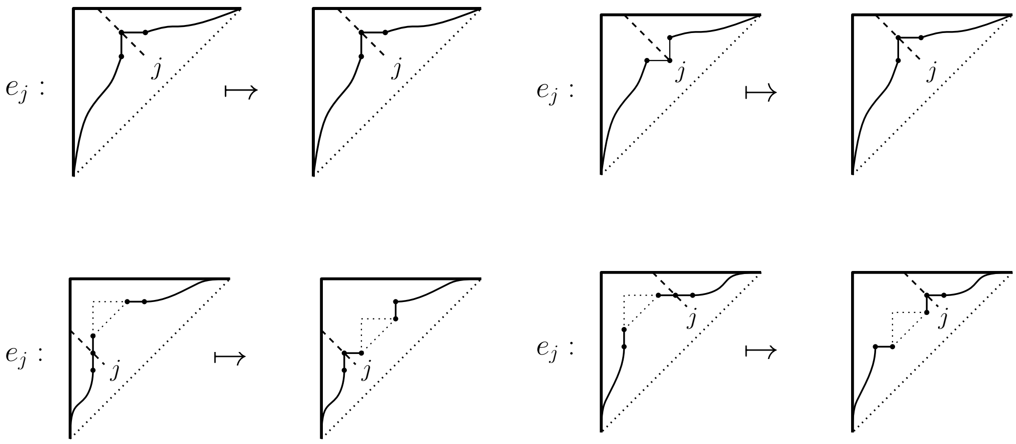


Figure 3.5: The action of e_j for $1 \leq j \leq 2n-1$ on Young diagrams corresponding to noncrossing matchings of size n.

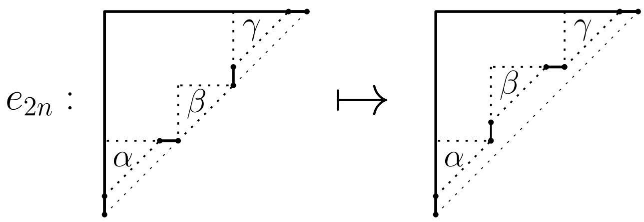


Figure 3.6: The action of e_{2n} on Young diagrams corresponding to noncrossing matchings of size n of the form $(\alpha)\beta(\gamma)$, where α, β, γ are noncrossing matchings.

Lemma 3.2.1. *1. For a noncrossing matching π of size n and $2 \leq j \leq 2n-2$, the preimage $e_j^{-1}(\pi)$ is a subset of $\{\sigma | \pi \nearrow_j \sigma\} \cup \{\sigma | \sigma \leq \pi\}$.*

2. Let $\alpha \in \mathrm{NC}_n$, $\beta, \gamma \in \mathrm{NC}_{n'}$ be noncrossing matchings such that there exists $2 \leq i \leq 2n'-2$ with $\beta \nearrow_i \gamma$. Then the preimage $e_{2n+i}^{-1}(\alpha\beta)$ is given by

$$e_{2n+i}^{-1}(\alpha\beta) = \{\alpha\sigma | \sigma \in e_i^{-1}(\beta)\}.$$

Proof. 1. If π has no arch between j and $j+1$, then $e_j^{-1}(\pi) = \emptyset$. Figure 3.6 displays the action of e_j on Young diagrams and implies the statement if π has an arch between j and $j+1$.

2. Let $\sigma \in e_{2n+i}^{-1}(\alpha\beta)$ and denote by x, y the integers which are connected in σ to $2n+i$ or $2n+i+1$ respectively. By definition of e_{2n+i} the noncrossing matchings $\alpha\beta$ and σ differ only in the arches between $2n+i, 2n+i+1, x, y$. The existence of a γ with $\beta \nearrow_i \gamma$ means there exists an arch in β with left-endpoint before i and right-endpoint after i. Hence there exists an arch in $\alpha\beta$ surrounding $2n+i$ and $2n+i+1$. Therefore x and y must be surrounded by this arch or they are the labels of the points connected by this arch. In both cases $x, y \geq 2n$ which implies σ can be written as $\alpha\sigma'$ with $e_i(\sigma') = \beta$. □

The *Temperley-Lieb algebra* with parameter $\tau = -(q+q^{-1})$ of size $2n$ is generated by the Temperley-Lieb operators e_i with $1 \leq i \leq 2n$ over $\mathbb{C}$. The elements e_i, e_j satisfy for all $1 \leq i, j \leq 2n$ the following relations

$$\begin{aligned} e_i^2 &= \tau e_i, \\ e_i e_j &= e_j e_i \qquad \text{if } 2 \leq |(i-j)| \leq 2n-2, \\ e_i e_{i\pm 1} e_i &= e_i. \end{aligned}$$

Throughout this chapter we interpret $e_i v$ for some vector $v \in \{f | f\colon \mathrm{NC}_n \to V\}$ and a vector space V always as the action of an element of the Temperley-Lieb algebra on the vector v, where the Temperley-Lieb operators act as permutations, i. e., $e_i((v_\pi)_{\pi \in \mathrm{NC}_n}) = (v_{e_i(\pi)})_{\pi \in \mathrm{NC}_n}$.

3.2.3 The (in-)homogeneous $O(\tau)$ loop model

A configuration of the *inhomogeneous $O(\tau)$ loop model*, sometimes also called *cylindrical loop percolation*, of size n is a tiling of $[0, 2n] \times [0, \infty)$ with plaquettes of side length 1 depicted in Figure 3.7. To obtain a cylinder we identify the half-lines $\{(0,t), t \geq 0\}$ and $\{(2n,t), t \geq 0\}$. In the following we assume that the configurations are filled randomly with the two plaquettes, where the probability to place the first plaquette of Figure 3.7 in column i is p_i with $0 < p_i < 1$ for all $1 \leq i \leq 2n$. If the probability does not depend on the column, i. e., $p_1 = \ldots = p_{2n}$, we call it the *homogeneous $O(\tau)$ loop model*. We parametrise the probabilities $p_i = \frac{qz_i - q^{-1}t}{qt - q^{-1}z_i}$ and set $\tau = q + q^{-1}$. The two plaquettes in Figure 3.7 are interpreted to consist of two paths. By concatenating the paths of a plaquette with the paths of the neighbouring plaquettes, we see that a configuration consists of noncrossing paths.

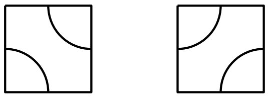


Figure 3.7: The two different plaquettes.

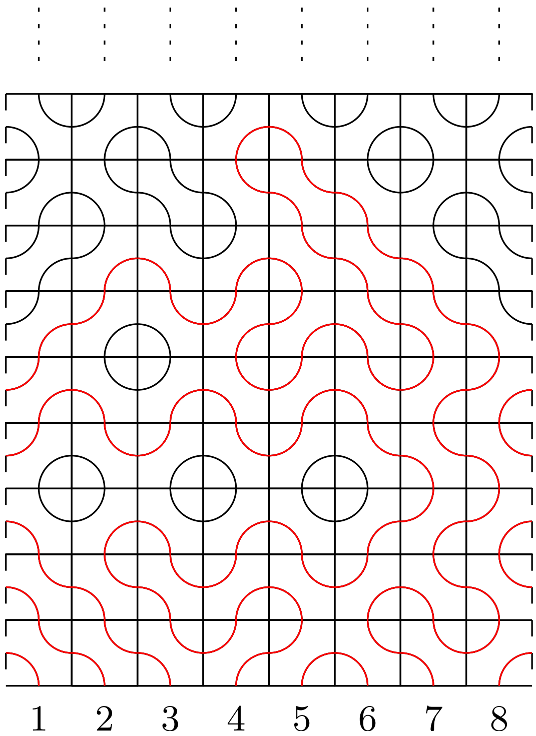


Figure 3.8: The beginning of a configuration of the inhomogeneous $O(\tau)$ loop model, where the paths starting and ending at the bottom are drawn in red.

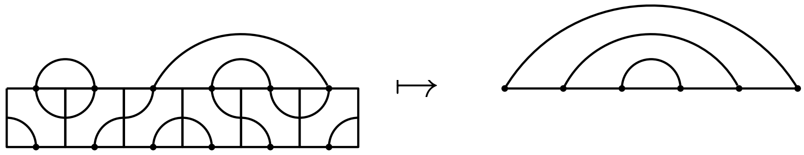


Figure 3.9: An example for a state transition of the noncrossing matching $\pi = ()(())$. The transition probability is $p_1(1-p_2)(1-p_3)p_4p_5(1-p_6)$.

Lemma 3.2.2. *With probability* 1 *all paths in a random configuration are finite.*

A proof for the homogeneous case can be found in [70, Lemma 1.6], the inhomogeneous case can be proven analogously. For a configuration C of the $O(\tau)$ loop model, we label the points $(i-\frac{1}{2},0)$ with i for $1 \leq i \leq 2n$. We define the *connectivity pattern* $\pi(C)$ as the noncrossing matching connecting i and j by an arch iff i and j are connected by paths in C. By the above lemma $\pi(C)$ is well defined for almost all cylindrical loop percolations C. For $\pi \in \mathrm{NC}_n$ denote by $\hat{\Psi}_\pi(t; z_1, \ldots, z_{2n})$ the probability that a configuration C has the connectivity pattern π and write $\hat{\Psi}_n(t; z_1, \ldots, z_{2n}) = (\hat{\Psi}_\pi(t; z_1, \ldots, z_{2n}))_{\pi \in \mathrm{NC}_n}$.

We define a Markov chain on the set NC_n of noncrossing matchings of size n in the following way. The transitions are given by putting $2n$ plaquettes below a noncrossing matching and simplify the paths to obtain a new noncrossing matching. An example is given in Figure 3.9. The probability of one transition is given by the product of the probabilities of placing the plaquettes, where placing the first plaquette of Figure 3.7 at the i-th position is p_i as before. We denote by $T_n(t; z_1, \ldots, z_{2n})$ the transition matrix of this Markov chain. By the Perron-Frobenius Theorem the matrix $T_n(t; z_1, \ldots, z_{2n})$ has 1 as an eigenvalue and the stationary distribution of the Markov chain is up to scaling the unique eigenvector with associated eigenvalue 1. Every configuration C of the inhomogeneous $O(\tau)$ loop model can be obtained uniquely by pushing all the plaquettes of a configuration C' one row up and filling the empty bottom row with plaquettes. Therefore the vector $\hat{\Psi}_n(t; z_1, \ldots, z_{2n})$ is the stationary distribution of this Markov chain and hence satisfies

$$T_n(t; z_1, \ldots, z_{2n})\hat{\Psi}_n(t; z_1, \ldots, z_{2n}) = \hat{\Psi}_n(t; z_1, \ldots, z_{2n}). \tag{3.1}$$

Theorem 3.2.3. *Let n be a positive integer and define the* Hamiltonian *as the linear map $\mathcal{H}_n := \sum_{j=1}^{2n} e_j$, where e_j is interpreted as an element of the Temperley-Lieb algebra. The specialised stationary distribution $\hat{\Psi}_n(t) = \hat{\Psi}_n(t; 1, \ldots, 1)$ satisfies for $\tau = 1$*

$$\mathcal{H}_n(\hat{\Psi}_n(t)) = 2n\hat{\Psi}_n(t). \tag{3.2}$$

Further $\hat{\Psi}_n(t)$ is independent of t and uniquely determined by (3.2).

A proof of this theorem can be found for example in [70, Appendix B], however note that the matrix H_n defined there is given by $2n \cdot \mathrm{Id} - \mathcal{H}_n$.

The following theorem was conjectured by Razumov and Stroganov in [65] and later proven by Cantini and Sportiello in [19]. It creates a connection between fully packed loop configurations and the stationary distribution of the homogeneous $O(1)$ loop model.

Theorem 3.2.4 (Razumov-Stroganov-Cantini-Sportiello Theorem). *Let $n \in \mathbb{N}$, set $q = e^{\frac{2\pi i}{3}}$ and $\hat{\Psi}_\pi = \hat{\Psi}_\pi(-q; 1, \ldots, 1)$. For all $\pi \in \mathrm{NC}_n$ holds*

$$\hat{\Psi}_\pi = \frac{A_\pi}{A_n}.$$

3.3 The vector space $W_n[z]$

3.3.1 The quantum Knizhnik-Zamolodchikov equations

In order to introduce the *quantum Knizhnik-Zamolodchikov equations*, or short qKZ-equations, we need to define the R-matrix and another operator S_i

$$\check{R}_i(u) = \frac{(qu - q^{-1})\,\mathrm{Id} + (u-1)e_i}{q - q^{-1}u},$$

$$S_i(z_1, \ldots, z_{2n}) = \prod_{k=1}^{i-1} \check{R}_{i-k}\left(\frac{z_{i-k}}{q^6 z_i}\right) \rho \prod_{k=1}^{2n-i} \check{R}_{2n-k}\left(\frac{z_{2n-k+1}}{z_i}\right),$$

for $1 \leq i \leq 2n$, where e_i is understood as an element of the Temperley-Lieb algebra and ρ is the rotation as defined in Subsection 3.2.2.

Proposition 3.3.1 ([79, Section 4.1]). *The R-matrices satisfy the* Yang-Baxter equation

$$\check{R}_i(z)\check{R}_{i+1}(zw)\check{R}_i(w) = \check{R}_{i+1}(w)\check{R}_i(zw)\check{R}_{i+1}(z),$$

for $1 \leq i \leq 2n-1$ and the unitary equation for $1 \leq i \leq 2n$

$$\check{R}_i(z)\check{R}_i\left(\frac{1}{z}\right) = \mathrm{Id}\,.$$

Denote by $\Psi_n = (\Psi_\pi)_{\pi \in \mathrm{NC}_n}$ a function in $z_1, \ldots, z_{2n}, q$. The level 1 *qKZ-equations* are a system of $2n$ equations

$$S_i(z_1, \ldots, z_{2n})\Psi_n(t; z_1, \ldots, z_{2n}) = \Psi_n(t; z_1, \ldots, q^6 z_i, \ldots, z_{2n}), \tag{3.3}$$

with $1 \leq i \leq 2n$. In the following we need the $2n+1$ equations

$$\check{R}_i\left(\frac{z_{i+1}}{z_i}\right)\Psi_n(t; z_1, \ldots, z_{2n}) = \Psi_n(t; z_1, \ldots, z_{i+1}, z_i, \ldots, z_{2n}), \tag{3.4a}$$

$$\rho^{-1}\Psi_n(t; z_1, \ldots, z_{2n}) = \Psi_n(t; z_2, \ldots, z_{2n}, q^6 z_1), \tag{3.4b}$$

where $1 \leq i \leq 2n$ in (3.4a).

Proposition 3.3.2 ([79, Section 4.1 and 4.3]). *1. The system of equations* (3.4) *implies the system of equations* (3.3).

2. *For $q = e^{\frac{2\pi i}{3}}$ and hence $\tau = 1$ holds $S_i(z_1, \ldots, z_{2n}) = T_n(z_i; z_1, \ldots, z_{2n})$. By using Lagrange interpolation one can show that* (3.3) *implies* (3.1)*. Since the solutions of* (3.1) *form a one dimensional vector space, the same is true for solutions of the system of equations* (3.3) *for $q = e^{\frac{2\pi i}{3}}$, if the set of solutions of* (3.3) *is not empty.*

3.3.2 Wheel polynomials

It turns out [26, Theorem 4] that for $q = e^{\frac{2\pi i}{3}}$ the components $\hat{\Psi}_\pi(t; z_1, \ldots, z_{2n})$ of the stationary distribution of the inhomogeneous $O(1)$ loop model are up to a common factor homogeneous polynomials in $z_1, \ldots, z_{2n}$ of degree $n(n-1)$ which are independent of t. In this section we characterise these homogeneous polynomials. In fact we characterise homogeneous solutions of degree $n(n-1)$ of (3.4a) and (3.4b) which are by Proposition 3.3.2 for $q = e^{\frac{2\pi i}{3}}$ also solutions of (3.1). The results presented here can be found in [26, 42, 43, 79, 80] and [70].

Definition 3.3.3. *Let n be a positive integer and q a variable. A homogeneous polynomial $p \in \mathbb{Q}(q)[z_1, \ldots, z_{2n}]$ of degree $n(n-1)$ is called* wheel polynomial *of order n if it satisfies the* wheel condition*:*

$$p(z_1, \ldots, z_{2n})|_{q^4 z_i = q^2 z_j = z_k} = 0, \tag{3.5}$$

for all triples $1 \leq i < j < k \leq 2n$. Denote by $W_n[z]$ the $\mathbb{Q}(q)$-vector space of wheel polynomials of order n.

Theorem 3.3.4 ([42, Section 4.2])**.** *The dual space $W_n[z]^*$ of $W_n[z]$ is given by*

$$W_n[z]^* = \bigoplus_{\pi \in \mathrm{NC}_n} \mathbb{Q}(q)\, \mathrm{ev}_\pi,$$

where ev_π is defined as $\mathrm{ev}_\pi : p(z_1, \ldots, z_{2n}) \mapsto p(q^{\epsilon_1(\pi)}, \ldots, q^{\epsilon_{2n}(\pi)})$ with $\epsilon_i(\pi) = -1$ iff an arch of π has a left-endpoint labelled with i and $\epsilon_i(\pi) = 1$ otherwise.

Define the linear maps $\mathbf{S}_k, \mathbf{D}_k : \mathbb{Q}(q)[z_1, \ldots, z_{2n}] \to \mathbb{Q}(q)[z_1, \ldots, z_{2n}]$ for $1 \leq k \leq 2n$ as

$$\mathbf{S}_k : \quad f(z_1, \ldots, z_{2n}) \mapsto f(z_1, \ldots, z_{k+1}, z_k, \ldots, z_{2n}), \tag{3.6}$$

$$\mathbf{D}_k : \quad f \mapsto \frac{q z_k - q^{-1} z_{k+1}}{z_{k+1} - z_k}(\mathbf{S}_k(f) - f). \tag{3.7}$$

By setting $\mathbf{D}_{k+2n} := \mathbf{D}_k$ we extend the definition of $\mathbf{D}_k$ to all integers k. The operators $\mathbf{D}_k$ are introduced as an abbreviation for $(q z_k - q^{-1} z_{k+1})\delta_k$, where $\delta_k = \frac{1}{z_{k+1} - z_k}(\mathbf{S}_k - \mathrm{Id})$ has been used before , e. g., in [79]. One can verify easily the following Lemma.

Lemma 3.3.5. *1. The space $W_n[z]$ of all wheel polynomials of order n is closed under the action of $\mathbf{D}_k$ for $1 \leq k \leq 2n-1$. If $q = e^{\frac{2\pi i}{3}}$ the vector space $W_n[z]$ is also closed under $\mathbf{D}_{2n}$.*

2. For all $1 \leq k \leq 2n$ and all polynomials $f, g \in \mathbb{Q}(q)[z_1, \ldots, z_{2n}]$ one has

$$\mathbf{D}_k(fg) = \mathbf{D}_k(f)\mathbf{S}_k(g) + f\mathbf{D}_k(g). \tag{3.8}$$

The following theorem describes a very important $\mathbb{Q}(q)$-basis of $W_n[z]$.

Theorem 3.3.6 ([79, Section 4.2])**.** *Set*

$$\Psi_{()_n}(z_1, \ldots, z_{2n}) := (q - q^{-1})^{-n(n-1)} \prod_{1 \leq i < j \leq n} (qz_i - q^{-1}z_j)(qz_{n+i} - q^{-1}z_{n+j}). \tag{3.9}$$

Define for two noncrossing matchings σ, π with $\sigma \nearrow_j \pi$

$$\Psi_\pi := \mathbf{D}_j(\Psi_\sigma) - \sum_{\tau \in e_j^{-1}(\sigma) \setminus \{\sigma, \pi\}} \Psi_\tau. \tag{3.10}$$

Then Ψ_π is well-defined for all $\pi \in \mathrm{NC}_n$ and satisfies

$$\Psi_{\rho^{-1}(\pi)}(z_1, \ldots, z_{2n}) = \Psi_\pi(z_2, \ldots, z_{2n}, q^6 z_1). \tag{3.11}$$

The set $\{\Psi_\pi, \pi \in \mathrm{NC}_n\}$ is a $\mathbb{Q}(q)$-basis of $W_n[z]$.

The noncrossing matchings τ which appear in the sum of (3.10) satisfy by Lemma 3.2.1 the relation $\tau < \pi$. Hence we can use (3.10) to calculate the basis Ψ_π of $W_n[z]$ recursively. The vector $\Psi_n = (\Psi_\pi)_{\pi \in \mathrm{NC}_n}$ satisfies (3.4a). This is true since we can reformulate (3.4a) as

$$e_i \Psi_n = \mathbf{D}_i(\Psi_n) - (q + q^{-1})\Psi_n, \tag{3.12}$$

for $1 \leq i \leq 2n-1$. Let $\sigma, \pi \in \mathrm{NC}_n$ with $\sigma \nearrow_i \pi$, then the σ component of both sides in (3.12) is

$$\Psi_\pi - (q + q^{-1})\Psi_\sigma + \sum_{\tau \in e_i^{-1}(\sigma) \setminus \{\sigma, \pi\}} \Psi_\tau = \mathbf{D}_i(\Psi_\sigma) - (q + q^{-1})\Psi_\sigma,$$

which is exactly (3.10). Since Ψ_n satisfies (3.4b) by Theorem 3.3.6, Proposition 3.3.2 states that Ψ_n is a solution of the qKZ equations and therefore for $\tau = 1$ a multiple of the stationary distribution of the inhomogeneous $O(1)$ loop model. By setting $z_1 = \ldots = z_{2n} = 1$ Theorem 3.2.4 implies $\Psi_\pi(1, \ldots, 1)|_{\tau=1} = cA_\pi$ for an appropriate constant c. Because of $\Psi_{()_n}(1, \ldots, 1)|_{\tau=1} = 1 = A_{()_n}$ by definition, and Theorem 3.2.4 we obtain the following theorem.

Theorem 3.3.7. *Set $q = e^{\frac{2\pi i}{3}}$ and let $\pi \in \mathrm{NC}_n$, then one has*

$$\begin{aligned} \Psi_\pi(z_1, \ldots, z_{2n}) &= A_n \cdot \hat{\Psi}_n(t; z_1, \ldots, z_{2n}), \\ \Psi_\pi(1, \ldots, 1) &= A_\pi, \end{aligned}$$

where A_n denotes the number of ASMs of size n.

The above theorem obviously implies that $\sum_{\pi\in \mathrm{NC}_n} \Psi_\pi(1,\ldots,1)|_{\tau=1} = A_n$ holds for $q = e^{\frac{2\pi i}{3}}$. In fact an even stronger statement is true. Recall that FPLs are in bijection to configurations of the six-vertex model and that $Z_n(q;\mathbf{x},\mathbf{y})$ denotes the partition function, i.e., a weighted enumeration of configurations of the six-vertex model which is defined in Subsection 2.2.1. The following theorem relates the partition function and wheel polynomials.

Theorem 3.3.8 ([26, Theorem 5])**.** *For $q = e^{\frac{2\pi i}{3}}$, the following holds*

$$\left((-1)^{n(n-1)}\left(\frac{1}{q}-q\right)^{-n^2}\prod_{i=1}^{2n} z_i^{\frac{n-1}{2}}\right) Z_n(\sqrt{q};\sqrt{z_1},\ldots,\sqrt{z_{2n}}) = \sum_{\pi\in\mathrm{NC}_n}\Psi_\pi.$$

Theorem 3.3.4 states that a wheel polynomial $p \in W_n[z]$ is uniquely determined by its evaluations at the points $(q^{\epsilon_1(\pi)},\cdots,q^{\epsilon_{2n}(\pi)})$ for $\pi \in \mathrm{NC}_n$. Since $\mathfrak{S}_{2n}$ operates transitively on the set of these points, the $\mathbb{Q}(q)$-vector space $W_n[z]^{\mathfrak{S}_{2n}}$ of symmetric wheel polynomials is one dimensional. If $q = e^{\frac{2\pi i}{3}}$ the partition function is symmetric, see for example [74], hence the vector space of symmetric wheel polynomials is generated over $\mathbb{Q}(q)$ by $\prod_{i=1}^{2n} z_i^{\frac{n-1}{2}} Z_n(\sqrt{q};\sqrt{z_1},\ldots,\sqrt{z_{2n}})$.

3.3.3 A new basis for $W_n[z]$

The following lemma is a direct consequence of the definitions of the $\mathbf{D}_i$'s and $\Psi_{()_n}$.

Lemma 3.3.9. *Let n be a positive integer, then one has*

1. $\mathbf{D}_i \circ \mathbf{D}_i = (q+q^{-1})\mathbf{D}_i$ *for* $1 \le i \le 2n$,
2. $\mathbf{D}_i \circ \mathbf{D}_j = \mathbf{D}_j \circ \mathbf{D}_i$ *for* $1 \le i,j \le 2n$ *with* $|i-j| > 1$,
3. $\mathbf{D}_{i+1} \circ \mathbf{D}_i \circ \mathbf{D}_{i+1} + \mathbf{D}_i = \mathbf{D}_i \circ \mathbf{D}_{i+1} \circ \mathbf{D}_i + \mathbf{D}_{i+1}$ *for* $1 \le i \le 2n$,
4. $\mathbf{D}_i(\Psi_{()_n}) = (q+q^{-1})\Psi_{()_n}$ *for* $i \notin \{n, 2n\}$.

In the following we write $\prod_{i=1}^k \mathbf{D}_i$ for the product $\mathbf{D}_1 \circ \ldots \circ \mathbf{D}_k$. Let π be a noncrossing matching with corresponding Young diagram $\lambda(\pi) = (\lambda_1,\ldots,\lambda_l)$, i.e., λ_i is the number of boxes of $\lambda(\pi)$ in the i-th row from top. We define the wheel polynomial D_π by the following algorithm. First write in every box of $\lambda(\pi)$ the number of the diagonal the box lies on. The wheel polynomial D_π is then constructed recursively by "reading" in the Young diagram $\lambda(\pi)$ the rows from top to bottom and in the rows all boxes from left to right and apply $\mathbf{D}_{\text{number in the box}}$ to the previous wheel polynomial, starting with $\Psi_{()_n}$, which is defined in (3.9). For π as in Figure 3.10 we obtain

$$D_\pi = (\mathbf{D}_{n-3} \circ \mathbf{D}_{n-2} \circ \mathbf{D}_n \circ \mathbf{D}_{n-1} \circ \mathbf{D}_{n+3} \circ \mathbf{D}_{n+2} \circ \mathbf{D}_{n+1} \circ \mathbf{D}_n)(\Psi_{()_n}).$$

Alternatively we can write D_π directly as

$$D_\pi = \left(\prod_{i=1}^{l}\prod_{j=1}^{\lambda_{l+1-i}} \mathbf{D}_{n+(i-l)+(\lambda_{l+1-i}-j)}\right)(\Psi_{()_n}). \tag{3.13}$$

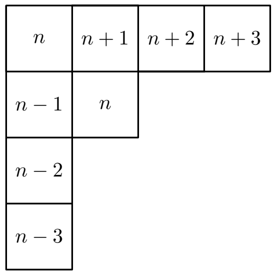


Figure 3.10: The numbers indicate the labels of the diagonals the boxes lie on.

Theorem 3.3.10. *The set of wheel polynomials $\{D_\pi | \pi \in \mathrm{NC}_n\}$ is a $\mathbb{Q}(q)$-basis of $W_n[z]$. Further Ψ_π is for $\pi \in \mathrm{NC}_n$ a linear combination of D_τ's with $\tau \leq \pi$ and the coefficient of D_π is 1.*

Proof. We prove the second statement by induction on the number of boxes of $\lambda(\pi)$. It is by definition true for $\pi = ()_n$, hence let the number $|\lambda(\pi)|$ be non-zero. Let σ be the noncrossing matching such that $\lambda(\sigma)$ is the Young diagram one obtains by deleting the rightmost box in the bottom row of $\lambda(\pi)$, and let i be the integer such that $\sigma \nearrow_i \pi$. Then Theorem 3.3.6 states

$$\Psi_\pi = \mathbf{D}_i \Psi_\sigma - \sum_{\tau \in e_i^{-1}(\sigma) \setminus \{\sigma, \pi\}} \Psi_\tau.$$

We use the induction hypothesis to express Ψ_τ and Ψ_σ as sums of $D_{\tau'}$ with $\tau' \leq \tau < \pi$ or $D_{\sigma'}$ with $\sigma' \leq \sigma < \pi$ respectively. The coefficient of D_σ in Ψ_σ is by the induction hypothesis equal to 1. Since all $\sigma' \leq \sigma$ satisfy the requirements of Lemma 3.3.11, this lemma implies the statement. By above arguments the set $\{D_\pi | \pi \in \mathrm{NC}_n\}$ is a $\mathbb{Q}(q)$-generating set for $W_n[z]$ of cardinality $\dim_{\mathbb{Q}(q)}(W_n[z])$, hence it is also a $\mathbb{Q}(q)$-basis. $\square$

The next lemma contains the technicalities which are needed to prove the above theorem.

Lemma 3.3.11. *Let $1 < i < 2n$ and $\sigma \in \mathrm{NC}_n$ such that the number of boxes on the i-th diagonal of $\lambda(\sigma)$ is less than the maximal possible number of boxes that can be placed there. Then $\mathbf{D}_i(D_\sigma) = D_\pi$ iff there exists a $\pi \in \mathrm{NC}_n$ with $\sigma \nearrow_i \pi$ or otherwise $\mathbf{D}_i(D_\sigma)$ is a $\mathbb{Q}(q)$-linear combination of D_τ's with $\tau \leq \sigma$.*

Proof. We use induction on the number of boxes of $\lambda(\sigma)$. We say that i appears in σ if there is a box in $\lambda(\sigma)$ which lies on the i-th diagonal.

1. Assume that i does not appear in σ. This implies that $i-1$ and $i+1$ can appear in σ, but not both at the same time.

   (a) Assume that $i+1, i-1$ do not appear in σ. By Lemma 3.3.9 $\mathbf{D}_i$ commutes with all the $\mathbf{D}$-operators appearing in D_σ. If $i \neq n$ Lemma 3.3.9 states $\mathbf{D}_i(\Psi_{()_n}) = (q+q^{-1})\Psi_{()_n}$ and hence $\mathbf{D}_i(D_\sigma) = (q+q^{-1})D_\sigma$. The case $i = n$ implies $\sigma = ()_n$ and hence $\mathbf{D}_i(D_\sigma) = D_{(()())_{n-2}}$.

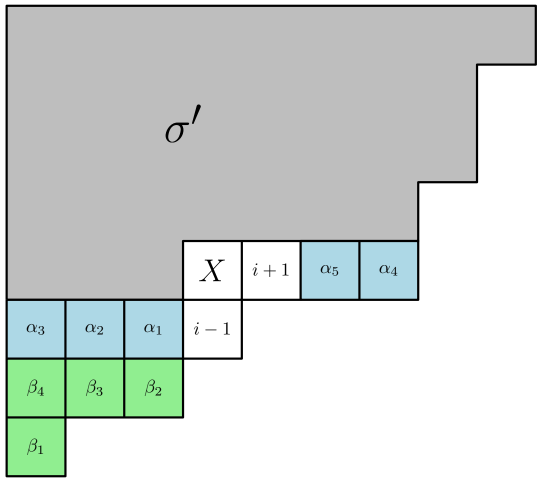


Figure 3.11: Schematic representation of $\lambda(\sigma)$ for σ as in the second case of the proof of Lemma 3.3.11 with $a = b = 1$.

(b) Let $i+1$ appear in σ. Then there is only one box on the $(i+1)$-th diagonal. This box is the leftmost box of the bottom row of $\lambda(\sigma)$. Let π be the noncrossing matching whose corresponding Young diagram is obtained by adding a box in a new row in $\lambda(\sigma)$, i. e., $\sigma \nearrow_i \pi$. By definition holds $D_\pi = \mathbf{D}_i(D_\sigma)$.

(c) Let $i-1$ appear in σ. Then there is only one box on the $(i-1)$-th diagonal. This box is the rightmost box of the first row of $\lambda(\sigma)$. Let π be the noncrossing matching whose corresponding Young diagram is obtained by adding a box in the first row in $\lambda(\sigma)$, i. e., $\sigma \nearrow_i \pi$. By Lemma 3.3.9 $\mathbf{D}_i$ commutes with all $\mathbf{D}$-operators in D_π except D_{i-1}, therefore holds $D_\pi = \mathbf{D}_i(D_\sigma)$.

2. Let i appear in σ. We consider the lowest box in the i-th diagonal and call it X. Let σ' be the noncrossing matching of size n whose corresponding Young diagram $\lambda(\sigma')$ consists of all boxes above and to the left of the box X, denote by α_i with $1 \leq i \leq A$ the boxes to the right of X and in the row below but excluding the boxes in the $(i+1)$-th and $(i-1)$-th diagonal and by β_i with $1 \leq i \leq B$ the remaining boxes at the bottom. A schematic picture is given in Figure 3.11. Using the previous definitions we can write D_σ as

$$D_\sigma = \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \mathbf{D}_{i-1}^{b} \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l} \circ \mathbf{D}_{i+1}^{a} \circ \mathbf{D}_i \right) (D_{\sigma'}), \tag{3.14}$$

where a, b are 0 or 1.

a) If $a = b = 0$ Lemma 3.3.9 (1,2) implies

$$\begin{aligned}\mathbf{D}_i D_\sigma &= \mathbf{D}_i \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l} \circ \mathbf{D}_i \right) (D_{\sigma'}) \\ &= \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l} \circ \mathbf{D}_i^2 \right) (D_{\sigma'}) \\ &= \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l} \circ ((q+q^{-1})\mathbf{D}_i) \right) (D_{\sigma'}) = (q+q^{-1}) D_\sigma.\end{aligned}$$

b) For $a = b = 1$, the operator $\mathbf{D}_i$ commutes with all $\mathbf{D}_{\beta_l}$. As Figure 3.11 shows and by the assumptions on σ there exists a noncrossing matching π with $\sigma \nearrow_i \pi$. Hence one has

$$\mathbf{D}_i(D_\sigma) = \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \mathbf{D}_i \circ \mathbf{D}_{i-1} \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l} \circ \mathbf{D}_{i+1} \circ \mathbf{D}_i \right) (D_{\sigma'}) = D_\pi.$$

c) For $a = 1, b = 0$ we obtain by Lemma 3.3.9 (2,3)

$$\begin{aligned}\mathbf{D}_i(D_\sigma) &= \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l} \circ \mathbf{D}_i \circ \mathbf{D}_{i+1} \circ \mathbf{D}_i \right) (D_{\sigma'}) \\ &= \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l} \right) ((\mathbf{D}_{i+1} \circ \mathbf{D}_i \circ \mathbf{D}_{i+1} + \mathbf{D}_i - \mathbf{D}_{i+1})(D_{\sigma'}))\,.\end{aligned}$$

By the induction hypothesis $((\mathbf{D}_{i+1} \circ \mathbf{D}_i \circ \mathbf{D}_{i+1} + \mathbf{D}_i - \mathbf{D}_{i+1})(D_{\sigma'}))$ is a linear combination of D_τ's with $\tau \leq \hat{\sigma}$ where $\hat{\sigma}$ is σ' with a box added on the i-th and $(i+1)$-st diagonal. Using again the induction hypothesis for the D_τ's with $\tau \leq \hat{\sigma}$ proves the claim.

d) Let $a = 0, b = 1$ and let $\hat{\sigma}$ be the noncrossing matching whose Young diagram consists of $\lambda(\sigma')$ and the boxes labelled with α_i for $1 \leq i \leq A$. Lemma 3.3.9 (2,3) implies

$$\begin{aligned}\mathbf{D}_i(D_\sigma) &= \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \mathbf{D}_i \circ \mathbf{D}_{i-1} \circ \mathbf{D}_i \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l} \right) (D_{\sigma'}) \\ &= \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ (\mathbf{D}_{i-1} \circ \mathbf{D}_i \circ \mathbf{D}_{i-1} + \mathbf{D}_i - \mathbf{D}_{i-1}) \right) (D_{\hat{\sigma}}).\end{aligned}$$

We finish the proof by using the induction hypothesis analogously to the above case. $\square$

Let $\pi \in \mathrm{NC}_n$ be a noncrossing matching given by $\pi = \pi_1\pi_2$ where π_i is a noncrossing matching of size n_i for $i = 1, 2$. We want to generalise D_π and Theorem 3.3.10 in the sense that we can write $\Psi_\pi = \Psi_{\pi_1\pi_2}$ as a linear combination of D_{τ_1,τ_2}

with $\tau_i \leq \pi_i$ for $i = 1, 2$. This will not be possible for Ψ_π but for $\Psi_{\rho^{n_2}\pi}$. Let the Young diagram corresponding to π_2 be given as $\lambda(\pi_2) = (\lambda_1, \dots, \lambda_l)$. The wheel polynomial D_{π_1,π_2} is then defined by the following algorithm. First we write in every box of $\lambda(\pi_2)$ the number of the diagonal the box lies on. The wheel polynomial D_{π_1,π_2} is then constructed recursively by "reading" in the Young diagram $\lambda(\pi_2)$ the rows from top to bottom and in the rows all boxes from left to right and apply $\mathbf{D}_{\text{number in the box}-n}$ to the previous wheel polynomial, starting with $D_{(\pi_1)_{n_2}}$, which is defined in (3.13). Remember that we have extended the definition of $\mathbf{D}_k$ to all integers via $\mathbf{D}_k = \mathbf{D}_{k+2n}$. We can express D_{π_1,π_2} also by the following formula

$$D_{\pi_1,\pi_2} := \left(\prod_{i=1}^{l} \prod_{j=1}^{\lambda_{l+1-i}} \mathbf{D}_{(i-l)+(\lambda_{l+1-i}-j)} \right) (D_{(\pi_1)_{n_2}}),$$

where n_2 is the size of π_2. For π_2 as in Figure 3.10 we obtain

$$D_{\pi_1,\pi_2} = (\mathbf{D}_{-3} \circ \mathbf{D}_{-2} \circ \mathbf{D}_0 \circ \mathbf{D}_{-1} \circ \mathbf{D}_3 \circ \mathbf{D}_2 \circ \mathbf{D}_1 \circ \mathbf{D}_0)\,(D_{(\pi_1)_{n_2}}).$$

Theorem 3.3.12. *Let π_1, π_2 and $\pi = \pi_1\pi_2$ be noncrossing matchings of size n_1, n_2 and $n = n_1 + n_2$ respectively and let q be a sixth root of unity which is not necessarily primitive. The wheel polynomial*

$$\Psi_{\rho^{n_2}(\pi_1\pi_2)}(z_1, \dots, z_{2n}) = \Psi_{\pi_1\pi_2}(z_{2n+1-n_2}, \dots, z_{2n}, z_1, \dots, z_{2n-n_2})$$

can be expressed as a linear combination of D_{τ_1,τ_2}'s where $\tau_i \leq \pi_i$ and the coefficient of D_{π_1,π_2} is 1.

Proof. We calculate $\Psi_{\rho^{n_2}(\pi_1\pi_2)}$ in three steps:

1. $\Psi_{(\pi_1)_{n_2}}$ is by Theorem 3.3.10 a linear combination of $D_{(\tau_1)_{n_2}}$'s with $\tau_1 \leq \pi_1$ and the coefficient of $D_{(\pi_1)_{n_2}}$ is 1.

2. Theorem 3.3.6 implies

$$\Psi_{\pi_1()_{n_2}} = \Psi_{\rho^{-n_2}((\pi_1)_{n_2})} = \Psi_{(\pi_1)_{n_2}}(z_{n_2+1}, \dots, z_{2n}, z_1, \dots, z_{n_2}).$$

3. Use the recursion (3.10) of Theorem 3.3.6 to obtain $\Psi_{\pi_1\pi_2}$ starting from $\Psi_{\pi_1()_{n_2}}$. By Lemma 3.2.1 the τ appearing in the sum in (3.10) are of the form $\pi_1\tau_2$ with $\tau_2 \leq \pi_2$.

The algorithm for calculating $\Psi_{(\pi_2)_{n_1}}$ and the third step of calculating $\Psi_{\pi_1\pi_2}$ differ by the initial condition – in the first case $\Psi_{()_n}$, in the second $\Psi_{\pi_1()_{n_2}}$ – and each $\mathbf{D}_i$ of the first algorithm is replaced by $\mathbf{D}_{i+n_2}$. Hence we can use Theorem 3.3.10 to express $\Psi_{\pi_1\pi_2}$ as a linear combination of $\hat{D}_{\tau_2}$ with $\tau_2 \leq \pi_2$, where $\hat{D}_{(\tau_2)_{n_1}}$ is obtained by taking $D_{(\tau_2)_{n_1}}$, changing every $\mathbf{D}_i$ to a $\mathbf{D}_{i+n_2}$ and replacing $\Psi_{()_n}$ by $\Psi_{\pi_1()_{n_2}}$. Together with the first two parts of the algorithm this implies that $\Psi_{\rho^{n_2}(\pi_1\pi_2)}$ is a linear combination of D_{τ_1,τ_2}'s with $\tau_i \leq \pi_i$ and the coefficient of D_{π_1,π_2} is 1. $\square$

Remark 3.3.13. *Let $\Psi_{\pi_i} = \sum_{\tau_i \leq \pi_i} \alpha_{\tau_i} D_{\tau_i}$ for $i = 1, 2$. The above proof implies*

$$\Psi_{\rho^{n_2}(\pi_1\pi_2)} = \sum_{\tau_1 \leq \pi_1, \tau_2 \leq \pi_2} \alpha_{\tau_1} \alpha_{\tau_2} D_{\tau_1,\tau_2}.$$

Hence gaining knowledge about

$$A_{\pi_1\pi_2} = \Psi_{\pi_1\pi_2}|_{z_1=\ldots=z_{2n}=1,q^3=1} = \Psi_{\rho^{n_2}(\pi_1\pi_2)}|_{z_1=\ldots=z_{2n}=1,q^3=1}$$

could be achieved by understanding the coefficients α_{τ_i} and the behaviour of D_{τ_1,τ_2} for $\tau_i \leq \pi_i$. However this seems to be very difficult.

3.4 Fully packed loops with a set of nested arches

In order to prove Theorem 3.1.1 we will need to calculate D_{π_1,π_2} at $z_1 = \ldots = z_{2(n_1+n_2)} = 1$ for two noncrossing matchings π_1, π_2. The following notations will simplify this task. We define

$$f(i,j) := \frac{qz_i - q^{-1}z_j}{q - q^{-1}}, \quad g(i) := \frac{q - q^{-1}z_i}{q - q^{-1}}, \quad h(i) := \frac{qz_i - q^{-1}}{q - q^{-1}},$$

for $1 \leq i \neq j \leq 2n$. Using these notations we obtain

$$\Psi_{()_n} = \prod_{1 \leq i < j \leq n} f(i,j) f(n+i, n+j).$$

One verifies the following lemma by simple calculation.

Lemma 3.4.1. *For $1 \leq i, j, k \leq 2n$ and $i \neq j$ one has*

1. $$\mathbf{D}_k(f(i,j)) = \begin{cases} (q+q^{-1}) f(k,k+1) & (i,j) = (k,k+1), \\ -(q+q^{-1}) f(k,k+1) & (i,j) = (k+1,k), \\ q f(k,k+1) & i = k;\ j \neq k+1, \\ -q f(k,k+1) & i = k+1;\ j \neq k, \\ -q^{-1} f(k,k+1) & j = k;\ i \neq k+1, \\ q^{-1} f(k,k+1) & j = k+1;\ i \neq k, \\ 0 & \{i,j\} \cap \{k,k+1\} = \emptyset, \end{cases}$$

2. $$\mathbf{D}_k(g(i)) = \begin{cases} -q^{-1} f(k,k+1) & i = k, \\ q^{-1} f(k,k+1) & i = k+1, \\ 0 & \textit{otherwise}, \end{cases}$$

3. $$\mathbf{D}_k(h(i)) = \begin{cases} q f(k,k+1) & i = k, \\ -q f(k,k+1) & i = k+1, \\ 0 & \textit{otherwise}. \end{cases}$$

4. Let m be a positive integer, then the following holds

$$\mathbf{D}_k(f(i,j)^m) = \mathbf{D}_k(f(i,j)) \sum_{l=0}^{m-1} f(i,j)^l \mathbf{S}_k(f(i,j)^{m-1-l}),$$

$$\mathbf{D}_k(g(i)^m) = \mathbf{D}_k(g(i)) \sum_{l=0}^{m-1} g(i)^l \mathbf{S}_k(g(i)^{m-1-l}),$$

$$\mathbf{D}_k(h(i)^m) = \mathbf{D}_k(h(i)) \sum_{l=0}^{m-1} h(i)^l \mathbf{S}_k(h(i)^{m-1-l}).$$

We further introduce the abbreviation

$$P(\alpha_{i,j}|\beta_i|\gamma_i) := \prod_{1\leq i\neq j\leq 2n} f(i,j)^{\alpha_{i,j}} \prod_{i=1}^{2n} g(i)^{\beta_i} h(i)^{\gamma_i},$$

where $\alpha_{i,j}, \beta_i, \gamma_i$ are non-negative integers for $1 \leq i \neq j \leq 2n$. Our goal is to obtain a useful expression for $\mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_m}(P(\alpha_{i,j}|\beta_i|\gamma_i))|_{z_1=\ldots=z_{2n}=1}$ for special values of $\alpha_{i,j}, \beta_i$ and γ_i. By using the previous lemma it is very easy to see that $\mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_m}(P(\alpha_{i,j}|\beta_i|\gamma_i))$ is a sum of products of the form $P(\alpha'_{i,j}|\beta'_i|\gamma'_i)$. The explicit form of this sum is easy to understand when only one $\mathbf{D}$-operator is applied but gets very complicated for more. It turns out that $\mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_m}(P(\alpha_{i,j}|\beta_i|\gamma_i))|_{z_1=\ldots=z_{2n}=1}$ is a polynomial in $\alpha_{i,j}, \beta_i$ and γ_i, which is stated in Lemma 3.4.3. The next example hints at the basic idea behind this fact.

Example 3.4.2. *Let $P = P(\alpha_{i,j}|\beta_i|\gamma_i)$ and $n = 1$. We calculate $\mathbf{D}_1(P)_{z_1=z_2=1}$ explicitly. By using Lemma 3.3.5 and Lemma 3.4.1 we obtain for $\mathbf{D}_1(P)$ the expression.*

$$\begin{aligned}
\mathbf{D}_1(P) =& \mathbf{D}_1\left(f(1,2)^{\alpha_{1,2}} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1} g(2)^{\beta_2} h(1)^{\gamma_1} h(2)^{\gamma_2}\right) \\
=& (q+q^{-1}) \sum_{t=0}^{\alpha_{1,2}-1} f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t} f(2,1)^t g(1)^{\beta_2} g(2)^{\beta_1} h(1)^{\gamma_2} h(2)^{\gamma_1} + \\
& -(q+q^{-1}) \sum_{t=0}^{\alpha_{2,1}-1} f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t} f(2,1)^t g(1)^{\beta_2} g(2)^{\beta_1} h(1)^{\gamma_2} h(2)^{\gamma_1} + \\
& -q^{-1} \sum_{t=0}^{\beta_1-1} f(1,2)^{\alpha_{1,2}+1} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1+\beta_2-t-1} g(2)^t h(1)^{\gamma_2} h(2)^{\gamma_1} + \\
& +q^{-1} \sum_{t=0}^{\beta_2-1} f(1,2)^{\alpha_{1,2}+1} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1+\beta_2-t-1} g(2)^t h(1)^{\gamma_2} h(2)^{\gamma_1} + \\
& +q \sum_{t=0}^{\gamma_1-1} f(1,2)^{\alpha_{1,2}+1} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1} g(2)^{\beta_2} h(1)^{\gamma_1+\gamma_2-t-1} h(2)^t + \\
& -q \sum_{t=0}^{\gamma_2-1} f(1,2)^{\alpha_{1,2}+1} f(2,1)^{\alpha_{2,1}} g(1)^{\beta_1} g(2)^{\beta_2} h(1)^{\gamma_1+\gamma_2-t-1} h(2)^t.
\end{aligned}$$

By evaluating this at $z_1 = z_2 = 1$ *we obtain:*

$$\mathbf{D}_1(P)|_{z_1=z_2=1} = (q+q^{-1})(\alpha_{1,2}-\alpha_{2,1}) + q^{-1}(\beta_2-\beta_1) + q(\gamma_1-\gamma_2),$$

which is a polynomial in the $\alpha_{i,j}, \beta_i, \gamma_i$.

The proof of Theorem 3.1.1 is achieved by using two main ingredients. First Theorem 3.3.12 allows us to express the wheel polynomial $\Psi_{(\pi_1)_m\pi_2}$ in a suitable basis and second Lemma 3.4.3 tells us what we have to expect when evaluating the basis at $z_1 = \ldots = z_{2N} = 1$.

Proof of Theorem 3.1.1. In the following we show that the number $A_{(\pi_1)_m\pi_2}$ of FPLs with link pattern $(\pi_1)_m\pi_2$ is a polynomial in m. Together with [21, Theorem 6.7], which states that $A_{(\pi_1)_m\pi_2}$ is a polynomial in m with requested degree and leading coefficient for large values of m, this proves Theorem 3.1.1.

Set $N = m + n_1 + n_2$ and $q = e^{\frac{2\pi i}{3}}$. By Theorem 3.3.6 and Theorem 3.3.7 one has

$$A_{(\pi_1)_m\pi_2} = \Psi_{(\pi_1)_m\pi_2}|_{z_1=\ldots=z_{2N}=1} = \Psi_{\rho^{n_2}((\pi_1)_m\pi_2)}|_{z_1=\ldots=z_{2N}=1}.$$

Theorem 3.3.12 implies that $\Psi_{\rho^{n_2}((\pi_1)_m\pi_2)}$ is a linear combination of $D_{(\tau_1)_m,\tau_2}$ with $\tau_i \leq \pi_i$ for $i = 1, 2$. By definition $D_{(\tau_1)_m,\tau_2}$ is of the form $\prod_{j=1}^{k} \mathbf{D}_{i_j}(\Psi_{()_N})$ with $k \leq |\lambda(\pi_1)| + |\lambda(\pi_2)|$ and $i_j \in \{1, \ldots, n_2-2, N-n_1+2, \ldots, N+n_1-2, 2N-n_2+2, \ldots, 2N\}$ for $1 \leq j \leq k$. The operator $\mathbf{D}_{i_j}$ acts for $1 \leq j \leq k$ trivially on z_i with $i \in I := \{n_2+1, \ldots, N-n_1, N+n_1+1, \ldots, 2N-n_2\}$. Hence one has

$$\left.\left(\prod_{j=1}^{k} \mathbf{D}_{i_j}(\Psi_{()_N})\right)\right|_{z_1=\ldots=z_{2N}=1} = \left.\left(\prod_{j=1}^{k} \mathbf{D}_{i_j}(\Psi_{()_N|\forall i\in I: z_i=1})\right)\right|_{\forall i\in\{1,\ldots,2N\}\setminus I: z_i=1}.$$

The polynomial $\Psi_{()_N}\big|_{z_i=1\forall i\in I}$ is a polynomial in the $2(n_1+n_2)$ variables z_i, where i is an element of $\{1, \ldots, 2N\} \setminus I$. For simplicity we substitute these remaining variables with $z_1, \ldots, z_{2(n_1+n_2)}$ whereby we keep the same order on the indices. Hence $\Psi_{()_N}\big|_{z_i=1\forall i\in I}$ can be written in the form $P = P(\alpha_{i,j}|\beta_i|\gamma_i)$ with

$$\alpha_{i,j} = \begin{cases} 1 & i < j \text{ and } (j \leq n_1+n_2 \text{ or } i > n_1+n_2), \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_i = \begin{cases} m & i \in \{n_2+1, \ldots, n_1+n_2, 2n_1+n_2+1, \ldots, 2(n_1+n_2)\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma_i = \begin{cases} m & i \in \{1, \ldots, n_2, n_1+n_2+1, \ldots, 2n_1+n_2\}, \\ 0 & \text{otherwise,} \end{cases}$$

Lemma 3.4.3 implies that $\prod_{j=1}^{k} \mathbf{D}_{i_j}(P)$ is a polynomial in m of degree at most $k \leq |\lambda(\pi_1)| + |\lambda(\pi_2)|$ which proves the statement. □

We conclude the proof of Theorem 3.1.1 by the following Lemma.

Lemma 3.4.3. *Let $P = P(\alpha_{i,j}|\beta_i|\gamma_i)$, m an integer and $i_1, \ldots, i_m \in \{1, \ldots, 2n\}$. There exists a polynomial $Q \in \mathbb{Q}(q)[y_1, \ldots, y_{2n(2n+1)}]$ with total degree at most m such that*

$$\mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_m}(P)|_{z_1=\ldots=z_{2n}=1} = Q((\alpha_{i,j}), (\beta_i), (\gamma_i)).$$

Proof. We prove the theorem by induction on m. The statement is trivial for $m = 0$, hence let $m > 0$ and set $k = i_m$. We can express $\mathbf{D}_k(P)$ as

$$\mathbf{D}_k P = \sum_{s \in S} a_s P_s, \tag{3.15}$$

for a finite set S of indices, $a_s \in \{\pm q, \pm q^{-1}, \pm(q + q^{-1})\}$ and $P_s = P(\alpha_{s;i,j}|\beta_{s;i}|\gamma_{s;i})$ for all $s \in S$. Indeed we can use iteratively the product rule for the operator $\mathbf{D}_k$, stated in Lemma 3.3.5, to split $\mathbf{D}_k(P)$ into a sum. Since this splitting depends on the order of the factors, we fix it to be

$$P = \prod_{i=1}^{2n} \prod_{\substack{j=1,\\ j \neq i}}^{2n} f(i,j)^{\alpha_{i,j}} \prod_{i=1}^{2n} g(i)^{\beta_i} \prod_{i=1}^{2n} h(i)^{\gamma_i}.$$

Lemma 3.4.1 implies that every summand is of the form $P_s = P(\alpha_{s;i,j}|\beta_{s;i}|\gamma_{s;i})$ and the coefficients a_s are as stated above, which verifies (3.15). In particular we can write $\mathbf{D}_k(P)$ more explicitly by using the above defined ordering of the factors and Lemma 3.3.5

$$\begin{aligned}
\mathbf{D}_k(P) = \mathbf{D}_k &\left(\prod_{1 \leq i \neq j \leq 2n} f(i,j)^{\alpha_{i,j}} \prod_{i=1}^{2n} g(i)^{\beta_i} h(i)^{\gamma_i} \right) \\
&= \sum_{1 \leq i \neq j \leq 2n} \prod_{\substack{1 \leq i' \neq j' \leq 2n \\ (i'<i) \vee (i'=i, j'<j)}} f(i',j')^{\alpha_{i',j'}} \times \mathbf{D}_k(f(i,j)^{\alpha_{i,j}}) \\
&\quad \times \mathbf{S}_k \left(\prod_{\substack{1 \leq i' \neq j' \leq 2n \\ (i'>i) \vee (i'=i, j'>j)}} f(i',j')^{\alpha_{i',j'}} \prod_{i'=1}^{2n} g(i')^{\beta_{i'}} h(i')^{\gamma_{i'}} \right) &\text{(3.16a)} \\
&\quad + \sum_{i=1}^{2n} \prod_{1 \leq i' \neq j' \leq 2n} f(i',j')^{\alpha_{i',j'}} \prod_{i'=1}^{i-1} g(i')^{\beta_{i'}} \times \mathbf{D}_k(g(i)^{\beta_i}) \\
&\quad \times \mathbf{S}_k \left(\prod_{i'=i+1}^{2n} g(i')^{\beta_{i'}} \prod_{i'=1}^{2n} h(i')^{\gamma_{i'}} \right) &\text{(3.16b)} \\
&\quad + \sum_{i=1}^{2n} \prod_{1 \leq i' \neq j' \leq 2n} f(i',j')^{\alpha_{i',j'}} \prod_{i'=1}^{2n} g(i')^{\beta_{i'}} \prod_{i'=1}^{i-1} h(i')^{\gamma_{i'}} \times \mathbf{D}_k(h(i)^{\gamma_i}) \\
&\quad \times \mathbf{S}_k \left(\prod_{i'=i+1}^{2n} h(i')^{\gamma_{i'}} \right). &\text{(3.16c)}
\end{aligned}$$

Using Lemma 3.4.1 we split up every summand in (3.16a) up into a sum of P_s's with $s \in S$ and say that these P_s originate from this very summand. We define $A_{i,j}$ for $1 \leq i \neq j \leq 2n$ to be the set consisting of all $s \in S$ such that P_s originates from the summand in (3.16a) with control variables i, j. Analogously, we define for $1 \leq i \leq 2n$, the sets B_i and C_i to consist of all $s \in S$ such that P_s originates from the summand with control variable i in (3.16b) or (3.16c) respectively. Hence we can write the set S as the disjoint union

$$S = \left(\bigcup_{1 \leq i \neq j \leq 2n} A_{i,j} \right) \cup \left(\bigcup_{1 \leq i \leq 2n} B_i \right) \cup \left(\bigcup_{1 \leq i \leq 2n} C_i \right).$$

Lemma 3.4.1 implies $\mathbf{D}_k(f(i,j)) = 0$ for $\{i,j\} \cap \{k, k+1\} = \emptyset$ and $\mathbf{D}_k(g(i)) = \mathbf{D}_k(h(i)) = 0$ for $i \notin \{k, k+1\}$. Therefore the sets $A_{i,j}, B_i, C_i$ are empty in these cases.

Let $1 \leq i \neq j \leq 2n$ be fixed with $\{i,j\} \cap \{k,k+1\} \neq \emptyset$ and let $\sigma \in \mathfrak{S}_{2n}$ be the permutation $\sigma = (k, k+1)$. Set $\Lambda_{i,j} = \{(i',j') : 1 \leq i' \neq j' \leq 2n, (i' < i) \vee (i' = i, j' < j)\}$. The definition of $A_{i,j}$ and Lemma 3.4.1 imply for all $(i',j') \notin \{(i,j), (\sigma(i), \sigma(j)), (k,k+1)\}$ and all $s \in A_{i,j}$:

$$\alpha_{s;i',j'} = \begin{cases} \alpha_{i',j'} & \text{for } \{i',j'\} \cap \{k,k+1\} = \emptyset, \\ & \quad \text{or } (i',j'), (\sigma(i'), \sigma(j')) \in \Lambda_{i,j}, \\ \alpha_{i',j'} + \alpha_{\sigma(i'),\sigma(j')} & \text{for } \{i',j'\} \cap \{k,k+1\} \neq \emptyset, \\ & \quad \text{and } (i',j') \in \Lambda_{i,j}, (\sigma(i'), \sigma(j')) \notin \Lambda_{i,j}, \\ 0 & \text{for } \{i',j'\} \cap \{k,k+1\} \neq \emptyset, \\ & \quad \text{and } (i',j') \notin \Lambda_{i,j}, (\sigma(i'), \sigma(j')) \in \Lambda_{i,j}, \\ \alpha_{\sigma(i'),\sigma(j')} & \text{for } \{i',j'\} \cap \{k,k+1\} \neq \emptyset, \\ & \quad \text{and } (i',j'), (\sigma(i'), \sigma(j')) \notin \Lambda_{i,j}. \end{cases}$$

If $(k,k+1) \notin \{(i,j), (\sigma(i), \sigma(j))\}$, the parameter $\alpha_{s;k,k+1}$ is given as the adequate value of the above case analysis added by 1. Further we obtain $\beta_{s;i'} = \beta_{\sigma(i')}$ and $\gamma_{s;i'} = \gamma_{\sigma(i')}$ for all $1 \leq i' \leq 2n$ and $s \in A_{i,j}$. By Lemma 3.4.1 the constant a_s is for all $s \in A_{i,j}$ determined by the corresponding constant of $\mathbf{D}_k(f(i,j))$ and hence not depending on s. The last statement of Lemma 3.4.1 implies that we can list the elements of $A_{i,j} = \{s_1, \ldots, s_{\alpha_{i,j}}\}$ such that we have the following description for the remaining parameters $\alpha_{s;i,j}$ and $\alpha_{s;\sigma(i),\sigma(j)}$:

$$\alpha_{s_t;i,j} = \begin{cases} \alpha_{i,j} + \alpha_{j,i} + 1 - t & i = k, j = k+1, \\ \alpha_{i,j} - t & i = k+1, j = k, \\ \alpha_{i,j} + \alpha_{\sigma(i),\sigma(j)} - t & \{i,j\} \cap \{k,k+1\} = \{k\}, \\ \alpha_{i,j} - t & \{i,j\} \cap \{k,k+1\} = \{k+1\}, \end{cases}$$

$$\alpha_{s_t;\sigma(i),\sigma(j)} = \begin{cases} \alpha_{i,j} + \alpha_{j,i} - \alpha_{s_t;i,j} & \{i,j\} = \{k,k+1\}, \\ \alpha_{i,j} + \alpha_{\sigma(i),\sigma(j)} - \alpha_{s_t;i,j} - 1 & \text{otherwise,} \end{cases}$$

with $1 \leq t \leq \alpha_{i,j}$. If $k = 2n$ the first two and last two cases in the description of $\alpha_{s_t;i,j}$ switch places, which is due to the fact that we identify $k+1$ with 1 for $k = 2n$.

There exists an analogue description for the sets B_i, C_i and $i \in \{k, k+1\}$ as above, whereas the only parameters that are non-constant on B_i or C_i respectively are given in the case of B_i by

$$\beta_{s_t;k} = \beta_k + \beta_{k+1} - t, \qquad \beta_{s_t;k+1} = t - 1,$$

with $1 \leq t \leq \beta_i$ and in the case of C_i by

$$\gamma_{s_t;k} = \gamma_k + \gamma_{k+1} - t, \qquad \gamma_{s_t;k+1} = t - 1,$$

with $1 \leq t \leq \gamma_i$. For $k = 2n$ the description of $\beta_{s_t;k}, \beta_{s_t;k+1}$ and $\gamma_{s_t;k}, \gamma_{s_t;k}$ are interchanged.

We know by induction that $\mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_{m-1}} \left(P(a_{i,j}|b_i|c_i)\right)|_{z_1=\ldots=z_{2n}=1}$ is a polynomial Q' of degree at most $m-1$ in $(a_{i,j}), (b_i)$ and (c_i). Since the operators $\mathbf{D}_i$ are linear we can write

$$\begin{aligned}
&\mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_m}(P)|_{z_1=\ldots=z_{2n}=1} \\
&\qquad = \mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_{m-1}} \left(\sum_{s \in S} a_s P(\alpha_{s;i,j}|\beta_{s;i}|\gamma_{s;i}) \right)\Bigg|_{z_1=\ldots=z_{2n}=1} \\
&\qquad = \sum_{s \in S} a_s \mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_{m-1}} \left(P(\alpha_{s;i,j}|\beta_{s;i}|\gamma_{s;i})\right)|_{z_1=\ldots=z_{2n}=1} \\
&\qquad = \sum_{s \in S} a_s Q((\alpha_{s;i,j}), (\beta_{s;i}), (\gamma_{s;i})). \qquad (3.17)
\end{aligned}$$

The description above implies that if we restrict ourselves to $s \in A_{i,j}$, $s \in B_i$ or $s \in C_i$ respectively, a_s is independent of s, the parameters $\alpha_{s;i',j'}, \beta_{s;i'}, \gamma_{s;i'}$ are constant for $(i', j') \neq (i, j), (\sigma(i), \sigma(j))$ or $i' \neq k, k+1$ respectively and otherwise depending linearly on a parameter t which runs from 1 up to the cardinality of the set $A_{i,j}$, B_i or C_i respectively. The fact, that for a polynomial $p(t)$ of degree d the sum $\sum_{x \leq t \leq y} p(t)$ is a polynomial in x and y of degree at most $d+1$, together with the previous statement imply that the sum

$$\sum_{s \in A_{i,j}} a_s Q((\alpha_{s;i,j}), (\beta_{s;i}), (\gamma_{s;i})),$$

and the analogous sums for $s \in B_i$ or $s \in C_i$ respectively are polynomials in $(\alpha_{i,j}), (\beta_i), (\gamma_i)$ of degree at most m for all $1 \leq i \neq j \leq 2n$. Therefore

$$\mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_m}(P)|_{z_1=\ldots=z_{2n}=1} = \sum_{s \in S} a_s Q((\alpha_{s;i,j}), (\beta_{s;i}), (\gamma_{s;i})),$$

is a polynomial in $(\alpha_{i,j}), (\beta_i), (\gamma_i)$ of degree at most m. $\square$

Chapter 4

Refined enumeration of alternating sign trapezoids

The content of the following chapter has been published in [5] and appeared in the proceedings of FPSAC 2017 [2].

4.1 Introduction

In this chapter we introduce and investigate a refined enumeration of ASTs using *centred Catalan sets*, objects enumerated by the Catalan numbers and Motzkin paths (Ayyer [8]). As we will see in Section 4.6, it makes sense to look at both refinements, even though the Motzkin path refinement is coarser, since we may deduce statements and make conjectures for this refinement that are not true for the centred Catalan set refinement.

The structure of ASTs and their associated centred Catalan set lead naturally to the introduction of *AS-trapezoids*. AS-trapezoids arise by generalising ASTs from a triangular to a trapezoidal shape, and have by definition the following property: by taking an AST of order n and putting an $(m, 2n)$-AS-trapezoid on top of it, one obtains an AST of order $m+n$. In addition, the associated centred Catalan set (resp. Motzkin path) of the AST of order $m+n$ is the concatenation of the centred Catalan set (resp. Motzkin path) associated with the AST of order n and the $(m, 2n)$-AS-trapezoid. In Lemma 4.3.2 we show that the converse is also true. In particular, every AST whose associated centred Catalan set (resp. Motzkin path) can be split into two parts can also be split into two parts: a smaller AST corresponding to the first centred Catalan set (resp. Motzkin path) and an AS-trapezoid corresponding to the second. Following [8], we define the weight function $w_l(S)$ (resp. $w_l(M)$) as the number of (n, l)-AS-trapezoids with associated centred Catalan set S of size $n-1$ (resp. Motzkin path M of length n). Exploiting the splitting property of ASTs into an AST and an AS-trapezoid leads to our first main result.

Proposition 4.1.1. *Let S_1, S_2 be centred Catalan sets and M_1, M_2 be Motzkin paths.*

We have

$$w_l(S_1 \circ S_2) = w_l(S_1) w_{l+2|S_1|-2}(S_2), \tag{4.1}$$

$$w_l(M_1 \circ M_2) = w_l(M_1) w_{l+2|M_1|}(M_2), \tag{4.2}$$

i.e., the weight functions are multiplicative.

We establish a bijection between the AS-trapezoids corresponding to a given centred Catalan set S to $(\mathbf{s}, \mathbf{t})$-trees, which were defined in [38], but had already appeared as *partial monotone triangles* in [33], and are a generalisation of monotone triangles. By analysing the structure of the $(\mathbf{s}, \mathbf{t})$-trees, we can deduce our second main result.

Theorem 4.1.2. *Let $S = S_1 \circ \ldots \circ S_k$ be a centred Catalan set and $S_1, \ldots, S_k$ its irreducible components. The weight $w_l(S)$ is a polynomial in l of degree $|\lambda(S)/\mu(S)| = \mathrm{area}(\mathbf{M}(S))$ with leading coefficient $\prod_{i=1}^{k} \frac{f^{\lambda(S_i)/\mu(S_i)}}{|\lambda(S_i)/\mu(S_i)|!}$, where $f^{\lambda/\mu}$ denotes the number of standard Young tableaux of skew shape λ/μ.*

Remarkably, there exists an analogous theorem for the refined enumeration of FPLs for a specific class of noncrossing matchings. This theorem was conjectured in [81] and proved in [4, 21]. A detailed explanation is provided in Remark 4.4.14.

Using the enumeration formula for $(\mathbf{s}, \mathbf{t})$-trees from [38], we can deduce a constant term formula for the refined enumeration of AS-trapezoids. Essentially, this formula generalises a constant term formula for refined ASTs [38, Theorem 2].

Theorem 4.1.3. *Let $S = \{s_1, \ldots, s_u, 0, s_{u+1}, \ldots, s_n\}$ be a centred Catalan set with $s_1 < \ldots < s_u \leq -1, 1 \leq s_{u+1} < \ldots < s_n$. The number $w_l(S)$ of (n, l)-AS-trapezoids with centred Catalan set S is equal to the constant term of*

$$\prod_{i=1}^{n} x_i^{-n-s_i} \prod_{i=u+1}^{n} x_i^{2-l}(1+x_i)^l \prod_{1 \leq i < j \leq n} (x_j - x_i)(1 + x_i + x_i x_j),$$

in $x_1, \ldots, x_n$.

The final section of this chapter describes the rational roots of the weight functions $w_l(S)$ and $w_l(M)$. Both seem to have a rich structure of rational roots that is currently not completely understood. We present our first analysis of their structure in the form of conjectures and partial results. The goal is to find a description of all rational roots using combinatorial methods, similar to [41] for the number of FPLs associated with certain families of link patterns.

The chapter is structured as follows. In Section 4.2 we complement the definitions of Chapter 2. In Section 4.3, we present a relation between the inner structure of an AS-trapezoid and the centred Catalan path it corresponds to. This allows us to deduce the splitting lemma (Lemma 4.3.2), which leads directly to our first main result. In Section 4.4, we relate AS-trapezoids to $(\mathbf{s}, \mathbf{t})$-trees and derive our second main result. Section 4.5 contains the derivation of the constant term identity for the refined enumeration of AS-trapezoids. Finally, in Section 4.6 we present conjectures and partial results concerning the rational roots of $w_l(S)$ and $w_l(M)$.

4.2 Preliminaries

4.2.1 Centred Catalan sets, Dyck and Motzkin paths

The definitions of centred Catalan sets, Dyck and Motzkin paths and their relations can be found in Section 2.1. Let l be an integer, we define the dilation operator $\mathfrak{s}_l : \mathbb{Z} \to \mathbb{Z}$ as

$$\mathfrak{s}_l(x) = \begin{cases} x+l & x>0, \\ 0 & x=0, \\ x-l & x<0. \end{cases}$$

By abuse of notation we write $\mathfrak{s}_l : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$, $\mathfrak{s}_l(A) = \{\mathfrak{s}_l(x) | x \in A\}$. The *concatenation* $S_1 \circ S_2$ of two centred Catalan sets S_1, S_2 is defined as

$$S_1 \circ S_2 := S_1 \cup \mathfrak{s}_{|S_1|-1}(S_2).$$

Example 4.2.1. *The concatenation of the centred Catalan sets* $S_1 = \{-1,0,1\}$ *and* $S_2 = \{-1,0,1,2\}$ *is*

$$S_1 \circ S_2 = \{-1,0,1\} \cup \mathfrak{s}_2(\{-1,0,1,2\}) = \{-3,-1,0,1,3,4\}.$$

We call a centred Catalan set S *irreducible* if there exist no centred Catalan sets S_1, S_2 of size at least 2 such that $S = S_1 \circ S_2$. It is not hard to convince oneself that every centred Catalan set can be written uniquely as the concatenation of irreducible centred Catalan sets. For two centred Catalan sets S_1, S_2, the Dyck path $\mathbf{D}(S_1 \circ S_2)$ is obtained by deleting the last step of $\mathbf{D}(S_1)$ and the first step of $\mathbf{D}(S_2)$, and concatenating both paths, note that one does not simply add the paths $\mathbf{D}(S_1)$ and $\mathbf{D}(S_2)$, see Figure 4.1.

Let M be a Motzkin path. We define area(M) as the area enclosed between the x-axis and the Motzkin path M. The concatenation of two Motzkin paths is given by attaching the second path at the end of the first one. A Motzkin path is called *irreducible* iff it cannot be written as a concatenation of non-empty Motzkin paths. It is easy to see that concatenating centred Catalan sets (resp. Motzkin paths) commutes with the map from centred Catalan sets to Motzkin paths, i.e., $\mathbf{M}(S_1 \circ S_2) = \mathbf{M}(S_1) \circ \mathbf{M}(S_2)$. Further every irreducible component of $\mathbf{M}(S)$ corresponds to an irreducible component of S and vice-versa. For an example see Figure 4.1.

4.2.2 Alternating sign triangles and trapezoids

The definitions of alternating sign triangles and trapezoids and their relation to centred Catalan sets and Motzkin paths are defined in Subsection 2.2.2. Let A be an AS-trapezoid, in the following proposition we prove that the set $\mathbf{S}(A)$ is a centred Catalan set.

Proposition 4.2.2. *Let* A *be an* (n,l)*-AS-trapezoid. The set* $\mathbf{S}(A)$ *is a centred Catalan set of size* $n+1$*. Conversely, for all centred Catalan sets* S *of size* $n+1$ *and positive integers* l *there exists an* (n,l)*-AS-trapezoid* A *with* $\mathbf{S}(A) = S$*.*

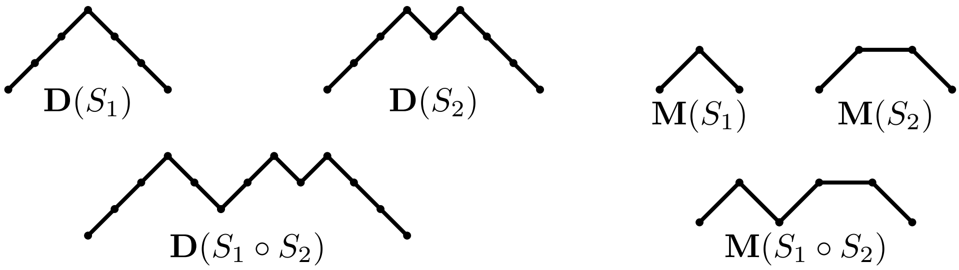


Figure 4.1: The Dyck and Motzkin paths of the centred Catalan set $S = \{-3,-1,0,1,3,4\}$ and its irreducible components $S_1 = \{-1,0,1\}$ and $S_2 = \{-1,0,1,2\}$.

Proof. The definition of an (n,l)-AS-trapezoid implies that the column-sums can be either 0 or 1. Since all row-sums are equal to 1, the sum over all entries in an (n,l)-AS-trapezoid is equal to n. Therefore, exactly n columns have column-sum equal to 1. The labels of the columns with column-sum equal to 1 lie in the set $\{-n,\dots,0\} \cup \{l,\dots,l+n\}$. Hence by the definition of the map $\mathbf{S}$, the set $\mathbf{S}(A)$ is an $(n+1)$-subset of $\{-n,\dots,n\}$. For all positive integers i with $i \leq n$ define $S_i(A)$ as the set of integers with $|j| < i$ such that the following three conditions hold.

- Zero is an element of $S_i(A)$ for all $i \leq n$.
- A negative integer j is an element of S_i iff the partial column-sum of elements below the $(i+1)$-st row in the $(j-1)$-st column is equal to 1.
- A positive integer j is an element of S_i iff the partial column-sum of elements below the $(i+1)$-st row in the $(j+l-1)$-st column is equal to 1.

If a column has a positive partial column-sum of elements below a certain row, it follows that it has a positive column-sum. Hence, we have the following relation between $\mathbf{S}_i(A)$ and $\mathbf{S}(A)$

$$\mathbf{S}_i(A) \subseteq \mathbf{S}(A) \cap \{-i,-i+1,\dots,i\},$$

for all $1 \leq i \leq n$. Since the partial column-sums can only have a value of 1, 0, or -1 and the sum of the partial column-sums of the elements below the $(i+1)$-st row is i and $0 \in S_i$, we obtain $|\mathbf{S}_i(A)| \geq i+1$. Hence we have

$$1+i \leq |\mathbf{S}_i(A)| \leq |\mathbf{S}(A) \cap \{-i,\dots,i\}|,$$

which implies the first claim.

Now let S be given. By definition we can choose a sequence $(s_i)_{1\leq i\leq n}$ such that $S = \{0\} \cup \{s_i : 1 \leq i \leq n\}$ and $0 < |s_i| \leq i$. We construct an (n,l)-AS-trapezoid A by setting all entries equal to 0 except in the i-th row the entry in column $s_i - 1$ if $s_i < 0$ or in in column $s_i + l - 1$ if $s_i > 0$ respectively, which we set to 1. Then A is an (n,l)-AS-trapezoid and satisfies $\mathbf{S}(A) = S$. $\square$

The following refinement of AS-trapezoids by Motzkin paths is a generalisation of the one for ASTs by Ayyer [8].

Corollary 4.2.3. *The map* $\mathbf{M}(A) := \mathbf{M}(\mathbf{S}(A))$ *is a surjection from* (n,l)*-AS-trapezoids to Motzkin paths of length* n*.*

4.3 The structure of AS-trapezoids

In the following we prove two technical lemmas. Together they immediately imply Proposition 4.1.1.

Lemma 4.3.1. *Let* S *be a centred Catalan set of size* $n+1$*,* A *an* (n,l)*-AS-trapezoid with* $\mathbf{S}(A) = S$ *and* $\mathbf{M}(S) = (m_1, \ldots, m_n)$*. Then the number of* 1 *entries of* A *in the* i*-th row is at most* $1+\sum_{j=1}^{i} m_j$*. Further, for* $l \geq 2$ *there exists an* (n,l)*-AS-trapezoid such that these bounds are sharp.*

Proof. We define an *allowed position* for a 1 in the i-th row of A as a position such that the next non-zero entry below is negative or all entries below are 0 and the label of the column corresponds to an element in S. Denote by a_i the number of allowed positions for a 1 in the i-th row. By definition of $\mathbf{M}(S)$ we have $m_1 = |\{-1,1\} \cap S| - 1$. This implies that $m_1 + 1$ entries of the first row are in a column corresponding to an element in S, i.e., we obtain $a_1 = m_1 + 1$. Since every 1 (resp. -1) in the i-th row cancels out (resp. adds) an allowed position for a 1 in the $(i+1)$-st row and there is one more 1 than -1 in every row, the number of allowed positions in the central $l + 2i - 1$ columns of the $(i+1)$-st row is $a_i - 1$. There are two new columns in row $i+1$ of which $|S \cap \{-i-1, i+1\}| = m_{i+1} + 1$ have a label corresponding to an element in S. Hence we obtain $a_{i+1} = a_i + m_{i+1}$ and therefore

$$a_i = a_1 + \sum_{j=2}^{i} m_i = 1 + \sum_{j=1}^{i} m_i.$$

We construct an (n,l)-AS-trapezoid A' with $\mathbf{S}(A') = S$ and a maximal number of 1 entries in a recursive manner. Place a 1 in all allowed positions in the i-th row and put a -1 between the 1 entries. This implies that in all columns the non-zero entries alternate. Since the allowed positions are either the leftmost or rightmost positions of a row or above a -1 from the row before, two allowed positions are by induction not direct neighbours. Therefore, the non-zero entries in each row are alternating. By the above formula, there is only one allowed position in the top row. If there exists a column in A' with a -1 as first non-zero entry from the top there would be a second allowed position in the top row, hence this cannot happen. An allowed position can appear in the central $l-1$ columns only above a -1. Hence the column-sum is zero for the central $l-1$ columns and the resulting array is an (n,l)-AS-trapezoid. $\square$

Lemma 4.3.2. *Let* S_1, S_2 *be centred Catalan sets,* A_1 *an* $(|S_1|-1,l)$*-AS-trapezoid and* A_2 *an* $(|S_2|-1, l+2|S_1|-2)$*-AS-trapezoid with* $\mathbf{S}(A_i) = S_i$ *for* $i = 1,2$*. By placing* A_2 *centred above* A_1*, we obtain an* $(|S_1 \circ S_2|-1,l)$*-AS-trapezoid with centred Catalan set* $S_1 \circ S_2$*. Moreover, every* $(|S_1 \circ S_2|-1,l)$*-AS-trapezoid* A *with* $\mathbf{S}(A) = S_1 \circ S_2$ *is of the above form.*

Proof. It follows immediately from the definitions of AS-trapezoids that this construction yields an $(|S_1|+|S_2|-2,l)$-AS-trapezoid with centred Catalan set $S_1 \circ S_2$. On the other hand, let A be an $(|S_1 \circ S_2|-1,l)$-AS-trapezoid with $\mathbf{S}(A)=S_1 \circ S_2$. We split A into a bottom part A_1 consisting of the first $|S_1|-1$ rows from the bottom and a top part A_2 consisting of the remaining rows. By Lemma 4.3.1, there is only one allowed position for a 1 in the top row of A_1. If A_1 has a column whose first non-zero entry from the top is negative, there would be a second allowed position for a 1 in the top row of A_1, hence this can not happen. The central $l-1$ columns of A have column-sum zero and the first non-zero entry from top is positive, hence the partial column-sum of the top $|S_2|-1$ rows is either 0 or 1. Since the first non-zero entry of every column of A_1 is also positive, the column-sums of the central $l-1$ columns of A_1 also have column-sum 0 or 1. Together with the above, this implies that the central $l-1$ columns of A_1 have column-sum 0 and therefore A_1 is an $(|S_1|-1,l)$-AS-trapezoid with $\mathbf{S}(A_1)=S_1$. One of the central $l+2|S_1|-3$ columns of A has a positive column-sum if its column label corresponds to an element in S_1, which implies that this column of A_1 has a positive column-sum. Therefore, the column-sums of the central $l+2|S_1|-3$ columns of A_2 are zero, which implies that A_2 is an $(|S_2|-1,l+2|S_1|-2)$-AS-trapezoid. □

Proof of Proposition 4.1.1. The theorem is a direct consequence of the above lemma. □

4.4 A refined enumeration of AS-trapezoids

4.4.1 A different perspective on AS-trapezoids

The aim of this section is to prove that the weight function $w_l(S)$ is a polynomial in l. We will need the following definition due to Fischer [33, 38].

Definition 4.4.1. *Let $1 \le u < v \le n$, $\mathbf{s}=(s_1,\ldots,s_u)$ be a weakly decreasing sequence of non-negative integers, $\mathbf{t}=(t_v,\ldots,t_n)$ a weakly increasing sequence of non-negative integers and $\mathbf{k}=(k_1,\ldots,k_n)$ an increasing sequence of integers. An $(\mathbf{s},\mathbf{t})$-tree is an array of integers of the following shape satisfying the properties listed below. We obtain the shape of an $(\mathbf{s},\mathbf{t})$-tree by starting with a triangular array of n centred rows where the i-th row from top has i entries. Then we delete the bottom s_i elements in the i-th north-east diagonal for $1 \le i \le u$ and the bottom t_j elements in the j-th south-east diagonal for $v \le j \le n$, see Figure 4.2.*

- *The entries are weakly increasing in the north-east and south-east directions, and strictly increasing in the east direction.*
- *For $1 \le i \le u$, the bottom entry of the i-th north-east diagonal is k_i.*
- *For $v \le i \le n$ the bottom entry of the i-th south-east diagonal is k_i.*
- *The entries in the bottom row are $k_{u+1},\ldots,k_{v-1}$.*

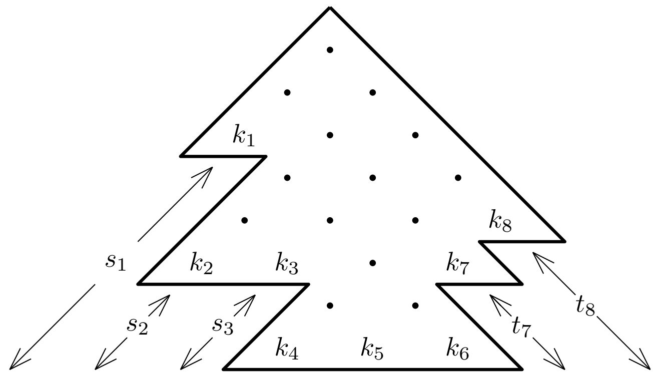


Figure 4.2: Schematic diagram of an $(\mathbf{s},\mathbf{t})$-tree.

Let f be a function in x. We define the following:

$$\begin{aligned} E_x(f)(x) :=& f(x+1) && \text{shift operator,} \\ \overline{\Delta}_x :=& E_x - \mathrm{Id} && \text{forward difference,} \\ \underline{\Delta}_x :=& \mathrm{Id} - E_x^{-1} && \text{backward difference.} \end{aligned}$$

Theorem 4.4.2 ([33, Section 4])**.** *Set*

$$M_n(x_1,\dots,x_n) := \prod_{1\le p<q\le n} (\mathrm{Id} + \overline{\Delta}_{x_p} \overline{\Delta}_{x_q} + \overline{\Delta}_{x_q}) \prod_{1\le p<q\le n} \frac{x_q - x_p}{q-p}. \tag{4.3}$$

The number of $(\mathbf{s},\mathbf{t})$*-trees with bottom entries* $\mathbf{k}=(k_1,\dots,k_n)$ *is given by*

$$\left(\prod_{i=1}^{u} (-\overline{\Delta}_{x_i})^{s_i} \prod_{i=v}^{n} \underline{\Delta}_{x_i}^{t_i} \right) M_n(x_1,\dots,x_n) \Bigg|_{\mathbf{x}=\mathbf{k}}. \tag{4.4}$$

Remark 4.4.3. *The above theorem is stated in [33] in the more general setting where* $\mathbf{k}$ *is a weakly increasing sequence. The notion of* $(\mathbf{s},\mathbf{t})$*-trees can be generalised to this setting using the notion of regular entries: An entry of an* $(\mathbf{s},\mathbf{t})$*-tree is called* regular *iff it has a south-east and a south-west neighbour. Then, the condition that the rows of an* $(\mathbf{s},\mathbf{t})$*-tree are strictly increasing is replaced by the condition that two adjacent regular entries in the same row must be different.*

Let $S=\{s_1,\dots,s_u,0,s_{u+1},\dots,s_n\}$ be a centred Catalan set of size $n+1$, $s_1 < \dots < s_u \le -1$ and $1 \le s_{u+1} < \dots < s_n$. Set $\mathbf{s}=(-s_1-1,\dots,-s_u-1), \mathbf{t}=(s_{u+1}-1,\dots,s_n-1)$ and $\mathbf{k}=(s_1+1,\dots,s_u+1,l+s_{u+1}-1,\dots,l+s_n-1)$. The following algorithm is a bijection between (n,l)-AS-trapezoids A with $\mathbf{S}(A)=S$ and $(\mathbf{s},\mathbf{t})$-trees with bottom entries $\mathbf{k}$, for an example see Figure 4.3. First we construct a triangular array T_A. We fill the i-th row from the bottom of T_A with the column labels of A, for which the first non-zero entry above the $(i-1)$-st row is positive. In doing so, we write the numbers in an increasing order from left to right. The bottom row of T_A is $\mathbf{k}$. Since there is one more 1 than -1 in every row of the trapezoid, the number of entries in a row of T_A is one less than the number of entries in the row

$$
\begin{array}{ccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & \\
 & & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & & \\
 & & & 1 & 0 & 0 & 0 & -1 & 0 & 1 & & &
\end{array}
\quad \leftrightarrow \quad
\begin{array}{ccccccc}
 & & & 4 & & & \\
 & & 2 & & 8 & & \\
 & -1 & & 4 & & \color{red}{8} & \\
\color{red}{-1} & & 0 & & 6 & & \color{red}{8}
\end{array}
$$

Figure 4.3: On the left is a $(4,6)$-AS-trapezoid and on the right the corresponding $(\mathbf{s},\mathbf{t})$-tree. The red entries are not part of the $(\mathbf{s},\mathbf{t})$-tree, but appear in its construction.

below. Furthermore, it is easy to see that the entries of T_A are weakly increasing in north-east and south-east direction. For $1 \leq i \leq u$, the column-sum of the (s_i+1)-st column in A is equal to 1. Since the first entry of A in this column is in row $(-s_i)$, the first $(-s_i)$ entries of the i-th north-east diagonal of T_A are equal to s_i+1. Hence we can delete $(-s_i-1)$ of these entries without losing information. Similarly, for $u+1 \leq i \leq n$ the first s_i entries of the i-th south-east diagonal of T_A will be $(l+s_i-1)$ and we can delete (s_i-1) of them. The resulting array T_A is an $(\mathbf{s},\mathbf{t})$-tree with bottom entries $\mathbf{k}$. Then, it is not difficult to see that every such $(\mathbf{s},\mathbf{t})$-tree with bottom entries $\mathbf{k}$ is of the form T_A. This proves the following proposition.

Proposition 4.4.4. *Let S be a centred Catalan set of size $n+1$. The map $A \mapsto T_A$ from (n,l)-AS-trapezoids with associated centred Catalan set S to $(\mathbf{s},\mathbf{t})$-trees with bottom entries $\mathbf{k}$ described above is a bijection.*

An $(\mathbf{s},\mathbf{t})$-tree with bottom entries $\mathbf{k}$ corresponding to an (n,l)-AS-trapezoid with associated centred Catalan set S will be be called an *(S,l)-tree* .

4.4.2 Equivalence classes of (S,l)-trees

In the following, we introduce an equivalence relation for AS-trapezoids and translate it to an equivalence relation for (S,l)-trees. The equivalence relation is more intuitively accessible in the AS-trapezoid picture, however, it is more convenient in the (S,l)-tree setting.

Definition 4.4.5. *Let n,l be positive integers and A,B two (n,l)-AS-trapezoids. For the central $l-1$ columns of A (resp. B), denote from left to right the columns which have non-zero entries by $c_1(A),\dots,c_f(A)$ (resp. $c_1(B),\dots,c_g(B)$). We say that A is equivalent to B iff the following holds.*

- *The columns with labels less than or equal to 0 are identical for A and B.*
- *The columns with labels greater than or equal to l are identical for A and B.*
- *For all $1 \leq i \leq f$, $f=g$ and $c_i(A)=c_i(B)$.*

We call the columns $c_1(A),\dots,c_f(A)$ the free columns *of A.*

Example 4.4.6. *Let A,B,C be the $(4,4)$-AS-trapezoids defined as below. The columns with labels less than or equal to 0 and the columns with labels greater than* *or equal to $l = 4$ are identical for A, B and C. The free columns are marked in red and green.*

$$A = \begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & {\color{red}0} & {\color{green}1} & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & {\color{red}1} & {\color{green}0} & 0 & 0 & 0 & \\
 & & 1 & 0 & 0 & {\color{red}0} & {\color{green}-1} & 0 & 1 & & \\
 & & & 1 & 0 & {\color{red}-1} & {\color{green}0} & 1 & & &
\end{array},$$

$$B = \begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & {\color{red}1} & {\color{green}0} & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & {\color{red}0} & {\color{green}1} & 0 & 0 & 0 & \\
 & & 1 & 0 & 0 & {\color{red}-1} & {\color{green}0} & 0 & 1 & & \\
 & & & 1 & 0 & {\color{red}0} & {\color{green}-1} & 1 & & &
\end{array},$$

$$C = \begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & {\color{red}0} & 0 & {\color{green}1} & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & {\color{red}1} & 0 & {\color{green}0} & 0 & 0 & 0 & \\
 & & 1 & 0 & {\color{red}0} & 0 & {\color{green}-1} & 0 & 1 & & \\
 & & & 1 & {\color{red}-1} & 0 & {\color{green}0} & 1 & & &
\end{array}.$$

The free columns of A and C are

$$c_1 = \begin{array}{c} 0 \\ 1 \\ 0 \\ -1 \end{array}, \qquad c_2 = \begin{array}{c} 1 \\ 0 \\ -1 \\ 0 \end{array},$$

hence A and C are equivalent. The free columns of B are c_2, c_1. The order of these columns is the opposite of the free columns of A, so the AS-trapezoids A and B are not equivalent.

Let A, B be (n, l)-AS-trapezoids. If A and B are equivalent, then $\mathbf{S}(A) = \mathbf{S}(B)$. This follows because the centred Catalan set $\mathbf{S}(A)$ depends only on the columns of A with labels less than or equal to 0 and columns with labels greater than or equal l, which are by definition fixed by the equivalence relation. Hence the equivalence relation on (n, l)-AS-trapezoids induces an equivalence relation for (S, l)-trees, which is given as follows.

Definition 4.4.7. *Let $T = (T_{i,j}), T' = (T'_{i,j})$ be two (S, l)-trees. We call T and T'* equivalent *iff the following holds.*

1. *For all $i, i', j, j', T_{i,j} < T_{i',j'}$ holds if and only if $T'_{i,j} < T'_{i',j'}$.*
2. *For all i, j if $T_{i,j} \leq 0$ or $T_{i,j} \geq l$, or $T'_{i,j} \leq 0$ or $T'_{i,j} \geq l$, then $T_{i,j} = T'_{i,j}$.*

We define the number of free columns *of an (S, l)-tree T as $|\{T_{i,j} : 0 < T_{i,j} < l\}|$. It follows from the definition that the number of free columns is constant on equivalence classes.*

Example 4.4.8. *Let A be the first $(4,4)$-AS-trapezoid in Example 4.4.6. The $(\{-2,-1,0,1,2\},4)$-tree T_A corresponding to A and the equivalence class $\overline{T_A}$ of T_A are, respectively:*

$$T_A = \begin{matrix} & & 3 & & \\ & 2 & & 3 & \\ -1 & & 2 & & 5 \\ & 0 & & 4 & \end{matrix} \quad \textit{and} \quad \overline{T_A} = \begin{matrix} & & b & & \\ & a & & b & \\ -1 & & a & & 5 \\ & 0 & & 4 & \end{matrix},$$

with $0 < a < b < 4$. The number of free columns of T_A is 2.

Lemma 4.4.9. *Let S be a centred Catalan set, $l > 0$ and T an (S,l)-tree with f free columns. Then the size of the equivalence class $\overline{T}$ of T is given by*

$$|\overline{T}| = \binom{l-1}{f}.$$

Proof. Let $x_1, \dots, x_f$ be entries in T such that $0 < x_1 < \dots < x_f < l$. Every element of the equivalence class $\overline{T}$ is uniquely described by the sequence of these values $(x_1, \dots, x_f)$. Hence

$$|\overline{T}| = |\{(x_1, \dots, x_f) : 0 < x_1 < \dots < x_f < l\}| = \binom{l-1}{f}. \qquad \square$$

Let $S = \{s_1, \dots, s_u, 0, s_{u+1}, \dots, s_n\}$ be a centred Catalan set of size $n+1$ with $s_1 < \dots < s_u < 0 < s_{u+1} < \dots < s_n$. We define two Young diagrams $\lambda(S), \mu(S)$ via

$$\lambda(S) = (u + 1 - s_{u+1}, \dots, n - s_n)',$$
$$\mu(S) = (-s_1 - u, \dots, -s_u - 1),$$

where λ' denotes the conjugate Young diagram, see Figure 4.4. Alternatively, one can describe $\lambda(S)$ and $\mu(S)$ as the smallest Young diagrams such that $\lambda(S)/\mu(S)$ is the area enclosed between the paths $\tilde{\lambda}(S) = (\tilde{\lambda}_i)_{1 \le i \le n}$ and $\tilde{\mu}(S) = (\tilde{\mu}_i)_{1 \le i \le n}$, which are defined in the following table.

	$\tilde{\lambda}_i$	$\tilde{\mu}_i$
$\{-i, i\} \subseteq S$	E	N
$-i \in S, i \notin S$	N	N
$i \in S, -i \notin S$	E	E
$-i, i \notin S$	N	E

Using this description, it is easy to see that $\mu(S)$ is included in $\lambda(S)$ and that the following holds

$$|\lambda(S)/\mu(S)| = \text{area}(\mathbf{M}(S)). \tag{4.5}$$

Remark 4.4.10. *Let S_1, S_2 be two centred Catalan sets, each of the form $S_i = \{s_{i,1}, \dots, s_{i,u_i}, 0, s_{i,u_i+1}, \dots, s_{i,n_i}\}$ with $s_{i,1} < \dots < s_{i,u_i} < 0 < s_{i,u_i+1} < \dots < s_{i,n_i}$ and $\lambda(S_i) = (\lambda_{i,j})_{1 \le j \le u_i}$ and $\mu(S_i) = (\mu_{i,j})_{1 \le j \le u_i}$ for $1 \le i \le 2$. It follows from*

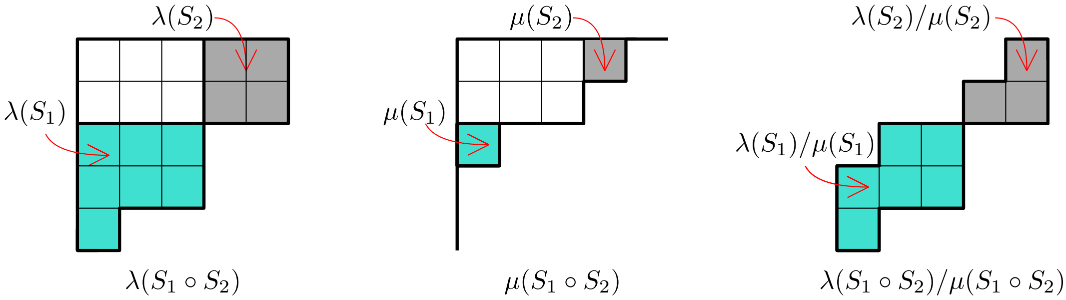


Figure 4.4: The (skew-shaped) Young diagrams corresponding to $S_1 = \{-4,-2,-1,0,1,3,4\}, S_2 = \{-3,-1,0,1,2\}$ and $S_1 \circ S_2$.

the definition of the Young diagrams λ, μ that the diagrams corresponding to the concatenation $S_1 \circ S_2$ are given by

$$\lambda(S_1 \circ S_2) = (\lambda_{2,1} + u_1, \ldots, \lambda_{2,u_2} + u_1, \lambda_{1,1}, \ldots, \lambda_{1,u_1}),$$
$$\mu(S_1 \circ S_2) = (\mu_{2,1} + u_1, \ldots, \mu_{2,u_2} + u_1, \mu_{1,1}, \ldots, \mu_{1,u_1}).$$

For an example, see Figure 4.4. Hence the size of the skew-shaped Young diagram is given by

$$|\lambda(S_1 \circ S_2)/\mu(S_1 \circ S_2)| = |\lambda(S_1)/\mu(S_1)| + |\lambda(S_2)/\mu(S_2)|.$$

Lemma 4.4.11. *Let S be an irreducible centred Catalan set of size n, then the following are true.*

1. *The number of free columns of an (S,l)-tree is at most $|\lambda(S)/\mu(S)|$.*
2. *For $l > |\lambda(S)/\mu(S)|$, the equivalence classes of (S,l)-trees are in bijection with the equivalence classes of $(S, |\lambda(S)/\mu(S)| + 1)$-trees.*
3. *For $l > |\lambda(S)/\mu(S)|$ the number of equivalence classes with maximal free columns is equal to $f^{\lambda(S)/\mu(S)}$, the number of standard Young tableaux of skew shape $\lambda(S)/\mu(S)$.*

Since the above lemma plays a fundamental role in the proof of Theorem 4.1.2 we want to illustrate it first by an example before proving it.

Example 4.4.12. *Let $S = \{-2,-1,0,1,2\}$, as in Example 4.4.8. Every (S,l)-tree T has the form*

$$T = \begin{array}{ccccc} & & c & & \\ & a & & d & \\ -1 & & b & & l+1 \\ & 0 & & l & \end{array},$$

with $a \leq b, c \leq d$. We show in the proof of Lemma 4.4.11 that the skew-shaped Young tableau $\lambda(S)/\mu(S)$ can be constructed by putting a box around the entries of T that are not fixed by definition (in our case, around a,b,c,d) and rotate the

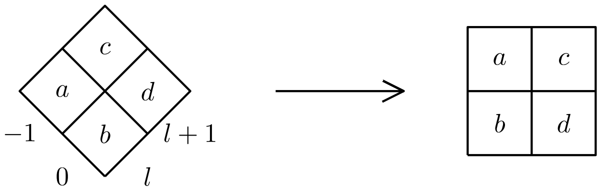


Figure 4.5: On the left: the general form of a $(4, l)$-AS-trapezoid with central Catalan set $S = \{-2, -1, 0, 1, 2\}$, where there is a box around every entry of the trapezoid that is not fixed by definition. On the right: the skew-shaped Young diagram $\lambda(S)/\mu(S)$ with $S = \{-2, -1, 0, 1, 2\}$.

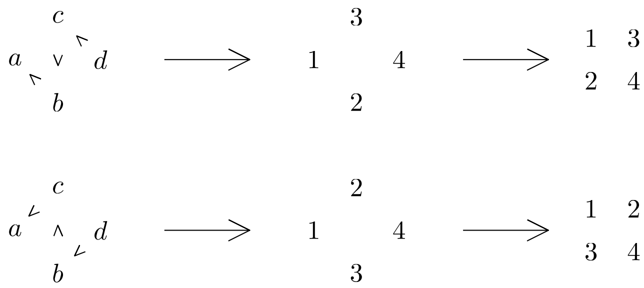


Figure 4.6: The transformation from the order relation between a, b, c and d to a standard Young diagram.

resulting shape by $45°$ *clockwise, see Figure 4.5. The number of free columns of* T *is* $\{a, b, c, d\} \cap \{1, 2 \ldots, l-1\} \leq 4$*. The equivalence class of* T *is fully determined by the order relation between the entries* a, b, c, d *and their exact value iff the value is in the set* $\{z \in \mathbb{Z} | z \leq 0 \text{ or } z \geq l\}$*. It is thus obvious that all possible equivalence classes appear for* $|\{1, \ldots, l-1\}| \geq 4$*, which is equivalent to* $l > 4$*. If* T *has a maximal number of free columns, the equivalence class of* T *is determined solely by the order relation between* a, b, c, d*; this can be either* $a < b < c < d$ *or* $a < c < b < d$*. We forget about all other entries of the AS-trapezoid and replace* a, b, c, d *by the numbers* $1, 2, 3, 4$ *according to their ordering. This is possible because we have a maximal number of free columns, which means that all values of* a, b, c, d *must be pairwise different. In the next step, we rotate the array by* $45°$ *clockwise and obtain a standard Young tableaux of skew shape* $\lambda(S)/\mu(S)$*, see Figure 4.6.*

Proof of Lemma 4.4.11. Let T be an (S, l)-tree. We draw a box around all entries of T that are not fixed by definition. (An example is on the left side of Figure 4.7.) Denote by b_i the number of boxes in the i-th row from the bottom of T. The definition of (S, l)-trees implies that

- there is no box in the bottom row,
- there is a box north-west of the leftmost box in row i iff $-i \in S$,
- there is a box north-east of the rightmost box in row i iff $i \in S$.

Hence, the outer shape is determined by the two paths λ' and μ', where the i-th steps λ'_i, μ'_i of λ', μ' are given by

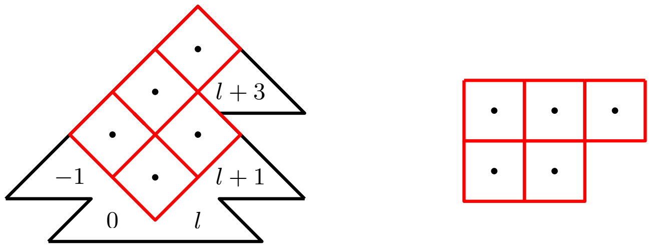


Figure 4.7: Schematic representation of an (S, l)-tree and the skew-shaped Young diagram $\lambda(S)/\mu(S)$ for $S = \{-2, -1, 0, 1, 2, -4\}$.

	λ'_i	μ'_i
$\{-i, i\} \subseteq S$	NE	NW
$-i \in S, i \notin S$	NW	NW
$i \in S, -i \notin S$	NE	NE
$-i, i \notin S$	NW	NE

By rotating the box complex by 45° clockwise, we obtain the skew-shaped diagram $\lambda(S)/\mu(S)$ whose entries are the entries of T not fixed by definition. Hence the maximal number of free columns is $|\lambda(S)/\mu(S)|$.

Let $l > |\lambda(S)/\mu(S)|$ and $\overline{T}$ be an equivalence class of (S, l)-trees with f free columns. Then there exists a representative T such that the entries that lie in a box have values between 1 and f. Denote by T' the $(S, |\lambda(S)/\mu(S)| + 1)$-tree which is obtained from T by subtracting $l - (|\lambda(S)/\mu(S)| + 1)$ from all positive entries in T which do not lie in a box. The map $\overline{T} \mapsto \overline{T'}$ is then a bijection of equivalence classes of (S, l)-trees to equivalence classes of $(S, |\lambda(S)/\mu(S)| + 1)$-trees.

The above argument implies that we can restrict ourselves to $l = |\lambda(S)/\mu(S)|+1$. For an equivalence class with a maximal number of free columns, all entries in the boxes are different. Since $l = |\lambda(S)/\mu(S)|+1$, the entries in the boxes are the integers $1, \ldots, |\lambda(S)/\mu(S)|$. The entries in an $(\mathbf{s}, \mathbf{t})$-tree are strictly increasing along rows from left to right and weakly increasing along north-east and south-east diagonals from bottom to top. Therefore, the filling of the skew-shaped Young diagram is strictly increasing along columns and rows, i.e., it is a skew-shaped standard Young tableau. Hence every equivalence class with maximal free columns corresponds to a standard Young tableau of skew shape $\lambda(S)/\mu(S)$. It is obvious that the converse is also true. □

We can now prove our second main result.

Proof of Theorem 4.1.2. By Proposition 4.1.1 and Remark 4.4.10, it suffices to assume that S is irreducible. As a consequence of Theorem 4.4.2 and Proposition 4.4.4, the number $w_l(S)$ of (S, l)-trees is a polynomial in l. Let $d(S)$ be the degree of $w_l(S)$. We show by induction on $\text{area}(\mathbf{M}(S))$ that $d(S) \leq \text{area}(\mathbf{M}(S))$. Write S in the form $S = \{s_1, \ldots, s_u, 0, s_{u+1}, \ldots, s_n\}$ with $s_1 < \ldots < s_u < 0 < s_{u+1} < \ldots < s_n$. The degree $d(S)$ is at most the degree of the polynomial $M_n(x_1, \ldots, x_n)$ minus the

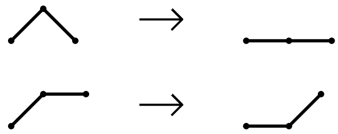


Figure 4.8: The two possibilities for changes between $M(S)$ and $M(S')$ at the positions i_0, i_0+1, i_0+2.

number of $\overline{\Delta}, \underline{\Delta}$ operators appearing in (4.4) and therefore

$$d(S) \leq \binom{n}{2} - \sum_{i=1}^{u}(-s_i - 1) - \sum_{i=u+1}^{n}(s_i - 1) = \binom{n}{2} - \sum_{i \in S \setminus \{0\}}(|i| - 1). \tag{4.6}$$

We claim that

$$\binom{n}{2} = \sum_{i \in S \setminus \{0\}}(|i| - 1) + \operatorname{area}(\mathbf{M}(S)), \tag{4.7}$$

which, together with (4.6), will imply that $d(S) \leq \operatorname{area}(\mathbf{M}(S))$. The area of $\mathbf{M}(S)$ is equal to 0 iff for all $1 \leq i \leq n$ either $i \in S$ or $-i \in S$, hence

$$\sum_{i \in S \setminus \{0\}}(|i| - 1) = \sum_{i=1}^{n}(i-1) = \binom{n}{2}.$$

If $\operatorname{area}(\mathbf{M}(S)) > 0$, denote by i_0 the largest integer with $1 \leq i_0 \leq n-1$ and $\{-i_0, i_0\} \subseteq S$. Then the centred Catalan set

$$S' := \begin{cases} (S \setminus \{i_0\}) \cup \{i_0 + 1\} & (i_0+1) \notin S, \\ (S \setminus \{i_0\}) \cup \{-(i_0+1)\} & -(i_0+1) \notin S, \end{cases}$$

is well defined. The paths $\mathbf{M}(S)$ and $\mathbf{M}(S')$ differ only in the i_0-th and (i_0+1)-st step, as shown in Figure 4.8. This implies that $\operatorname{area}(\mathbf{M}(S)) = \operatorname{area}(\mathbf{M}(S')) + 1$. On the other hand, the sum over all $i \in S \setminus \{0\}$ in (4.7) is one less than the sum over all $i \in S' \setminus \{0\}$, which proves (4.7).

By Lemma 4.4.9, an equivalence class of (S, l)-trees with f free columns contributes $\binom{l-1}{f}$ to $w_l(S)$. Lemma 4.4.11 states that there are $f^{\lambda(S)/\mu(S)}$ equivalence classes with $|\lambda(S)/\mu(S)| = \operatorname{area}(\mathbf{M}(S))$ free columns. Hence $w_l(S)$ has degree $\operatorname{area}(\mathbf{M}(S))$ and the leading coefficient is $\frac{f^{\lambda(S)/\mu(S)}}{|\lambda(S)/\mu(S)|!}$. □

Since $w_l(M) = \sum_{S:\mathbf{M}(S)=M} w_l(S)$, the following is a direct consequence of the above theorem.

Corollary 4.4.13. *Let $M = M_1 \circ \ldots \circ M_k$ be a Motzkin path and $M_1, \ldots, M_k$ its irreducible components. The weight function $w_l(M)$ is a polynomial in l of degree* $\operatorname{area}(M)$ *with leading coefficient*

$$\prod_{i=1}^{k} \sum_{S:\mathbf{M}(S)=M_i} \frac{f^{\lambda(S)/\mu(S)}}{\operatorname{area}(M_i)!}. \tag{4.8}$$

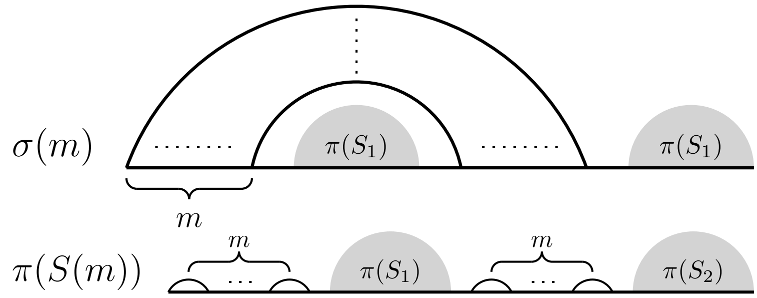


Figure 4.9: Graphical representation of the noncrossing matchings $\sigma(m)$ in the top and $\pi(S(m))$ in the bottom.

It is unknown to the author whether (4.8) can be simplified.

Remark 4.4.14. *For a centred Catalan set of size n, denote by $\pi(S)$ the noncrossing matching of size n corresponding to $\mathbf{D}(S)$. Let S_1, S_2 be two irreducible centred Catalan sets of size n_1 and n_2, respectively, and let m be a positive integer. We define $S(m) = \{0,1,\ldots,m\} \cup \mathfrak{s}_m(S_1) \cup \{n_1+m, n_1+m+1, \ldots, n_1+2m+1\} \cup \mathfrak{s}_{n_1+2m+1}(S_2)$. The noncrossing matching $\pi(S(m))$ consists of $\pi(S_1)$ enclosed by m 'small arches' on every side concatenated with $\pi(S_2)$, see Figure 4.9. By Proposition 4.1.1 and the fact that $w(\{0,1,\ldots,m\}) = 1$, the weight function $w(S(m))$ can be written as*

$$\begin{aligned} w(S(m)) &= w_2(S(m)) \\ &= w_2(\{0,1,\ldots m\}) w_{2(m+1)}(S_1) w_{2(n_1+m+1)}(\{0,1,\ldots m+1\}) w_{2(n_1+2m+3)}(S_2) \\ &= w_{2m}(S_1) w_{2(n_1+2m+3)}(S_2), \end{aligned}$$

Theorem 4.1.2 implies that $w(S(m))$ is a polynomial in m of degree $|\lambda(S_1)/\mu(S_1)| + |\lambda(S_2)/\mu(S_2)|$ with leading coefficient

$$\frac{2^{|\lambda(S_1)/\mu(S_1)|+|\lambda(S_2)/\mu(S_2)|} f^{\lambda(S_1)/\mu(S_1)} f^{\lambda(S_2)/\mu(S_2)}}{|\lambda(S_1)/\mu(S_1)|! \cdot |\lambda(S_2)/\mu(S_2)|!}. \tag{4.9}$$

An analogous theorem, Theorem 3.1.1, exists for fully packed loops (FPLs). Denote by $\sigma(m) := (\pi(S_1))_m \pi(S_2)$ the noncrossing matching which consists of $\pi(S_1)$ enclosed by m 'nested arches' concatenated with $\pi(S_2)$, see Figure 4.9. Then Theorem 3.1.1 states that the number of FPLs associated to $\sigma(m)$ is a polynomial in m of degree $|\lambda(\pi(S_1))| + |\lambda(\pi(S_2))|$ with leading coefficient

$$\frac{f^{\lambda(\pi(S_1))} f^{\lambda(\pi(S_2))}}{|\lambda(\pi(S_1))|! \cdot |\lambda(\pi(S_2))|!}. \tag{4.10}$$

This is remarkable for three reasons. First, the refinements of ASTs and FPLs by Catalan objects are of different nature, however we have in both cases a polynomiality theorem. Second, the degree and the leading coefficient of the polynomials are almost the same, with the important difference that the degree and the leading coefficient are linked to standard Young tableaux in the FPL case, whereas they are linked to *skew-shaped standard Young tableaux in the AST case. Third, the two refinements seem to be in some sense 'dual': for one, the polynomiality results differ by the dual notions of 'nested arches' and 'small arches'. Then, the number of FPLs linked to a noncrossing matching is minimal if it consists of nested arches and is maximal if it consists of small arches. Furthermore, computer experiments suggest that for a centred Catalan set S the weight $w_l(S)$ is maximal if $\pi(S)$ consists of nested arches.*

4.5 A constant term expression for AS-trapezoids

Using Theorem 4.4.2, we will derive a constant term expression for $w_l(S)$. Our proof follows the same steps as the proof of Theorem 7 in [38]. The following identities will be needed later. The first is an easy consequence of the definitions of the operators E_x, $\overline{\Delta}_x$, and $\underline{\Delta}_x$, while the second is stated in [30, Lemma 5].

$$\mathrm{Id} + \overline{\Delta}_y + \overline{\Delta}_x \overline{\Delta}_y = E_x E_y (\mathrm{Id} - \underline{\Delta}_x + \underline{\Delta}_x \underline{\Delta}_y) = E_y (\mathrm{Id} + \overline{\Delta}_x \underline{\Delta}_y), \tag{4.11}$$

$$M_n(x_1, \ldots, x_n) = (-1)^{n-1} M_n(x_2, \ldots, x_n, x_1 - n). \tag{4.12}$$

Proof of Theorem 4.1.3. We set $\mathbf{s} = \{-s_1 - 1, \ldots, -s_u - 1\}, \mathbf{t} = \{s_{u+1} - 1, \ldots, s_n - 1\}, \mathbf{k} = \{s_1 + 1, \ldots, s_u + 1, l + s_{u+1} - 1, \ldots, l + s_n - 1\}$. Since (n, l)-alternating sign trapezoids correspond to $(\mathbf{s}, \mathbf{t})$-trees with bottom entries $\mathbf{k}$, Theorem 4.4.2 states

$$\begin{aligned} w_l(S) &= \left(\prod_{i=1}^{u} (-\overline{\Delta}_{x_i})^{-s_i-1} \prod_{i=u+1}^{n} \underline{\Delta}_{x_i}^{s_i-1} \right) M_n(x_1, \ldots, x_n) \Bigg|_{\mathbf{x}=\mathbf{k}} \\ &= \left(\prod_{i=1}^{u} (-\overline{\Delta}_{x_i})^{-s_i-1} E_{x_i}^{s_i+1} \prod_{i=u+1}^{n} \underline{\Delta}_{x_i}^{s_i-1} E_{x_i}^{l+s_i-1} \right) M_n(x_1, \ldots, x_n) \Bigg|_{\mathbf{x}=\mathbf{0}} \\ &= \left(\prod_{i=1}^{u} (-\underline{\Delta}_{x_i})^{-s_i-1} \prod_{i=u+1}^{n} \overline{\Delta}_{x_i}^{s_i-1} E_{x_i}^{l} \right) M_n(x_1, \ldots, x_n) \Bigg|_{\mathbf{x}=\mathbf{0}}. \end{aligned} \tag{4.13}$$

Using (4.11) and (4.12), we can write $M_n(x_1, \ldots, x_n)$ as

$$\begin{aligned} M_n(x_1, \ldots, x_n) &= (-1)^{u(n-1)} M_n(x_{u+1}, \ldots, x_n, x_1 - n, \ldots, x_u - n) \\ &= (-1)^{u(n-1)} \prod_{i=1}^{u} E_{x_i}^{-n} \prod_{1 \le i < j \le u} E_{x_i} E_{x_j} (\mathrm{Id} - \underline{\Delta}_{x_i} + \underline{\Delta}_{x_i} \underline{\Delta}_{x_j}) \\ &\times \prod_{u+1 \le i < j \le n} (\mathrm{Id} + \overline{\Delta}_{x_j} + \overline{\Delta}_{x_i} \overline{\Delta}_{x_j}) \prod_{i=1}^{u} \prod_{j=u+1}^{n} E_{x_i} (\mathrm{Id} + \underline{\Delta}_{x_i} \overline{\Delta}_{x_j}) \\ &\times (-1)^{u(n-1)} \prod_{1 \le i < j \le n} \frac{x_j - x_i}{j - i} = \mathrm{OP} \prod_{i=1}^{u} E_{x_i}^{-1} \det_{1 \le i, j \le n} \left(\binom{x_i}{j-1} \right), \end{aligned} \tag{4.14}$$

with

$$\mathrm{OP} = \prod_{1\le i<j\le u} (\mathrm{Id} - \underline{\Delta}_{x_i} + \underline{\Delta}_{x_i}\underline{\Delta}_{x_j}) \\ \times \prod_{u+1\le i<j\le n} (\mathrm{Id} + \overline{\Delta}_{x_j} + \overline{\Delta}_{x_i}\overline{\Delta}_{x_j}) \prod_{i=1}^{u}\prod_{j=u+1}^{n} (\mathrm{Id} + \underline{\Delta}_{x_i}\overline{\Delta}_{x_j}).$$

Using (4.13) and (4.14), we obtain the following expression for $w_l(S)$

$$w_l(S) = \mathrm{OP} \prod_{i=1}^{u} (-\underline{\Delta}_{x_i})^{-s_i-1} E_{x_i}^{-1} \prod_{i=u+1}^{n} \overline{\Delta}_{x_i}^{s_i-1} E_{x_i}^{l} \det_{1\le i,j\le n}\left(\binom{x_i}{j-1}\right)\Bigg|_{\mathbf{x}=\mathbf{0}}$$
$$= (-1)^{u+\sum_{i=1}^{u} s_i} \mathrm{OP} \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{u} \binom{x_i+s_i}{\sigma(i)+s_i} \prod_{i=u+1}^{n} \binom{x_i+l}{\sigma(i)-s_i}\Bigg|_{\mathbf{x}=\mathbf{0}}. \quad (4.15)$$

Since $(-\underline{\Delta}_x)\binom{x+s}{j+s}(-1)^s = E_s^{-1}\binom{x+s}{j+s}(-1)^s$ and $\overline{\Delta}_x\binom{x+l}{j-s} = E_s\binom{x+l}{j-s}$, we can replace OP by OP$'$, where OP$'$ is OP with every $-\underline{\Delta}_{x_i}$ replaced by $E_{s_i}^{-1}$ and every $\overline{\Delta}_{x_j}$ replaced by E_{s_j}. As a consequence of this, we can now evaluate (4.15) at $x_1, \dots, x_n = 0$ and obtain

$$w_l(S) = \mathrm{OP}'(-1)^{u+\sum_{i=1}^{u} s_i} \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{u} \binom{s_i}{\sigma(i)+s_i} \prod_{i=u+1}^{n} \binom{l}{\sigma(i)-s_i}.$$

In order to evaluate this expression we use the following identity. Let $f : \mathbb{Z}^n \to \mathbb{R}$ be a function such that the generating function $F(x_1, \dots, x_n) := \sum_{a\in\mathbb{Z}^n} f(a)x^a$ is a Laurent series and let $p(x_1, \dots, x_n)$ be a Laurent polynomial. Then

$$p(E_{a_1}, \dots, E_{a_n}) f(a) = \mathrm{CT}_{x_1,\dots,x_n} \prod_{i=1}^{n} x_i^{-a_i} p(x_1^{-1}, \dots, x_n^{-1}) F(x_1, \dots, x_n), \quad (4.16)$$

where $CT_{x_1,\dots,x_n}$ denotes the constant term in $x_1, \dots, x_n$. This identity can be proven easily if p is a monomial. The general statement follows immediately since p is a linear combination of monomials. We apply (4.16) for the function

$$f(s_1, \dots, s_n) := (-1)^{u+\sum_{i=1}^{u} s_i} \\ \times \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{u} \binom{s_i}{\sigma(i)+s_i} \prod_{i=u+1}^{n} \binom{l}{\sigma(i)-s_i}.$$

By using

$$\sum_{s\in\mathbb{Z}} (-1)^s \binom{s}{\sigma+s} x^s = (-x)^{-1}(-1-x^{-1})^{\sigma-1},$$
$$\sum_{s\in\mathbb{Z}} \binom{l}{\sigma-s} x^s = x^{\sigma-l}(1+x)^l.$$

we obtain for F

$$\begin{aligned}
F(x_1,\ldots,x_n) &= (-1)^u \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) \\
&\quad \times \prod_{i=1}^{u} (-x_i)^{-1}(-1-x_i^{-1})^{\sigma(i)-1} \prod_{i=u+1}^{n} x_i^{\sigma(i)-l}(1+x_i)^l \\
&= (-1)^u \det_{1\le i,j\le n}\left(\begin{cases} (-x_i)^{-1}(-1-x_i^{-1})^{j-1} & i\le u \\ x_i^{j-l}(1+x_i)^l & i>u \end{cases}\right) \\
= \prod_{i=1}^{u} x_i^{-u} \prod_{i=u+1}^{n} x_i^{1-l}(1+x_i)^l & \prod_{\substack{1\le i<j\le u,\\ u+1\le i<j\le n}} (x_j-x_i) \prod_{i=1}^{u}\prod_{j=u+1}^{n}(1+x_i^{-1}+x_j).
\end{aligned}$$

In the last step, we evaluated the determinant using the Vandermonde determinant evaluation. Hence we obtain

$$\begin{aligned}
w_l(S) = \mathrm{CT}_{x_1,\ldots,x_n} & \prod_{i=1}^{u} x_i^{-n-s_i} \\
& \prod_{i=u+1}^{n} x_i^{-n-l-s_i+2}(1+x_i)^l \prod_{1\le i<j\le n} (x_j-x_i)(1+x_i+x_ix_j). \quad \square
\end{aligned}$$

4.6 Roots of $w_l(S)$ and $w_l(M)$

Studies of $w_l(S)$ and $w_l(M)$ for small centred Catalan sets S and Motzkin paths M show that $w_l(S)$ and $w_l(M)$ often have rational roots, where almost all of these are even integer roots. In Section 4.7, we list $w_l(S), w_l(M)$ for all irreducible centred Catalan sets up to size six and irreducible Motzkin paths up to length five. This section contains results and conjectures based on a first analysis of a slightly larger data set. The conjectures can be divided into three types.

- Conjecture 4.6.1 predicts which rational numbers appear as roots of weight functions.
- The conjectures in Subsection 4.6.1 state relations between the rational roots of $w_l(S)$ and the beginning of the Motzkin path $\mathbf{M}(S)$.
- In Subsection 4.6.2, a conjecture connects the rational roots of $w_l(M)$ to the ending of the Motzkin path M.

The ultimate goal is a description of all rational roots of $w_l(S)$ and $w_l(M)$ similar to [41, Conjecture 1.2].

Conjecture 4.6.1. *1. Let M be an irreducible Motzkin path of length $n \ge 8$. The rational roots of $w_l(M)$ lie in $\{-1,-2,\ldots,-2n+2\}$, and for every integer x in this set there exists a Motzkin path M such that x is a root of $w_l(M)$.*

2. *Let S be an irreducible centred Catalan set of size $n \geq 11$. The rational roots of $w_l(S)$ lie in $\{-1, -2, \ldots, -2n+4, -\frac{n^2-5n+7}{(n-3)}\}$, and for every rational number x in this set there exists a centred Catalan set S such that x is a root of $w_l(S)$. Furthermore, $-\frac{n^2-5n+7}{(n-3)}$ is a root of $w_l(S)$ iff $S = \{-n+2, -1, 0, 1 \ldots, n-3\}$ or $S = \{-n+3, -n+4, \ldots, -1, 0, 1, n-2\}$.*

4.6.1 Rational roots of $w_l(S)$

Conjecture 4.6.2. *Let S be a centred Catalan set. Then*

$$\{-i, \ldots, i\} \subseteq S \quad \Leftrightarrow \quad \prod_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} (l+1+3k)_{i-2k} \text{ divides } w_l(S),$$

where $(x)_k := x(x+1)\cdots(x+k-1)$ is the Pochhammer symbol.

For Motzkin paths, the above conjecture implies that the weight function has certain integer roots that depend on the number of consecutive north-east steps at the beginning of the path. We can prove the above conjecture for $i = 1, 2$, however our proof technique does not extend to $i > 2$.

Proposition 4.6.3. *Let S be a centred Catalan set. Then the following holds:*

$$\{-1, 1\} \subseteq S \Leftrightarrow (l+1) \mid w_l(S),$$
$$\{-2, -1, 1, 2\} \subseteq S \Leftrightarrow (l+1)(l+2) \mid w_l(S).$$

Proof. By Proposition 4.1.1, the weight function $w_l(S)$, where $S = S_1 \circ S_2$, factorises into $w_l(S) = w_l(S_1) w_{l+2|S_1|-2}(S_2)$. Hence, it suffices to prove the proposition for irreducible S. The proof is based on the following fact. Let $p(x)$ be a polynomial in x and define

$$P(x) = \begin{cases} \sum_{i=0}^{x} p(i) & x \geq 0, \\ 0 & x = -1, \\ -\sum_{i=x+1}^{-1} p(i) & x < 0. \end{cases} \tag{4.17}$$

Then $P(x)$ is again a polynomial in x. Let $\{-1, 1\} \subset S$. Then, an (S, l)-tree has the form

$$\begin{array}{ccccc} \ddots & & a & & \iddots \\ & 0 & & l & \end{array},$$

where $0 \leq a \leq l$. Let $f(l, x)$ denote the number of (S, l)-trees such that the entry a in the second row from the bottom is equal to x. By Theorem 4.4.2, $f(l, x)$ is a polynomial in l and x. The weight $w_l(S)$ is given by

$$w_l(S) = \sum_{x=0}^{l} f(l, x).$$

Hence for $l = -1$, (4.17) implies

$$w_{-1}(S) = \sum_{x=0}^{-1} f(-1, x) = 0.$$

Assume $(l+1)|w_l(S)$. Since S is irreducible and $w_l(S)$ has degree at least one, $\{-1, 0, 1\}$ must be a subset of S, which proves the first claim.

Let $\{-2, -1, 1, 2\} \subset S$. Then, an (S, l)-tree has the form

$$\begin{array}{ccccccccc} \ddots & & & a & & b & & & \iddots \\ & & -1 & & c & & l+1 & & \\ & & & 0 & & l & & & \end{array},$$

with $-1 \le a \le c \le b \le l+1$, $0 \le c \le l$ and $a < b$. Let $f(l, x, y)$ denote the number of (S, l)-trees where the entries a, b in the third row from the bottom are equal to x and y respectively. The weight function is given by

$$w_l(S) = \sum_{c=0}^{l} \left(\sum_{a=-1}^{c} \sum_{b=c+1}^{l+1} f(l, a, b) + \sum_{a=-1}^{c-1} f(l, a, c) \right)$$

For $l = -2$, the sum $\sum_{c=0}^{-2}$ is equal to $-\sum_{c=-1}^{-1}$. Thus, for the weight function we obtain

$$\begin{aligned} w_{-2}(S) = -&\left(\sum_{a=-1}^{-1} \sum_{b=0}^{-1} f(-2, a, b) + \sum_{a=-1}^{-2} f(-2, a, -1) \right) \\ &= -\left(\sum_{a=-1}^{-1} 0 + 0 \right) = 0. \end{aligned}$$

Assume $\{-2, -1, -1, 2\}$ is not a subset of S. We can extend our definition of an (S, l)-tree using the generalisation of (4.17). The enumeration then becomes a weighted enumeration, where the weight of an (S, l)-tree is (-1) to the power of encounters of entries $x_{i,j} > x_{i,j+1}$. Hence an $(S, -2)$-tree has the form

$$\begin{array}{ccccc} \ddots & & \vdots & & \iddots \\ & -1 & & -1 & \\ & & 0 & & -2 \end{array}, \qquad \begin{array}{ccccc} \ddots & & \vdots & & \iddots \\ & & -1 & & -1 \\ & 0 & & -2 & \end{array},$$

where the left corresponds to an $(S, -2)$-tree with $\{-2, -1, 0, 1\} \subseteq S$ and the right to $\{-1, 0, 1, 2\} \subseteq S$, and their weight is -1. By deleting the bottom row and adding 1 to all entries, we obtain an $(S', 0)$-tree with $S' = \{i+1 | i \in S, i < -2\} \cup \{-1, 0, 1\} \cup \{i - 1, i \in S, i > 2\}$. Hence in both cases

$$-w_{-2}(S) = |w_0(S')|$$

holds. By the definition of an (S, l)-tree, it is obvious that for $l \ge 0$ there is at least one (S, l)-tree, hence $w_0(S') \ne 0$, proving the claim. $\square$

4.6.2 Rational roots of $w_l(M)$

Proposition 4.6.4. *Let M be a Motzkin path of length n that ends with a south-east step, and let M' denote the Motzkin path of length $n+1$ obtained by putting an east step in front of the last step of M. The weight $w_l(M')$ is given by*

$$w_l(M') = (l + 2n)w_l(M).$$

Proof. Let A' be an $(n+1, l)$-AS-trapezoid with $\mathbf{M}(A') = M'$. If A' does not have a -1 in the second to last row from the top, the last two rows have the form

$$\begin{array}{cccccccc} 0 & 1 & 0\cdots 0 & 0 & 0\cdots 0 & 0 \\ & 0 & 0\cdots 0 & 1 & 0\cdots 0 & \end{array}$$

up to horizontal and vertical reflection of the inner $l+2n-1$ columns. By reflecting the top two rows in such a way that one obtains the above form, and deleting the top row, we obtain an (n, l)-AS-trapezoid A with $\mathbf{M}(A) = M$. Now assume A' has a -1 entry in its second row from the top. Then the two top rows have the form

$$\begin{array}{cccccccccc} 0 & 0 & 0\cdots 0 & 1 & 0\cdots 0 & 0 & 0\cdots 0 & 0 \\ & 1 & 0\cdots 0 & -1 & 0\cdots 0 & 1 & 0\cdots 0 & \end{array}$$

up to horizontal reflection. If we delete the top row and the leftmost 1 and -1 in the second row from the top, we again obtain an (n, l)-AS-trapezoid A with $\mathbf{M}(A) = M$. Conversely, let A be an (n, l)-AS-trapezoid. We can construct 4 AS-trapezoids A' with no -1 in the second row from the top and $l + 2(n-2)$ AS-trapezoids A' with one -1 entry in the second row from the top, such that $\mathbf{M}(A') = M'$. This proves the claim. □

Conjecture 4.6.5. *Let M be a Motzkin path of length $n+k$. We denote by $M = (\ldots, e_1, \ldots, e_k)$ that the last k steps of M are $e_1, \ldots, e_k$. Then the following hold.*

$$M = (\ldots, 1, 0, -1, -1) \Rightarrow (l + 2n + 5)|w_l(M),$$

$$M = (\ldots, 1, 0, 0, -1, -1) \Rightarrow (l + 2n + 7)|w_l(M),$$

$$M = (\ldots, 1, 1, -1, -1, -1) \Rightarrow (l + 2n + 2)(l + 2n + 7)(l + 2n + 8)|w_l(M),$$

$$M = (\ldots, 1, 1, -1, 0, -1, -1) \Rightarrow (l + 2n + 2)(l + 2n + 7)|w_l(M),$$

$$M = (\ldots, 1, 1, -1, 0, 0, -1, -1) \Rightarrow (l + 2n + 2)|w_l(M),$$

$$M = (\ldots, 1, 1, -1, 0, 0, 0, -1, -1) \Rightarrow (l + 2n + 2)|w_l(M),$$

$$M = (\ldots, 1, 0, 1, -1, -1, -1) \Rightarrow (l + 2n + 8)|w_l(M),$$

$$M = (\ldots, 1, 1, 0, -1, -1, -1) \Rightarrow (l + 2n + 2)(l + 2n + 8)^2|w_l(M),$$

$$M = (\ldots, 1, 1, 0, -1, 0, -1, -1) \Rightarrow (l + 2n + 2)|w_l(M),$$

$$M = (\ldots, 1, 1, 0, -1, 0, 0, -1, -1) \Rightarrow (l + 2n + 2)|w_l(M),$$

$$M = (\ldots, 1, 1, 0, -1, 0, 0, 0, -1, -1) \Rightarrow (l + 2n + 2)|w_l(M),$$

$$M = (\ldots, 1, 1, 0, 0, -1, -1, -1) \Rightarrow (l + 2n + 2)(l + 2n + 10)|w_l(M).$$

It seems that the list of conjectures of the above kind can be continued as long as one likes. Experiments suggest that the steps $e_1, \ldots, e_k$ at the end must satisfy $e_1 = 1$ and $\sum_{i=1}^{k} e_i = -1$, i.e., the Motzkin path before these steps "ends" at height 1. Further, it is interesting that all rational roots of the shortest Motzkin path with one of the above endings, with the exception of the path with the $7, 9, 12$-th ending, are explained by the above conjecture and Conjecture 4.6.2.

4.7 Tables for $w_l(S)$ and $w_l(M)$

The following two tables list the weight functions of all irreducible centred Catalan sets of size less than 6, up to the reflection $S \mapsto \{-s : s \in S\}$, and the weight functions of irreducible Motzkin paths up to length 5.

S	$w_l(S)$
$\{0,1\}$	1
$\{-1,0,1\}$	$(l+1)$
$\{-1,0,1,2\}$	$\frac{1}{2}(l+1)(l+4)$
$\{-2,-1,0,1,2\}$	$\frac{1}{12}(l+1)(l+2)(l+6)(l+7)$
$\{-3,-1,0,1,2\}$	$\frac{1}{6}(l+1)(l+6)(2l+7)$
$\{-1,0,1,2,3\}$	$\frac{1}{6}(l+1)(l+5)(l+6)$
$\{-2,-1,0,1,2,3\}$	$\frac{1}{144}(l+1)(l+2)(l+7)(l^3+23l^2+168l+360)$
$\{-2,-1,0,1,2,4\}$	$\frac{1}{24}(l+1)(l+2)(l+6)(l+7)(l+8)$
$\{-3,-1,0,1,2,3\}$	$\frac{1}{24}(l+1)(l^4+25l^3+226l^2+864l+1176)$
$\{-1,0,1,2,3,4\}$	$\frac{1}{24}(l+1)(l+6)(l+7)(l+8)$
$\{-2,-1,0,1,3,4\}$	$\frac{1}{24}(l+1)(l+6)(3l^2+37l+92)$
$\{-3,-1,0,1,2,4\}$	$\frac{1}{24}(l+1)(l+6)(5l^2+55l+132)$
$\{-4,-1,0,1,2,3\}$	$\frac{1}{24}(l+1)(l+6)(l+8)(3l+13)$

M	$w_l(M)$
(0)	2
$(1,-1)$	$(l+1)$
$(1,0,-1)$	$(l+1)(l+4)$
$(1,1,-1,-1)$	$\frac{1}{12}(l+1)(l+2)(l+6)(l+7)$
$(1,0,0,1)$	$(l+1)(l+4)(l+6)$
$(1,1,0,-1,-1)$	$\frac{1}{72}(l+1)(l+2)(l+7)(l^3+23l^2+168l+360)$
$(1,1,-1,0,-1)$	$\frac{1}{12}(l+1)(l+2)(l+6)(l+7)(l+8)$
$(1,0,1,-1,-1)$	$\frac{1}{12}(l+1)(l^4+25l^3+226l^2+864l+1176)$
$(1,0,0,0,-1)$	$(l+1)(l+4)(l+6)(l+8)$

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Chapter 5

A new determinant for the Q-enumeration of alternating sign matrices

5.1 Introduction

The starting point of this chapter is the following theorem.

Theorem 5.1.1. *Let k be an integer and n be a positive integer and denote by*

$$d_{n,k}(x,q) := \det_{1 \le i,j \le n} \left(\binom{x+i+j-2}{j-1} \frac{1-(-q)^{j-i+k}}{1+q} \right).$$

Then the $(q^{-1}+2+q)$-enumeration of alternating sign matrices is equal to $d_{n,1}(0,q)$.

The above determinant appeared first in [37] for $k=1$ and was shown by Fischer to count the number of ASMs for $x=0, k=1$ and q being a primitive third root of unity. By introducing the variable x, which was suggested in [37], and the integer parameter k in the determinant, we are able to write $d_{n,k}(x,q)$ as a closed product formula for arbitrary k and q being a primitive second root of unity (Theorem 5.4.1), primitive third root of unity (Theorem 5.4.2), primitive fourth root of unity (Theorem 5.4.4) or primitive sixth root of unity (Theorem 5.4.6), which was conjectured for $k=1$ in [37]. For $q=1$ we provide in Theorem 5.4.8 a factorisation of the determinant as a polynomial in x. Compared to other known determinantal formulas for the Q-enumeration of ASMs, for which the evaluation is rather complicated, the evaluation of the determinant considered in this chapter turns out to be easy and thus leading immediately to the known formulas for the 0-,1-,2- and 3-enumeration of alternating sign matrices by setting $x=0$.

In Theorem 5.3.5 we prove a general factorisation result which states that the determinant $d_{n,k}(x,q)$ factors for arbitrary q into a power of q, a polynomial $p_{n,k}(x) \in \mathbb{Q}[x]$ which factorises into linear factors and a polynomial $f_{n,k}(x,q) \in \mathbb{Q}[x,q]$ which is given recursively. Theorem 5.4.1 implies further that all linear factors in x of

$d_{n,k}(x,q)$ are covered in $p_{n,k}(x)$. For $k=1$, Theorem 5.3.5 implies that the determinant $d_{n,k}(x,q)$ can be written as a product of two Laurent polynomials in q with coefficients in $\mathbb{Q}[x]$, which was conjectured in [37]. As a direct consequence we obtain that the generating function of ASMs with respect to the number of -1 is a product of two polynomials in $Q=(q^{-1}+2+q)$, which was conjectured in [60, Conjecture 4] and first proven in [51].

Surprisingly the determinant $d_{n,k}(x,q)$ is connected to the famous determinant by Andrews [6] and its generalisation by Ciucu, Eisenkölbl, Krattenthaler and Zare [23] in the following way

$$d_{n,3-k}(x,q^2) = q^{-n} \det_{1\leq i,j\leq n}\left(\binom{x+i+j-2}{j-1}+q^k\delta_{i,j}\right), \tag{5.1}$$

where q is a sixth root of unity. This fact was first conjectured for $k=2,4$ in [37] and is remarkable because of two reasons. Firstly, for q being a primitive sixth root of unity and k an integer the evaluation of the determinants have a closed product formula. This fact is easy to prove for the left hand side of (5.1) by using the Desnanot-Jacobi identity which is also known as the *Condensation method*. For the right hand side this method is however not applicable and the proof relies on the *method of identification of factors*. Secondly, the determinant by Ciucu et al. is a weighted enumeration of cyclically symmetric lozenge tilings of hexagons with a triangular hole. These are a "one parameter generalisation" of DPPs, where q corresponds to the weight and x is the side length of the hole. Very recently Fischer proved in [36] that this determinant also enumerates alternating sign trapezoids, which are one parameter generalisations of ASTs. Hence the determinant $d_{n,k}(x,q)$ suggests a one parameter refinement for ASMs and might be of help in finding one.

This chapter is structured in the following way. In Section 5.2 we follow the steps of [37, 39] and prove Theorem 5.1.1. Section 5.3 contains a description of the factorisation of the determinant $d_{n,k}(x,q)$ for general q which allows us to prove that the $(q^{-1}+2+q)$-enumeration of ASMs is a product of two polynomials. In Section 5.4 we present the product formulas for $d_{n,k}(x,q)$ for q being a primitive second, third, fourth or sixth root of unity and present a factorisation for $q=1$. This leads to new proofs for the 1-,2- and 3-enumeration of ASMs. Finally in Section 5.5 we relate the determinant $d_{n,k}(x,q)$ to the Andrews determinant and its generalisation by Ciucu, Eisenkölbl, Krattenthaler and Zare. The final section of this chapter contains a list of specialisations of $d_{n,k}(x,q)$ which turn out to be known enumeration formulas.

5.2 A determinantal formula for the number of ASMs

We remind the reader of the following notations

$$\begin{aligned} E_x(f)(x) &:= f(x+1) && \text{shift operator,}\\ \overline{\Delta}_x &:= E_x - \mathrm{Id} && \text{forward difference,}\\ \underline{\Delta}_x &:= \mathrm{Id} - E_x^{-1} && \text{backward difference.} \end{aligned}$$

We denote by $\mathcal{AS}_{x_1,\dots,x_n} f(x_1,\dots,x_n)$ the antisymmetriser of f with respect to the variables $x_1,\dots,x_n$ which is defined as

$$\mathcal{AS}_{x_1,\dots,x_n} f(x_1,\dots,x_n) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) f(x_{\sigma(1)},\dots,x_{\sigma(n)}).$$

Further we are using at various points the multi-index notation, i. e., a bold variable $\mathbf{x}$ refers to a vector $\mathbf{x}=(x_1,\dots,x_n)$ and $\mathbf{x}^{\mathbf{a}}$ is defined as $\mathbf{x}^{\mathbf{a}} := \prod_{i=1}^n x_i^{a_i}$.

This section is used to deduce the determinantal expression for the Q-enumeration of ASMs. It is a revision of results in [39] and generalises results in [37]. The starting point is a weighted version of the operator formula for monotone triangles.

Theorem 5.2.1 ([32, Theorem 1])**.** *The generating function of monotone triangles with bottom row $\mathbf{k}=(k_1,\dots,k_n)$ with respect to the Q-weight, which is defined in Section 2.2.1, is given by evaluating the polynomial*

$$M_n(x_1,\dots,x_n) := \prod_{1\le i<j\le n} \left(Q\,\mathrm{Id} + (Q-1)\overline{\Delta}_{x_i} + \overline{\Delta}_{x_j} + \overline{\Delta}_{x_i}\overline{\Delta}_{x_j}\right) \prod_{1\le i<j\le n} \frac{x_j-x_i}{j-i},$$

at $\mathbf{x}=\mathbf{k}$.

Following [39], we can rewrite the operator formula as a constant term expression.

Proposition 5.2.2 ([39, Proposition 10.1])**.** *The number of monotone triangles with bottom row $\mathbf{k}=(k_1,\dots,k_n)$ with respect to the Q-weight is the constant term of*

$$\mathcal{AS}_{x_1,\dots,x_n}\left(\prod_{i=1}^n (1+x_i)^{k_i} \prod_{1\le i<j\le n} (Q+(Q-1)x_i+x_j+x_ix_j)\right) \prod_{1\le i<j\le n} (x_j-x_i)^{-1}.$$

The following theorem allows us to transfer the antisymmetriser in the above proposition into a determinant; the theorem is a variation by Fischer of Equation (D.3) in [42].

Theorem 5.2.3 ([37, Theorem 16])**.** *Let $f(x,y)=qx-q^{-1}y$ and $h(x,y)=x-y$, then holds*

$$\mathcal{AS}_{w_1,\dots,w_n} \frac{\prod_{1\le i<j\le n} f(w_i,w_j)}{\prod_{1\le i\le j\le n} h(w_j,y_i) f(w_i,y_j)} = \frac{\det_{1\le i,j\le n}\left(\frac{1}{f(w_i,y_j)h(w_i,y_j)}\right)}{\prod_{1\le i<j\le n} h(y_j,y_i)}.$$

By setting $w_i = \frac{x_i+1+q^{-1}}{x_i+1+q}$, $Q=(q^{-1}+2+q)$ and taking the limit of $y_i\to 1$ for all $1\le i\le n$, the above theorem becomes

$$\frac{(-1)^{\frac{n(n+1)}{2}}}{(q-q^{-1})^{\frac{n(n+3)}{2}}} \times \mathcal{AS}_{x_1,\dots,x_n}\left(\frac{\prod_{1\le i<j\le n}(Q+(Q-1)x_i+x_j+x_ix_j)\prod_{i=1}^n (x_i+1+q)^2}{\prod_{i=1}^n (1+x_i)^{n+1-i}}\right)$$

$$= q^n \lim_{y_1,\dots,y_n\to 1} \det_{1\le i,j\le n}\left(\frac{1}{\left(y_j-\frac{x_i+1+q^{-1}}{x_i+1+q}\right)\left(y_j-q^2\frac{x_i+1+q^{-1}}{x_i+1+q}\right)}\right) \prod_{1\le i<j\le n} (y_j-y_i)^{-1}.$$

Hence, by Proposition 5.2.2, the $(q^{-1}+2+q)$-enumeration of ASMs is given by

$$\mathrm{CT}_{x_1,\ldots,x_n}\left((-1)^{\frac{n(n+1)}{2}}q^n(q-q^{-1})^{\frac{n(n+3)}{2}}\prod_{i=1}^{n}(1+x_i)^{n+1}(x_i+1+q)^{-2}\right.$$
$$\times\lim_{y_1,\ldots,y_n\to 1}\left(\det_{1\le i,j\le n}\left(\frac{1}{\left(y_j-\frac{x_i+1+q^{-1}}{x_i+1+q}\right)\left(y_j-q^2\frac{x_i+1+q^{-1}}{x_i+1+q}\right)}\right)\right.$$
$$\left.\left.\times\prod_{1\le i<j\le n}(x_j-x_i)^{-1}(y_j-y_i)^{-1}\right)\right), \tag{5.2}$$

where $CT_{x_1,\ldots,x_n}$ denotes the constant term in $x_1,\ldots,x_n$. Using the partial fraction decomposition $\frac{1}{(y-a)(y-b)}=\frac{1}{a-b}\left(\frac{1}{y-a}-\frac{1}{y-b}\right)$ we can rewrite the determinant in (5.2) as

$$(1-q^2)^{-n}\prod_{i=1}^{n}(x_i+1+q)^2(x_i+1+q^{-1})^{-1}$$
$$\det_{1\le i,j\le n}\left(\frac{1}{y_j(x_i+1+q)-(x_i+1+q^{-1})}-\frac{1}{y_j(x_i+1+q)-q^2(x_i+1+q^{-1})}\right).$$

The following lemma allows us to evaluate the limit in (5.2).

Lemma 5.2.4 ([15, Eq. (43)-(47)]). *Let $f(x,y)=\sum_{i,j\ge 0}c_{i,j}x^iy^j$ be a formal power series in x and y, then holds*

$$\lim_{\substack{x_1,\ldots,x_n\to x\\ y_1,\ldots,y_n\to y}}\frac{\det_{1\le i,j\le n}(f(x_i,y_j))}{\prod_{1\le i<j\le n}(x_j-x_i)(y_j-y_i)}=\det_{0\le i,j\le n-1}\left([u^iv^j]f(x+u,y+v)\right),$$

where $[u^iv^j]f(x+u,y+v)$ denotes the coefficient of u^iv^j in $f(x+u,y+v)$.

By the above lemma we need to calculate the coefficient of x^iy^j in

$$\frac{1}{(y+1)(x+1+q)-(x+1+q^{-1})}-\frac{1}{(y+1)(x+1+q)-q^2(x+1+q^{-1})}. \tag{5.3}$$

Using the geometric series expansion in x and y of (5.3), we obtain for the coefficient of x^iy^j

$$\binom{j}{i}(1+q)^{-i-1}(q-1)^{-j-1}(-1)^jq^{j+1}-\sum_{k=i}^{i+j}(-1)^k\binom{k}{j}\binom{j}{i-k+j}\frac{(1+q)^{k-i-j-1}}{(1-q)^{j+1}}.$$

Putting the above together, it follows that the $(q^{-1}+2+q)$-enumeration of ASMs of size n is given by

$$(1+q)^{-n}q^{-\binom{n}{2}}\det_{0\le i,j\le n-1}\left(\binom{j}{i}(-1)^jq^{j+1}+\sum_{k=0}^{n-1}\binom{k+i}{j}\binom{j}{k}(-1-q)^{k+i-j}\right).$$

Finally we use the following identity which is due to Fischer [37] and can be proven using basic properties of the binomial coefficient

$$\binom{j}{i}(-1)^j q^{j+1} + \sum_{k=0}^{n-1} \binom{k+i}{j}\binom{j}{k}(-1-q)^{k+i-j}$$
$$= \sum_{k=0}^{n-1}(-1)^i \binom{i}{k}\binom{-k-1}{j}(q^{j+1}(-1)^k + q^k(-1)^j).$$

Hence the $(q^{-1}+2+q)$-enumeration of ASMs is

$$\det\left(\left(\binom{i}{j}(-1)^{i+j}\right)_{0\leq,i,j\leq n-1} \times \left(\binom{i+j}{j}\frac{1-(-q)^{j-i+1}}{1+q}\right)_{0\leq i,j\leq n-1}\right)$$
$$= \det_{1\leq i,j\leq n}\left(\binom{i+j-2}{j-1}\frac{1-(-q)^{j-i+1}}{1+q}\right), \tag{5.4}$$

which proves Theorem 5.1.1.

5.3 A generalised $(q^{-1}+2+q)$-enumeration

We introduce to the determinant in (5.4) a variable x, as suggested in [37], and further a parameter $k \in \mathbb{Z}$. The determinant of our interest is $d_{n,k}(x,q) := \det(D_{n,k}(x,q))$, where $D_{n,k}(x,q)$ is defined by

$$D_{n,k}(x,q) := \left(\binom{x+i+j-2}{j-1}\frac{1-(-q)^{j-i+k}}{1+q}\right)_{1\leq i,j\leq n}.$$

The evaluation of this determinant will generally follow two steps. The first step is to guess a formula for $d_{n,k}(x,q)$ using a computer algebra system. The second step is to use induction and the Desnanot-Jacobi Theorem, which is sometimes also called *condensation method.*

Theorem 5.3.1 (Desnanot-Jacobi). *Let n be a positive integer, A an $n \times n$ matrix and denote by $A^{i_1,\cdots,i_k}_{j_1,\cdots,j_k}$ the submatrix of A in which the $i_1,\cdots,i_k$-th rows and $j_1,\cdots,j_k$-th columns are omitted. Then holds*

$$\det A \det A^{1,n}_{1,n} = \det A^1_1 \det A^n_n - \det A^n_1 \det A^1_n.$$

Deleting the first or last row and the first or last column of the matrix $D_{n,k}(x,q)$

can be expressed as follows

$$\begin{aligned}
\det(D_{n,k}(x,q)_1^1) &= \det(D_{n-1,k}(x+2,q))\binom{x+n}{n-1}, \\
\det(D_{n,k}(x,q)_n^n) &= \det(D_{n-1,k}(x,q)), \\
\det(D_{n,k}(x,q)_{1,n}^{1,n}) &= \det(D_{n-2,k}(x+2,q))\binom{x+n-1}{n-2}, \\
\det(D_{n,k}(x,q)_n^1) &= \det(D_{n-1,k+1}(x+1,q)), \\
\det(D_{n,k}(x,q)_1^n) &= \det(D_{n-1,k-1}(x+1,q))\binom{x+n-1}{n-1}.
\end{aligned} \tag{5.5}$$

We will prove it in the case of $D_{n,k}(x,q)_1^1$, the other cases follow analogously. By definition we have

$$\begin{aligned}
\det(D_{n,k}(x,q)_1^1) &= \det_{2\le i,j\le n}\left(\binom{x+i+j-2}{j-1}\frac{1-(-q)^{j-i+k}}{1+q}\right) \\
&= \det_{1\le i,j\le n-1}\left(\binom{x+i+j}{j}\frac{1-(-q)^{j-i+k}}{1+q}\right).
\end{aligned}$$

Since the i-th row is divisible by $(x+i+1)$ we can pull it out for all $1\le i\le n-1$. Further we pull out j^{-1} from the j-th column for $1\le j\le n-1$ and obtain

$$\begin{aligned}
\det(D_{n,k}(x,q)_1^1) &= \prod_{i=1}^{n}\frac{(x+i+1)}{i}\det_{1\le i,j\le n-1}\left(\binom{x+i+j}{j-1}\frac{1-(-q)^{j-i+k}}{1+q}\right) \\
&= \binom{x+n}{n-1}\det(D_{n-1,k}(x+2,q)).
\end{aligned}$$

By applying (5.5) to Theorem 5.3.1 we obtain

$$\begin{aligned}
&(n-1)d_{n,k}(x,q)d_{n-2,k}(x+2,q) \\
&= (x+n)d_{n-1,k}(x,q)d_{n-1,k}(x+2,q)-(x+1)d_{n-1,k+1}(x+1,q)d_{n-1,k-1}(x+1,q).
\end{aligned} \tag{5.6}$$

We define for the rest of this section a_j for $j\in\mathbb{Z}$ as

$$a_j = \frac{1-(-q)^j}{1+q} = \begin{cases} \sum_{i=0}^{j-1}(-q)^i & j>0, \\ 0 & j=0, \\ -\sum_{i=1}^{-j}(-q)^{-i} & j<0. \end{cases}$$

The entries of the matrix $D_{n,k}(x,q)$ are polynomials in x and Laurent polynomials in q and hence the same is true for the determinant $d_{n,k}(x,q)$, i.e., $d_{n,k}(x,q)\in\mathbb{Q}[q,q^{-1},x]$.

Lemma 5.3.2. *Let n, k be positive integers, then the determinant $d_{n,k}(x,q)$ is as an element in $\mathbb{Q}[q,q^{-1},x]$ divisible by*

$$\prod_{l=0}^{\lfloor \frac{n-k-1}{2} \rfloor} (x+k+2l+1).$$

Proof. Let l be a non-negative integer. The (i,j)-th entry of $D_{n,k}(x,q)$ is divisible by $(x+k+2l+1)$ if $i \le k+2l+1$ and $i+j \ge k+2l+3$ or if $i = j+k$. Now choose l with $0 \le l \le \lfloor \frac{n-k-1}{2} \rfloor$ and set $x = -k-2l-1$. Then the (i,j)-th entry of $D_{n,k}(x,q)$ is equal to 0 for all $k+l+1 \le i \le k+2l+1 \le n$ and $j \ge l+1$. Therefore the $(k+l+1)$-st up to the $(k+2l+1)$-st row are linearly dependent and the determinant is henceforth equal to 0. $\square$

Lemma 5.3.3. *Let n, k be non-negative integers, then the following identity holds*

$$d_{n,-k}(x,q) = (-1)^{n(k-1)} q^{-nk} d_{n,k}(x,q).$$

Proof. Let $\sigma \in S_n$ be a permutation, then holds

$$\begin{aligned}\prod_{i=1}^{n} \binom{x+i+\sigma(i)-2}{\sigma(i)-1} &= \prod_{i=1}^{n} \frac{(x+i+\sigma(i)-2)!}{(\sigma(i)-1)!(x-1-i)!} \\ &= \prod_{i=1}^{n} \frac{(x+i+\sigma(i)-2)!}{(i-1)!(x-1-\sigma(i))!} = \prod_{i=1}^{n} \binom{x+\sigma(i)+i-2}{i-1}.\end{aligned}$$

This implies that we can replace the binomial coefficient $\binom{x+i+j-2}{j-1}$ in the determinant $d_{n,k}(x,q)$ with $\binom{x+i+j-2}{i-1}$ without changing the determinant. By using this fact and pulling out the factor $-(-q)^j$ from the j-th column and $(-q)^{-i-k}$ from the i-th row for all rows and columns, we obtain

$$\begin{aligned}d_{n,-k}(x,q) &= \det_{1\le i,j\le n}\left(\binom{x+i+j-2}{j-1} \frac{1-(-q)^{j-i-k}}{1+q} \right) \\ &= \prod_{i=1}^{n} (-q)^{-i-k} \prod_{j=1}^{n} (-1)(-q)^j \det_{1\le i,j\le n}\left(\binom{x+i+j-2}{i-1} \frac{-(-q)^{i+k-j}+1}{1+q} \right) \\ &= (-1)^{n(k-1)} q^{-nk} \det\left(D_{n,k}(x,q)^T\right) = (-1)^{n(k-1)} q^{-nk} d_{n,k}(x,q).\end{aligned}$$

$\square$

Corollary 5.3.4. *Let n be an odd positive integer, then holds $d_{n,0}(x,q) = 0$.*

With the above two lemmas at hand we can prove the following structural theorem.

Theorem 5.3.5. *The determinant $d_{n,k}(x,q)$ has the form*

$$d_{n,k}(x,q) = q^{c_q(n,k)} p_{n,k}(x) f_{n,k}(x,q), \tag{5.7}$$

with

$$p_{n,k}(x) = \prod_{i=1}^{n-1} \frac{\lfloor \frac{i}{2} \rfloor !}{i!} \prod_{i=0}^{\left\lfloor \frac{n-|k|-1}{2} \right\rfloor} (x + |k| + 2i + 1),$$

$$c_q(n,k) = \begin{cases} 0 & k > 0, n \leq k, \\ n\,k & k < 0, n \leq -k, \\ -\sum_{i=1}^{n-k} \left\lfloor \frac{i}{2} \right\rfloor & \text{otherwise,} \end{cases}$$

and $f_{n,k}(x,q)$ *being a polynomial in* x *and* q *satisfying for positive* k *the recursions*

$$f_{n,-k}(x,q) = (-1)^{n(k+1)} f_{n,k}(x,q), \tag{5.8}$$

$$f_{2n,0}(x,q) f_{2n-2,0}(x+2,q) = -f_{2n-1,1}(x+1,q)^2, \tag{5.9}$$

$$f_{2n,0}(x,q) f_{2n,0}(x+2,q) = f_{2n,1}(x+1,q)^2, \tag{5.10}$$

$$\begin{aligned} f_{n,k}(x,q) = {} & \frac{1}{f_{n-2,k}(x+2,q) \left(\frac{n-1}{2}\right)^{[n \in 2\mathbb{N}+1]}} \\ & \times \Big((x+n)^{[n \in \mathbb{N} \setminus (2\mathbb{N}+k+1)]} \left(q(x+n+1)\right)^{[n \in 2\mathbb{N}+k+2]} f_{n-1,k}(x,q) f_{n-1,k}(x+2,q) \\ & - (x+1) f_{n-1,k-1}(x+1,q) f_{n-1,k+1}(x+1,q) \Big), \end{aligned} \tag{5.11}$$

where [statement] *is* 1 *if the statement is true and* 0 *otherwise.*

Proof. Since $d_{n,k}(x,q)$ is a polynomial in x and a Laurent polynomial in q we can write $d_{n,k}(x,q)$ as in (5.7) where $f_{n,k}(x,q)$ is a rational function in x and q. In Lemma 5.3.2 we proved that $\frac{d_{n,k}(x,q)}{p_{n,k}(x)}$ is a polynomial in x. As already stated, we write $d_{n,k}(x,q)$ as

$$d_{n,k}(x,q) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} \binom{x+i+\sigma(i)-2}{\sigma(i)-1} a_{k+\sigma(i)-i}, \tag{5.12}$$

with

$$a_j = \begin{cases} \sum_{i=0}^{j-1} (-q)^i & j > 0, \\ 0 & j = 0, \\ -\sum_{i=1}^{-j} (-q)^{-i} & j < 0. \end{cases}$$

Let $\sigma \in \mathfrak{S}_n$, the exponent of the smallest power of q that appears in the summand associated to σ in (5.12) is

$$\sum_{i: k+\sigma(i)-i<0} (k + \sigma(i) - i). \tag{5.13}$$

It is an easy proof for the reader to show that the minimum of (5.13) for all $\sigma \in \mathfrak{S}_n$ is exactly $c_q(n,k)$. Hence $q^{-c_q(n,k)} d_{n,k}(x,q)$ is a polynomial in q implying that $f_{n,k}(x,q)$

is a polynomial in x and q.

It remains to prove that $f_{n,k}$ satisfies the equations (5.8) – (5.11), where (5.8) is a direct consequence of Lemma 5.3.3. For the others we use induction on n. Setting $k=0$ and using (5.6) and the induction hypothesis immediately implies (5.9) and (5.10). Now let $k \geq 1$. Using the induction hypothesis, (5.6) becomes

$$d_{n,k}(x,q) q^{c_q(n-2,k)} \prod_{i=1}^{n-3} \frac{\lfloor \frac{i}{2} \rfloor !}{i!} \prod_{i=0}^{\lfloor \frac{n-k-3}{2} \rfloor} (x+k+2i+3) f_{n-2,k}(x+2,q)$$
$$= \prod_{i=1}^{n-2} \left(\frac{\lfloor \frac{i}{2} \rfloor !}{i!} \right)^2 \times \frac{1}{(n-1)} \Bigg((x+n) q^{2c_q(n-1,k)}$$
$$\times \prod_{i=0}^{\lfloor \frac{n-k-2}{2} \rfloor} (x+k+2i+1)(x+k+2i+3) f_{n-1,k}(x,q) f_{n-1,k}(x+2,q)$$
$$- (x+1) q^{c_q(n-1,k-1)+c_q(n-1,k+1)} \prod_{i=0}^{\lfloor \frac{n-k-1}{2} \rfloor} (x+k+2i+1)$$
$$\times \prod_{i=0}^{\lfloor \frac{n-k-3}{2} \rfloor} (x+k+2i+3) f_{n-1,k-1}(x+1,q) f_{n-1,k+1}(x+1,q) \Bigg).$$

After cancellation we obtain

$$d_{n,k}(x,q) f_{n-2,k}(x+2,q) = \frac{q^{c_q(n,k)} p_{n,k}(x)}{\left(\frac{(n-1)}{2}\right)^{[n \in 2\mathbb{N}+1]}}$$
$$\times \Big((x+n)^{[n \in \mathbb{N} \setminus (2\mathbb{N}+k+1)]} (x+n+1)^{[n \in 2\mathbb{N}+k+2]} q^{[n \in 2\mathbb{N}+k+2]} f_{n-1,k}(x,q) f_{n-1,k}(x+2,q)$$
$$- (x+1) f_{n-1,k-1}(x+1,q) f_{n-1,k+1}(x+1,q) \Big).$$

By replacing $d_{n,k}(x,q)$ by $q^{c_q(n,k)} p_{n,k}(x) f_{n,k}(x,q)$, this implies the last recursion (5.11). □

Computer experiments suggest that $f_{n,k}(x,q)$ is a polynomial with integer coefficients and that the leading coefficient is either 1 or -1, where we order the monomials $x^a q^b$ with respect to the lexicographic order of (a,b). We can not prove this, but we can prove the following related statement.

Proposition 5.3.6. *The leading coefficient of $f_{n,1}(x,q)$ is 1, where we order the monomials $x^a q^b$ with respect to the reverse lexicographic order of (a,b).*

Proof. In order to prove this, we first calculate the leading coefficient of the highest power of q in $d_{n,1}(x,q)$ which is a polynomial in x and then calculate the leading coefficient of this polynomial. We will need the determinantal evaluations

$$\det_{1\le i,j\le n}\left(\binom{x+i+j-2}{a+j-1}\right)=\prod_{j=1}^{n}\frac{(j-1)!}{(a+j-1)!}\prod_{i=1}^{a}(x+j-i),$$

$$\det_{1\le i,j\le n}\left((-1)^{i+j}\binom{x+i+j-2}{a+j-1}\right)=\prod_{j=1}^{n}\frac{(j-1)!}{(a+j-1)!}\prod_{i=1}^{a}(x+j-i), \tag{5.14}$$

which can be proven by pulling out common factors in the rows and columns and by using the Vandermonde determinant evaluation. As already stated in (5.12) the determinant $d_{n,1}(x,q)$ can be written as

$$d_{n,1}(x,q)=\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^{n}\binom{x+i+\sigma(i)-2}{\sigma(i)-1}a_{1+\sigma(i)-i}. \tag{5.15}$$

Let $\sigma\in\mathfrak{S}_n$ and $i_1,\dots,i_l$ be the rows such that $\sigma(i)-i<0$, then the following is the exponent of the highest q power that appears in the summand associated to σ in (5.15)

$$-l+\sum_{i:\sigma(i)-i>0}(\sigma(i)-i)=-l+\sum_{j=1}^{l}(i_j-\sigma(i_j))\le -l+l(n-l).$$

It is obvious that there exists a $\sigma\in\mathfrak{S}_n$ such that the above inequality is sharp. The maximal exponent is reached for $l=\frac{n-1}{2}$ if $n\equiv 1 \mod 2$ or in the two cases $l=\frac{n-2}{2}$ or $l=\frac{n}{2}$ for $n\equiv 0 \mod 2$. First, let $n\equiv 1 \mod 2$ and hence $l=\frac{n-1}{2}$. The maximal q power is reached for all $\sigma\in\mathfrak{S}_n$ with $i_j=n+1-j$ and $\sigma(i_j)\le l$ for all $1\le j\le l$. Hence the coefficient of the maximal q power in $d_{n,1}(x,q)$ can be written as

$$\det_{1\le i,j\le l}\left(\binom{x+n-l+i+j-2}{j-1}\right)\det_{1\le i,j\le n-l}\left((-1)^{i+j+l}\binom{x+i+j+l-2}{l+j-1}\right).$$

By (5.14) the leading coefficient of the above determinants as polynomials in x is equal to

$$\prod_{i=1}^{n-l}\frac{(i-1)!}{(l+i-1)!}.$$

By simple manipulations and using $l=\frac{n-1}{2}$ this transforms to

$$\prod_{i=1}^{n-1}\frac{\lfloor\frac{i}{2}\rfloor!}{i!},$$

which proves the claim for odd n.

Now let $n=2a$, then the maximal q power is obtained for $l=a$ or $l=a-1$. Analogously to the above case we can write the coefficient of the highest power of q

as

$$\det_{1\le i,j\le a}\left(\begin{cases}0 & i=1\wedge j=a,\\ \binom{x+a+i+j-2}{j-1} & \text{otherwise,}\end{cases}\right)\det_{1\le i,j\le a}\left((-1)^{i+j+a}\binom{x+i+j+a-2}{a+j-1}\right)$$
$$+\det_{1\le i,j\le a-1}\binom{x+a+i+j-1}{j-1}$$
$$\times\det_{1\le i,j\le a+1}\left(\begin{cases}0 & i=a+1\wedge j=1,\\ (-1)^{i+j+a-1}\binom{x+i+j+a-3}{a+j-2} & \text{otherwise,}\end{cases}\right).$$

We can rewrite this by using Laplace expansion as

$$\det_{1\le i,j\le a}\binom{x+a+i+j-2}{j-1}\det_{1\le i,j\le a}\left((-1)^{i+j+a}\binom{x+i+j+a-2}{a+j-1}\right)$$
$$+(-1)^a\binom{x+n-1}{a-1}\det_{1\le i,j\le a-1}\binom{x+a+i+j-1}{j-1}$$
$$\times\det_{1\le i,j\le a}\left((-1)^{i+j+a}\binom{x+i+j+a-2}{a+j-1}\right)$$
$$+\det_{1\le i,j\le a-1}\binom{x+a+i+j-1}{j-1}\det_{1\le i,j\le a+1}\left((-1)^{i+j+a-1}\binom{x+i+j+a-3}{a+j-2}\right)$$
$$-(-1)^a\binom{x+n-1}{a-1}\det_{1\le i,j\le a-1}\binom{x+a+i+j-1}{j-1}$$
$$\times\det_{1\le i,j\le a}\left((-1)^{i+j+a}\binom{x+i+j+a-2}{a+j-1}\right).$$

The second and fourth product cancel each other and the degree of x is by (5.14) in the first product a^2 and for the third product a^2-1. Hence the leading coefficient is

$$\prod_{i=1}^{a}\frac{(i-1)!}{(a+i-1)!}=\prod_{i=1}^{n-1}\frac{\lfloor\frac{i}{2}\rfloor!}{i!},$$

where the equality is an easy transformation. □

One could extend the above proof to the calculation of the leading coefficient of $f_{n,k}(x,q)$ for arbitrary positive k where we order the monomials $x^a q^b$ with respect to the reverse lexicographic order of (a,b). However the leading coefficient of $f_{n,k}(x,q)$ will not be equal to ±1 for $k\neq 1$.

It seems that the polynomial $f_{n,k}(x,q)$ is for general k irreducible over $\mathbb{Q}[x,q]$. While this appears to be impossible to prove, Theorem 5.4.1 implies that $p_{n,k}(x)$ is maximal in $f_{n,k}(x,q)$, i.e., there exists no non-trivial polynomial $p'(x)\in\mathbb{Q}[x]$ dividing $f_{n,k}(x,q)$. For small values of k on the other side we could prove a factorisation of $f_{n,k}(x,q)$ which was already suggested to exist in [37].

Proposition 5.3.7. *The function $f_{n,k}(x,q)$ has in the special case for $k = 0, 1$ the form*

$$f_{n,0}(x,q) = \begin{cases} 0 & n \equiv 1 \mod 2, \\ (-1)^{\frac{n}{2}} F_{\frac{n}{2}}(x+1,q)^2 & n \equiv 0 \mod 2, \end{cases}$$
$$f_{n,1}(x,q) = F_{\lfloor \frac{n+1}{2} \rfloor}(x,q) F_{\lfloor \frac{n}{2} \rfloor}(x+2,q),$$

where $F_n(x,q)$ is a polynomial in x and q over $\mathbb{Q}$.

Proof. Corollary 5.3.4 and the equations (5.9) and (5.10) imply the statement. □

For $k = 2$ computer experiments still suggest a decomposition of $f_{n,k}(x,q)$ into two or three factors, depending on the parity of n, whereas for $k \geq 3$ the polynomial $f_{n,k}(x,q)$ seems to be irreducible over $\mathbb{Q}[x,q]$. In order to prove the factorisation for $k = 2$ one would need to show that the rational function

$$\frac{q(x+2n+1)(x+2n+2)F_n(x,q)F_n(x+4,q) - nF_{n+1}(x,q)F_{n-1}(x+4,q)}{F_n(x+2,q)},$$

is a polynomial in x and q. Further if one could guess the resulting polynomial, one would obtain a recursion for $F_n(x,q)$.

The following corollary is a direct consequence of Theorem 5.3.5 and the above proposition.

Corollary 5.3.8. *Set $Q = (q^{-1} + 2 + q)$ and define $\tilde{p}_n(q)$ as the Laurent polynomial*

$$\tilde{p}_{2n}(q) := 2q^{-\binom{n}{2}} \prod_{i=1}^{n-1} \frac{1}{(i+1)_i} F_n(0,q),$$
$$\tilde{p}_{2n+1}(q) := \frac{1}{2} q^{-\binom{n}{2}} \prod_{i=1}^{n} \frac{1}{(i)_i} F_n(2,q),$$

where $(x)_j := x(x+1)\cdots(x+j-1)$ is the Pochhammer symbol. Then the Q-enumeration of ASMs $A_n(Q)$ is given by

$$A_{2n}(Q) = 2\tilde{p}_{2n}(q)\tilde{p}_{2n+1}(q),$$
$$A_{2n+1}(Q) = \tilde{p}_{2n+1}(q)\tilde{p}_{2n+2}(q).$$

It is an easy proof to the reader that the Laurent polynomials $\tilde{p}_n(q)$ are actually polynomials in Q. The above corollary was actually conjectured in [60, Conjecture 4] and was first proven in [51].

5.4 The 0-,1-,2-,3- and 4-enumeration of ASMs

In the following we prove factorisations of the determinant $d_{n,k}(x,q)$ where q is a primitive first, second, third, fourth or sixth root of unity. As a consequence of these factorisations we obtain the known formulas for the 1-,2- and 3-enumeration of ASMs. The following table shows the connection between the specialisation of q and the weighted enumeration of ASMs.

0-enumeration:	$q=-1$	(primitive second root of unity),
1-enumeration:	$q=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$	(primitive third root of unity),
2-enumeration:	$q=\pm i$	(primitive fourth root of unity),
3-enumeration:	$q=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$	(primitive sixth root of unity),
4-enumeration:	$q=1$	(primitive first root of unity).

The factorisations in the following theorems can be proven (except for $q=-1$) by induction on n together with (5.6) and Corollary 5.3.4 and cancellation of terms. The case $q=-1$ however is proven solely by using row manipulations. We remind the reader of the definition of the Pochhammer symbol $(x)_j := x(x+1)\cdots(x+j-1)$.

Theorem 5.4.1. *For $q=-1$ holds*

$$d_{n,1}(x,-1) = \left(2\left\lfloor\frac{n+1}{2}\right\rfloor - 1\right)!!\prod_{i=1}^{\left\lfloor\frac{n}{2}\right\rfloor}(x+2i).$$

Proof. The limit of $d_{n,1}(x,q)$ for q to -1 is

$$\lim_{q\to -1} d_{n,k}(x,q) = \left(\binom{x+i+j-2}{j-1}(j-i+1)\right)_{1\le i,j\le n}.$$

The following identity of matrices can be shown by using a variant of the Chu–Vandermonde identity.

$$\begin{aligned}\left((-1)^{i+j}\binom{i-1}{j-1}\right)_{1\le i,j\le n} &\times \left(\binom{x+i+j-2}{j-1}(j-i+1)\right)_{1\le i,j\le n}\\ &= \left(\delta_{i,j}-\delta_{i,j+1}\left(\frac{j}{x+j}\right)^{[j\equiv 0(2)]}\right)_{1\le i,j\le n}\\ &\qquad\times\left(\binom{x+j-1}{j-i} j^{[i\equiv 1(2)]}\left((x+j)\right)^{[i\equiv 0(2)]}\right)_{1\le i,j\le n}.\end{aligned}$$

The closed product formula of $d_{n,1}(x,-1)$ is implied by taking the determinant on both sides. □

Theorem 5.4.2. *Let q be a primitive third root of unity, then holds*

$$
\begin{aligned}
d_{n,6k+1}(x,q) =& 2^{\lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor} \prod_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(i-1)!}{(n-i)!} \prod_{i \geq 0} \left(\frac{x}{2} + 3i + 1 \right)_{\lfloor \frac{n-4i}{2} \rfloor} \\
& \times \prod_{i \geq 0} \left(\frac{x}{2} + 3i + 3 \right)_{\lfloor \frac{n-4i-3}{2} \rfloor} \\
& \times \prod_{i \geq 0} \left(\frac{x}{2} + n - i + \frac{1}{2} \right)_{\lfloor \frac{n-4i-1}{2} \rfloor} \left(\frac{x}{2} + n - i - \frac{1}{2} \right)_{\lfloor \frac{n-4i-2}{2} \rfloor}, \\
d_{n,6k+2}(x,q) =& 2^{\lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor} 3^{-\lfloor \frac{n}{2} \rfloor} (1-q)^n \prod_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(i-1)!}{(n-i)!} \\
& \times \prod_{i \geq 0} \left(\frac{x}{2} + n - i \right)_{\lfloor \frac{n-4i}{2} \rfloor} \left(\frac{x}{2} + n - i \right)_{\lfloor \frac{n-4i-3}{2} \rfloor} \\
& \times \prod_{i \geq 0} \left(\frac{x}{2} + 3i + \frac{3}{2} \right)_{\lfloor \frac{n-4i-1}{2} \rfloor} \left(\frac{x}{2} + 3i + \frac{5}{2} \right)_{\lfloor \frac{n-4i-2}{2} \rfloor}, \\
d_{n,6k+3}(x,q) =& (-q)^n 2^{\lfloor \frac{n+1}{2} \rfloor \lfloor \frac{n+2}{2} \rfloor - \lfloor \frac{n}{2} \rfloor} \prod_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(i-1)!}{(n-i)!} \\
& \times \prod_{i \geq 0} \left(\frac{x}{2} + 3i + 2 \right)_{\lfloor \frac{n-4i-1}{2} \rfloor} \left(\frac{x}{2} + 3i + 2 \right)_{\lfloor \frac{n-4i-2}{2} \rfloor} \\
& \times \prod_{i \geq 0} \left(\frac{x}{2} + 2 \left\lfloor \frac{n}{2} \right\rfloor - i + \frac{1}{2} \right)_{\lfloor \frac{n-4i}{2} \rfloor} \\
& \times \prod_{i \geq 0} \left(\frac{x}{2} + 2 \left\lfloor \frac{n-1}{2} \right\rfloor - i + \frac{3}{2} \right)_{\lfloor \frac{n-4i-3}{2} \rfloor}, \\
d_{n,6k+4}(x,q) =& q^{-\frac{n}{2}} \, d_{n,6k+2}(x,q), \\
d_{n,6k+5}(x,q) =& q^{-n} \, d_{n,6k+1}(x,q), \\
d_{2n+1,6k}(x,q) =& 0, \\
d_{2n,6k}(x,q) =& q^{\frac{n}{2}} 2^{n^2} \prod_{i=1}^{n} \frac{(i-1)!}{(2n-i)!} \prod_{i \geq 0} \left(\frac{x}{2} + 2n - i \right)_{n-2i-1}^2 \\
& \times \prod_{i \geq 0} \left(\frac{x}{2} + 3i + \frac{1}{2} \right)_{n-2i} \left(\frac{x}{2} + 3i + \frac{7}{2} \right)_{n-2(i+1)}.
\end{aligned}
$$

Corollary 5.4.3. *The number A_n of ASMs of size n is given by*

$$
A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.
$$

Proof. Reorder the terms of the product formula of $d_{n,1}(0,q)$, where q is a primitive third root of unity. $\square$

Theorem 5.4.4. *Let q be a primitive fourth root of unity. Then the following holds*

$$\begin{aligned}
d_{2n+1,4k}(x,q) &= 0,\\
d_{2n,4k}(x,q) &= 2^n \prod_{i=1}^{2n-1} \frac{4^{\lfloor \frac{i}{2} \rfloor} \lfloor \frac{i}{2} \rfloor !}{i!} q^n \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{x}{2} + 2i + \frac{1}{2} \right)_{2n-4i+1}\\
&\quad \times \prod_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\frac{x}{2} + 2i + \frac{1}{2} \right)_{2n-4i-1},\\
d_{n,4k+1}(x,q) &= 2^{\lfloor \frac{n}{2} \rfloor} \prod_{i=1}^{n-1} \frac{4^{\lfloor \frac{i}{2} \rfloor} \lfloor \frac{i}{2} \rfloor !}{i!} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{x}{2} + i \right)_{n-2i+1},\\
d_{n,4k+2}(x,q) &= \prod_{i=1}^{n-1} \frac{4^{\lfloor \frac{i}{2} \rfloor} \lfloor \frac{i}{2} \rfloor !}{i!} (2q^{-1})^{\frac{n}{2}} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{x}{2} + i + \frac{1}{2} \right)_i \prod_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(\frac{x}{2} + i + \frac{1}{2} \right)_i,\\
d_{n,4k+3}(x,q) &= (-q)^n d_{n,4k+1}(x,q).
\end{aligned}$$

Corollary 5.4.5. *The 2-enumeration $A_n(2)$ of ASMs of size n is given by*

$$A_n(2) = 2^{\binom{n}{2}}.$$

Proof. Reorder the terms of the product formula of $d_{n,1}(0,q)$, where q is a primitive fourth root of unity. $\square$

Theorem 5.4.6. *Let q be a primitive sixth root of unity and k an integer. Then the following holds*

$$\begin{aligned}
d_{2n+1,3k}(x,q) &= 0,\\
d_{2n,3k}(x,q) &= q^{2n} c(2n) \prod_{i=0}^{n-1} (x+1+3i) \prod_{i=1}^{n-1} (x+3i)_{2(n-i)},\\
d_{n,3k+1}(x,q) &= c(n) \prod_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} (x+2+3i)_{n-1-2i},\\
d_{n,3k+2}(x,q) &= q^{-n} d_{n,3k+1}(x,q),
\end{aligned}$$

with

$$c(n) = \begin{cases} 3^{\frac{(n-2)n}{4}} \prod_{i=0}^{n-1} \frac{\lfloor \frac{i}{2} \rfloor !}{i!} & n \text{ is even},\\ 3^{\frac{(n-1)^2}{4}} \prod_{i=0}^{n-1} \frac{\lfloor \frac{i}{2} \rfloor !}{i!} & \text{otherwise.} \end{cases}$$

Corollary 5.4.7. *The* 3*-enumeration* $A_n(3)$ *of ASMs of size* n *is given by*

$$\begin{aligned} A_{2n+1}(3) &= 3^{n(n+1)} \prod_{i=1}^{n} \frac{(3i-1)!^2}{(n+i)!^2}, \\ A_{2n+2}(3) &= 3^{n(n+2)} \frac{n!}{(3n+2)!} \prod_{i=1}^{n+1} \frac{(3i-1)!^2}{(n+i)!^2}. \end{aligned}$$

Proof. Reorder the terms of the product formula of $d_{n,1}(0,q)$, where q is a primitive sixth root of unity. □

Theorem 5.4.8. *For* $q=1$ *holds*

$$\begin{aligned} d_{n,2k+1}(x,1) &= \prod_{i=2}^{\lfloor \frac{n}{2} \rfloor} (2i-1)^{-(n+1-2i)} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (x+2i) p_n(x) p_{n-1}(x), \\ d_{2n,2k}(x,1) &= (-1)^n \prod_{i=2}^{n} (2i-1)^{-(2n+1-2i)} \prod_{i=1}^{n} (x+2i-1) p_{2n}(x-1)^2, \\ d_{2n+1,2k}(x,1) &= 0, \end{aligned}$$

where $p_n(x)$ *is a polynomial in* x *satisfying the following recursion*

$$\begin{aligned} p_1(x) &= 1, \\ p_3(x) &= 2x+5, \\ p_{2n}(x) &= p_{2n-1}(x+2), \\ p_{2n+1}(x) &= ((x+2n+1)(x+2n+2) p_{2n-1}(x) p_{2n-1}(x+4) \\ &\quad -(x+1)(x+2) p_{2n-1}(x+2)^2) \left(2n p_{2n-3}(x+4)\right)^{-1}. \end{aligned}$$

Computer experiments suggest that the polynomials $p_n(x)$ are irreducible over $\mathbb{Q}$.

5.5 Connections to the Andrews determinant

In [37] Fischer conjectured that $d_{n,1}(x,q)$ is connected to the determinant

$$\det_{1 \le i,j \le n} \left(\binom{x+i+j-2}{j-1} + q\delta_{i,j} \right). \tag{5.16}$$

It was shown by Andrews [6] that for $q=1$ the above determinant has a closed product formula and that it counts for $x=2$ the number of descending plane partitions (DPPs) with parts less than n. In [23], it was shown by Ciucu et al. that the above determinant is equal to the weighted enumeration of cyclically symmetric lozenge tilings of a hexagon with side lengths $n, n+x, n, n+x, n, n+x$ and a central hole of side length x; these objects generalise DPPs. Further they proved that the evaluation of the determinant can be expressed by a closed product formula if q is a sixth root of unity. Comparing the factorisations of $d_{n,k}(x,q)$ and of the above determinant implies

$$d_{n,3-k}(x,q^2) = q^{-n} \det_{1\le i,j\le n}\left(\binom{x+i+j-2}{j-1} + q^k\delta_{i,j}\right),$$

where q is a primitive sixth root of unity. This implies that all factorisations of the determinant in (5.16) are covered by the determinant $d_{n,k}(x,q)$, where q is a third root of unity.

Very recently it was shown in [36] by Fischer that for $q=1$ the determinant in (5.16) also enumerates $(n,x-1)$-AS-trapezoids, which generalise alternating sign triangles (ASTs). This implies that not only DPPs and ASTs (and hence ASMs) are equinumerous but also a "one parameter generalisation" of DPPs and ASTs. An interpretation of x in $d_{n,k}(x,q)$, e.g. in form of a one parameter generalisation of ASMs, would be very interesting and could give important insight on the nature of the equinumerousity between ASMs, DPPs and ASTs. Such possible interpretations of x will be studied in a forthcoming work.

5.6 Enumeration formulas connected to $d_{n,k}(x,q)$

It turns out that some specialisations of x,q,k in $d_{n,k}(x,q)$ are known enumeration formulas. In the following we list those specialisations we found. In particular there is always a combinatorial interpretation for $d_{n,k}(0,q)$ where k is an integer and q is a sixth root of unity but not equals to -1. All formulas can be proven by using induction on n. We notate by ζ_l the l-th root of unity $\zeta_l = e^{\frac{2\pi i}{l}}$ and further use the notation $A^{(1)}_{QT}(4n,x)$ of [52].

$$\zeta_6^{-n} d_{2n,0}(0,\zeta_3) = \left(A^{(1)}_{QT}(4n,1)\right)^2 = (\#(\text{ASMs of size } n))^4, \tag{5.17}$$

$$\zeta_4^{-n} d_{2n,0}(0,\zeta_4) = \left(A^{(1)}_{QT}(4n,2)\right)^2, \tag{5.18}$$

$$\zeta_3^{-n} d_{2n,0}(0,\zeta_6) = \left(A^{(1)}_{QT}(4n,3)\right)^2 = 3^{n(n-1)}\,(\#(\text{ASMs of size } n))^2, \tag{5.19}$$

$$d_{n,1}(0,-1) = (\text{0-enumeration of ASMs of order } n) = n!, \tag{5.20}$$

$$d_{n,1}(0,\zeta_3) = \zeta_6^{n-1} d_{n-1,3}(2,\zeta_3) = \#(\text{ASMs of size } n), \tag{5.21}$$

$$\begin{aligned} d_{n,1}(0,\zeta_4) &= (\text{2-enumeration of ASMs of order } n)\\ &= \#(\text{perfect matchings of an order } n \text{ Atzec diamond})\\ &= \#(\text{Gelfand-Tsetlin patterns with bottom row } 1,2,\dots,n) = 2^{\binom{n}{2}}, \end{aligned} \tag{5.22}$$

$$d_{n,1}(0,\zeta_6) = \#(\text{3-enumeration of ASMs of size } n), \tag{5.23}$$

$$\zeta_{12}^{n} d_{n,2}(0,\zeta_3) = \sqrt{3}^{[n\equiv 1 \mod 2]} \left(\#(\text{half turn symmetric ASMs of size } n)\right)^2, \quad (5.24)$$

$$\zeta_4^n (1-\zeta_4)^n d_{n,2}(0,\zeta_4) = \text{Total dimension of the homology of a free 2-step nilpotent Lie algebra of rank n}^1, \quad (5.25)$$

$$\begin{aligned}\zeta_6^{n-1} d_{n-1,3}(0,\zeta_3) &= \#(\text{cyclic symmetric plane partitions in an } n-\text{cube})\\ &= \#(\text{half turn symmetric ASMs of size } 2n)/\#(\text{ASMs of size } n),\end{aligned} \quad (5.26)$$

$$\begin{aligned}\zeta_6^{n-1} d_{2(n-1),2}(1,\zeta_3) &= \frac{\zeta_6^{n+1}}{\sqrt{-3}} d_{2n-1,2}(-1,\zeta_3)\\ &= \#(\text{ASMs with two U-turn sides of size } 4n),\end{aligned} \quad (5.27)$$

$$\frac{\zeta_6^{n+1}}{\sqrt{-3}} d_{2n-1,2}(-2,\zeta_3) = \#(\text{quarter turn symmetric ASMs of size } 4n), \quad (5.28)$$

$$\zeta_4^n d_{2n,2}(1,\zeta_4) = \frac{\zeta_8^{2n+1}}{\sqrt{2}} d_{2n+1,2}(-1,\zeta_4) = 4^{n^2}, \quad (5.29)$$

$$\frac{\zeta_8^{2n-1}}{\sqrt{2}} d_{2n-1,2}(1,\zeta_4) = 4^{n(n+1)}. \quad (5.30)$$

[1]See [44, Theorem 1.1]. This sequence further coincides with tilings of a half-hexagon of side length n with glued sides, see [27, Eq. (3.5)].

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