BACHELORARBEIT

Titel der Bachelorarbeit

The Kakeya Problem

Verfasser

Florian Fürnsinn

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Betreuer:	ao. Univ.-Prof. Mag. Dr. Leonhard Summerer

Abstract

This thesis provides a historical overview as well as the solution to the Kakeya needle problem, a geometric problem that asks for the minimal area of a figure in which a needle can be rotated continuously. It is proved using so called Besikovitch sets, that this can be done within an arbitrarily small area. Moreover this thesis introduces the basic notions of fractal geometry, such as the Minkowski and Hausdorff dimension. It also contains a proof of the Kakeya conjecture in two dimensions, which states that the dimension of a n-dimensional Besikovich set is equal to n. Finally it briefly discusses the progress in higher dimensions, where the conjecture still remains unsolved.

Contents

1 Introduction

Take an (idealized) needle of length one and put it, with the pointy end facing downwards, inside a box. Now you want to rotate the needle inside the box, without lifting it into the air, such that its pointy end is showing to the top afterwards. How big must the box be? This, rather easy looking problem, the Kakeya problem, has kept mathematicians occupied for quite some time at the beginning of the twentieth century. It lead to new open problems and provided answers to different problems. Let us have a look at it.

The first possible shape of the box, that comes to mind, is a circle with radius $\frac{1}{2}$. Obviously you can rotate the needle inside of it, but can we do better? Indeed we can, e.g. take an equilateral triangle with height one. It has smaller area than the circle and figure 1 shows how to rotate the needle within. In fact we will see in the next few chapters, that we can even do much better than that.

In section 2 we will briefly discuss the history of the Kakeya problem. The next two sections, 3 and 4, are reserved for mathematical "machinery" needed to deal with it, especially from the field of measure theory. We will develop the concept of the Minkowsky dimension and the Hausdorff dimension. In section 5 we will eventually construct so called Kakeya or Besikovitch sets and briefly discuss their application. At last we will have a look at the so called Kakeya conjecture, which is about the dimension of these sets. The two-dimensional version of this conjecture is solved, and we will discuss a proof. In higher dimensions the question is still open, there we will briefly discuss progress towards a proof.

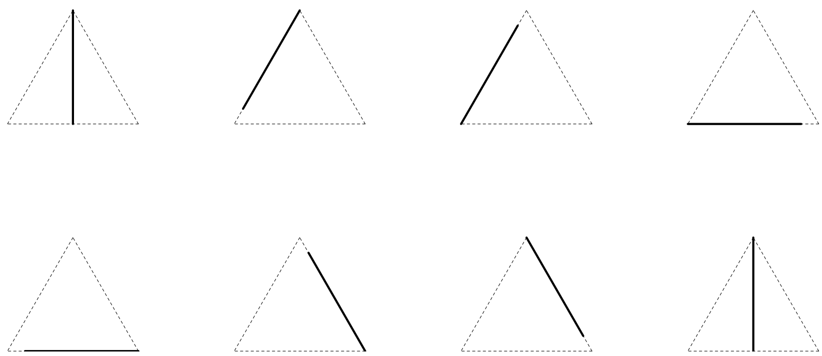

Figure 1: Rotating a needle inside an equilateral triangle

2 History of the Problem

In 1917 Sōichi Kakeya proposed the problem to find the minimal area of a convex set in which a one dimensional needle can be rotated continuously [Kak17]. This question was first answered by Julius Pál in 1921 by showing that the set is an equilateral triangle with height 1, hence the area is $\frac{1}{\sqrt{3}}$ [Pál21]. Naturally the question arose, what the answer is, if the convexity condition was dropped. At first it was believed, that in this case the figure of minimal area is the inside of a deltoid curve of area $\frac{\pi}{8}$, but in 1928 Abram S. Besicovitch showed, that the area can be made arbitrarily small. In 1917 he had been working on some problems concerning Riemann integrals and constructed a set of measure 0, which contained a rotated copy of the unit interval in every direction. His work had been published in some Russian journal and had remained unrecognised due to the political instability of Russia at that time. He republished his work and Pál noticed that his construction also solved Kakeya's quenstion [Bes28].

A further question was to understand the size of a so called Besicovitch set, i.e. a subset of $\mathbb{R}^n$ containing a unit line segment in every direction, even in more detail. In 1970 Roy O. Davies proved, that a Besicovitch set in the plain has to have Hausdorff Dimension 2 [Dav71]. The Kakeya conjecture states, that the generalisation of this to higher dimensions holds true, i.e. that an n-dimensional Besikovitch set has Hausdorff dimension n. Until today there has been a lot of work on this, but so far no proof has been given.

Besicovitch sets are sometimes used to provide counterexamples in analysis, especially in harmonic analysis. Starting with the initial problem on Riemann integrals, there are some applications of those sets. A remarkable result is the disprove of the Ball multiplier conjecture by Charles Fefferman, published in 1971 [Fef71].

3 Mathematical Prerequisites

3.1 Measure Theory

In the following sections we will need basic notions and results from measure theory. Here a short overview over the definitions and theorems needed later will be given. For further details as well as proofs to the theorems provided in the section below, see Roland Zweimüllers lecture notes [Zwe19] and Gerald Teschl's book on real analysis [Tes20].

First we need to define the system of sets, on which a measure can be defined.

Definition 3.1. Let X be a set. A *σ-algebra* is a subset $\Sigma \subseteq \mathcal{P}(X)$ of the power set of X, with the following properties:

i) $X \in \Sigma$.

ii) If $A \in \Sigma$, then $A^c \in \Sigma$, where A^c denotes the complement of A in X.

iii) If $A_n \in \Sigma$ for $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} A_n \in \Sigma$.

A subset $H \subseteq \mathcal{P}(X)$ is called a *semi-ring*, if

i) $\emptyset \in H$.

ii) If $A, B \in H$, then $A \cap B \in H$.

iii) If $A, B \in H$, then there are $C_1, \dots, C_n$, such that $A \setminus B = \bigcup_{k=1}^n C_n$.

One can easily prove that the intersection of σ-algebras forms itself a σ-algebra. This motivates the following definition:

Definition 3.2. Let $\mathcal{E} \subseteq \mathcal{P}(X)$. Then we call

$$\Sigma(\mathcal{E}) = \bigcap_{\mathcal{E}\subseteq\Sigma\subseteq\mathcal{P}(X)} \Sigma,$$

the *σ-algebra generated by $\mathcal{E}$*, where the intersection is over all σ-algebras over X, which contain $\mathcal{E}$.

Let X be a topological space with topology $\mathcal{O}$. Then $\Sigma(\mathcal{O})$ is called the *Borel σ-algebra* of X and its elements are called *Borel sets*.

Let us now define measures and related concepts.

Definition 3.3. A *pre-measure* $\mu : H \to [0, \infty]$ is a mapping from a semi ring H into the extended non-negative reals, that fulfils the following properties:

i) $\mu(\emptyset) = 0$

ii) For disjoint sets $A_i \in H$, where $i \in \mathbb{N}$, such that $\bigcup_{i\in\mathbb{N}} A_i \in H$, μ is σ-additive, i.e.

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i=0}^{\infty} \mu(A_i).$$

Further, a *measure* is a pre-measure defined on a σ-algebra and a *probability measure* P is a measure, for which $P(X) = 1$. A measure defined on a σ-algebra, that includes all Borel sets, is called *Borel measure*.

An *outer measure* is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$, such that:

i) $\mu^*(\emptyset) = 0$

ii) If $A \subseteq B \subseteq X$, then $\mu^*(A) \leq \mu^*(B)$.

iii) For $A_i \subseteq X$ for $i \in \mathbb{N}$, then

$$\mu^*\left(\bigcup_{i\in\mathbb{N}} A_I\right) \leq \sum_{i=0}^{\infty} \mu^*(A_i)$$

If X is a topological space, μ^* is called *metric* if for all $A, B \subseteq X$ with $\inf_{a\in A, b\in B} d(a,b) > 0$ we have that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Now we want to be able to extend pre-measures from a semi ring to measures. To this end we need to define the outer measure generated by a pre-measure and the concept of measurable sets, with respect to an outer measure.

Definition 3.4. Let μ be a pre-measure on a semi ring H. For $A \subseteq X$ let

$$\mu^*(A) = \inf\left\{\sum_{n=0}^{\infty} \mu(A_n) \middle| \{A_n\}_{n\in\mathbb{N}} \text{ is a countable cover of } A \text{ with sets from } H\right\}$$

be the *outer measure generated by* μ.

For an arbitrary outer measure μ^* let

$$\Sigma(\mu^*) = \{A \subseteq X | \mu^*(B) = \mu^*(A \cap B) + \mu^*(A^c \cap B) \quad \forall B \subseteq X\},$$

the set of μ^**-measurable sets.*

One needs to check, that these definitions indeed make sense.

Proposition 3.5. *For every pre-measure μ the generated outer measure μ^* is indeed an outer measure with $\mu^*|_H = \mu$. Moreover the set of μ^*-measurable sets form a σ-algebra for all outer measures μ^* and $\mu^*|_{\Sigma(\mu^*)}$ is a measure.*

Now we have everything to formulate two results, which we will need later on. The first one is the famous extension theorem of Carathéodory.

Theorem 3.6 (Carathéodory's extension theorem)**.** *Let $H \subseteq \mathcal{P}(X)$ be a semi ring, $\mu : H \to [0, \infty]$ a pre-measure and μ^* the outer measure generated by μ. Then $\Sigma(H) \subseteq \Sigma(\mu^*)$ and $\mu^*|_{\Sigma(\mu^*)}$ is an extension of μ to $\Sigma(\mu^*)$.*

The second one provides a characterisation of the outer measures μ^*, for which all Borel sets are μ^*-measurable.

Theorem 3.7. *The function μ^* is a metric outer measure if and only if all Borel sets are μ^*-measurable.*

With these theorems we can also introduce the n-dimensional Lebesgue measure $\mathcal{L}^n$. Consider the collection of semi-open cuboids $[a_1, b_1)\times, \ldots, \times[a_n, b_n) \subseteq \mathbb{R}^n$. They form a semi ring and we may define a pre-measure on them by letting $\mathcal{L}^n([a_1, b_1)\times, \ldots, \times[a_n, b_n)) = (b_1 - a_1) \cdot \ldots \cdot (b_n - a_n)$. By Carathéodory's extension theorem we may extend $\mathcal{L}^n$ to a measure on $\Sigma(\mathcal{L}^{n*})$. As $\mathcal{L}^{n*}$ is a metric outer measure, all Borel sets are measurable.

3.2 Graph theory

Later in the proof of Frostman's lemma (Lemma 6.3) we will need an important result from graph theory, the Max-Flow-Min-Cut theorem. It is usually stated in flow networks with a source and a sink, but it can be formulated for trees as well. The version we need is found in Bishop's and Peres' book [BP17], also as preparation for the proof of Frostman's lemma.

Definition 3.8. A *tree* is a graph in which each two vertices are connected by one path. A *rooted tree* is a tree with one special vertex, the *root*.

Let $\Gamma = (V, E)$ be a rooted tree. We denote by v_0 the root of Γ. For a vertex $v \in V$ let $|v|$ be the *depth of* v, i.e. the distance from v_0 to v and let v' be the unique neighbour of v of depth $|v| - 1$.

A *capacity function* on a Graph $\Gamma = (V, E)$ is a mapping $C : E \to [0, \infty)$, which assigns to each edge a non-negative real number. A *legal flow* on a rooted tree with capacity function C is a mapping $f : E \to [0, \infty)$, such that $f(e) \leq C(e)$ for all $e \in E$ and

$$f(v'v) = \sum_{\{w|(v,w)\in E, w\neq v'\}} f(vw).$$

The value of a flow $\|f\|$ is defined as

$$\|f\| = \sum_{|v|=1} f(v_0 v).$$

A cut set of a rooted tree is a subset $D \subseteq E$, such that D intersects every maximal path from the root.

With all these concepts we can formulate the Max-Flow-Min-Cut Theorem for rooted trees.

Theorem 3.9 (Max-Flow-Min-Cut Theorem)**.** *Let* Γ *be a rooted tree with capacity* C*. Then there is a maximal legal flow* f *in the sense that* $\|f\| \geq \|g\|$ *for all other legal flows* g*. Moreover*

$$\max_{f \text{ legal flow}} \|f\| = \inf_{D \text{ cut-set}} C(D) = \inf_{D \text{ cut-set}} \sum_{e\in D} C(e).$$

4 Fractal Geometry

For establishing the basics of fractal geometry we will follow Sascha Troscheit's lecture notes [Tro20] and Christopher J. Bishop's and Yuval Peres' book [BP17]. We will introduce two of the most commonly used concepts of dimensions, namely the Hausdorff and the Minkowski dimension.

The main idea is, that the Lebesgue measure is not always a suitable concept to characterize the size of a set. For example take the famous Koch snowflake curve, see figure 2. It is constructed by taking a equilateral triangle and in each step replacing the middle third of each line by two sides of an equilateral triangle. Its two dimensional Lebesgue measure is zero, however, it is a line of infinite length within a compact set. So intuitively it should neither be one nor two dimensional and we need a finer concept of dimension, which we will provide here. Note that most of dimension theory extends to metric spaces as well, however we will stick with $\mathbb{R}^n$ here.

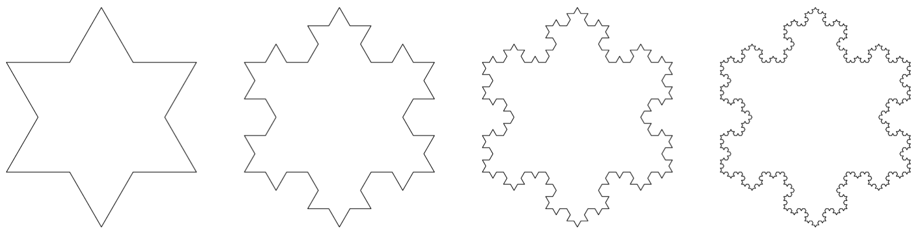

Figure 2: The first few iterations of the Koch Snowflake

4.1 Minkowski Dimension

The idea behind the Minkowski dimension or box-counting dimension is to count how many balls of equal radius r are needed to cover a set. We want this number to be proportional to r^{-d}, where d is our dimension. This motivates the following definition.

Definition 4.1. Let $X \subseteq \mathbb{R}^n$ be a bounded set and let $N_r(X)$ be the minimal number of balls of radius r needed to cover X. We then define the *lower* and *upper Minkowski dimension* respectively as follows:

$$\underline{\dim}_M(X) = \liminf_{r \searrow 0} \frac{\log N_r}{-\log r} \quad \text{and} \quad \overline{\dim}_M(X) = \limsup_{r \searrow 0} \frac{\log N_r}{-\log r}.$$

If both these quantities are equal to each other, we set

$$\dim_M(X) = \underline{\dim}_M(X) = \overline{\dim}_M(X)$$

and call them *Minkowski dimension* of X.

Actually there is a certain degree of freedom, how to define the quantity $N_r(X)$.

Proposition 4.2. *Replacing N_r in the definition for the Minkowski dimension by any of the following quantities, does not change $\dim_M(X)$:*

a) Smallest number of sets of diameter less than r needed to cover X.

b) Smallest number of axis parallel cubes needed to cover X.

c) Number of cubes of a grid with mesh size r that intersect X.

d) Largest number of disjoint balls of radius r with centres in X.

e) Largest number of points within X, such that they are r-separated, i.e. no two of them are closer than r to each other.

Proof. Let M_r be any of the quantities from the list. We want to show that there are constants A, B, such that $AM_r \leq N_r \leq BM_r$. Then we get

$$\begin{aligned}\liminf_{r\searrow 0} \frac{\log M_r}{-\log r} &= \liminf_{r\searrow 0} \frac{\log M_r + \log A}{-\log r} = \liminf_{r\searrow 0} \frac{\log AM_r}{-\log r} \leq \liminf_{r\searrow 0} \frac{\log N_r}{-\log r} \\ &\leq \liminf_{r\searrow 0} \frac{\log BM_r}{-\log r} = \liminf_{r\searrow 0} \frac{\log M_r}{-\log r}\end{aligned}$$

for the lower Minkowski dimension and an analogous result for the upper one.

We will only prove that a) is equivalent here, the rest follows in a similar way. Let M_r be the smallest number of sets of diameter less than r needed to cover X. Each of those sets can be enclosed in a ball of radius r, which then form a cover of X. Therefore $N_r \leq M_r$. On the other hand, the unit ball in dimension r is compact. Therefore it can be covered by finitely many balls of diameter 1. Assume this is possible with C such balls. Then each ball of radius r can be covered by C balls of diameter r, thus $M_r \leq CN_r$ and we are done. $\square$

Next we discuss the main properties of the Minkowski dimension, as well as its weaknesses.

Proposition 4.3. *Let $X, Y \subseteq \mathbb{R}^n$. The Minkowski dimension has the following properties:*

(1) If $X \subseteq Y$ then $\underline{\dim}_M(X) \leq \underline{\dim}_M(Y)$ as well as $\overline{\dim}_M(X) \leq \overline{\dim}_M(Y)$.

(2) $0 \leq \underline{\dim}_M(X) \leq \overline{\dim}_M(X) \leq n$.

(3) The upper Minkowski dimension is finitely stable, i.e.

$$\overline{\dim}_M(X \cup Y) = \max\left\{\overline{\dim}_M(X), \overline{\dim}_M(Y)\right\}.$$

(4) If X is open, then the Minkowski dimension of X exists and $\dim_M(X) = n$.

(5) If X is finite, then its Minkowski dimension exists and we have $\dim_M(X) = 0$.

(6) If $f : X \to \mathbb{R}^n$ is bi-Lipschitz, i.e. there is $c > 0$, such that

$$c^{-1}|x - y| \leq |f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}^n$, then

$$\underline{\dim}_M(f(X)) = \underline{\dim}_M(X) \quad \textit{and} \quad \overline{\dim}_M(f(X)) = \overline{\dim}_M(X).$$

Proof.

(1) Any cover with balls of radius smaller than r of Y also covers X. Thus $N_r(Y) \geq N_r(X)$ for all r.

(2) As for $r \leq 1$ the term $\frac{\log N_r}{-\log r}$ is always positive we get $0 \leq \underline{\dim}_M(X)$. Because the limes inferior is always smaller than the limes superior, the next inequality follows. Finally X has to be contained in some ball B_R of radius R, as it is bounded. It is easy to check, that there is a constant C, such that B_R can be covered by Cr^{-d} balls of radius r for all r. Thus

$$\overline{\dim}_M(X) = \limsup_{r\searrow 0} \frac{\log N_r}{-\log r} \leq \limsup_{r\searrow 0} \frac{\log Cr^{-d}}{-\log r} = d$$

(3) We may compute:

$$\overline{\dim}_M(X \cup Y) = \limsup_{r \searrow 0} \frac{\log N_r(X \cup Y)}{-\log r} \leq \limsup_{r \searrow 0} \frac{\log N_r(X) + N_r(Y)}{-\log r}$$
$$\leq \limsup_{r \searrow 0} \frac{\log\left(2\max\{N_r(X), N_r(Y)\}\right)}{-\log r} \leq \max\left\{\overline{\dim}_M(X), \overline{\dim}_M(Y)\right\}$$

(4) If X is open, then there is an open ball B_R contained in X. It is clear that the maximal number of r-separated points within B_R grows like r^{-n}, so there is C, such that $N_r(B_R) \geq Cr^{-n}$. Therefore

$$\underline{\dim}_M(X) \geq \underline{\dim}_M(B_R) \geq \liminf_{r \searrow 0} \frac{\log N_r(B_R)}{-\log r} \geq \liminf_{r \searrow 0} \frac{\log\left(Cr^{-n}\right)}{-\log r} = n$$

(5) Let N be the number of points in X. Then X can be covered by N balls of radius r for all r, thus the Minkowski dimension is 0.

(6) First we show, that if $|f(x)-f(y)| \leq c|x-y|$, it follows that $\underline{\dim}_M(f(X)) \leq \underline{\dim}_M(X)$ and $\overline{\dim}_M(f(X)) \leq \overline{\dim}_M(X)$. To this end we note that for every cover $\{U_i\}_{i \in I}$ of F with sets with diameter less than r, also $\{U_i \cap X\}_{i \in I}$ is such a cover. Therefore $\{f(U_i \cap X)\}_{i \in I}$ is a cover of $f(X)$ with sets of diameter smaller than cr. As already seen in the proof of Proposition 4.2 there is a constant A, such that every ball of diameter c can be covered of A balls of diameter r. Thus $N_r(f(X)) \leq AN_{cr}(f(X)) \leq AN_r(X)$ and as the constant has no effect in the definition of the Minkowski dimension we get the desired result.

Now note that $c^{-1}|x-y| \leq |f(x)-f(y)|$ ensures that f is injective. So we may let $F : f(X) \to X$ be its inverse mapping and note that $F(f(X)) = X$. We have $|F(x)-F(y)| \leq c|x-y|$ and the first part of the proof gives us $\underline{\dim}_M(f(X)) \geq \underline{\dim}_M(X)$ and $\overline{\dim}_M(f(X)) \geq \overline{\dim}_M(X)$.

□

A major drawback of the Minkowski dimension is that it is finitely stable, but not countably stable. In fact there are countable sets with positive Minkowski dimension, such as the set $\left\{\frac{1}{n} | n \in \mathbb{N}\right\} \cup \{0\}$, which has Minkowski dimension $\frac{1}{2}$. Thus there is a need to introduce another concept of dimensions which overcomes these problems, the Hausdorff dimension. Its definition, however, is a bit more involved.

4.2 Hausdorff Measure and Dimension

Definition 4.4. Let $X \subseteq \mathbb{R}^n$ and $\alpha > 0$. The *α-dimensional Hausdorff content* of X is defined as

$$\mathcal{H}^\alpha_\infty(X) = \inf\left\{\sum_{i \in I} |U_i|^\alpha \,\middle|\, X \subseteq \bigcup_{i \in I} U_i\right\},$$

where the infimum is taken of all families of open sets $\{U_i\}_{i \in I}$ covering X and $|S|$ denotes the diameter of a set S. We define the *Hausdorff dimension* of X as

$$\dim_H(X) = \inf\{\alpha | \mathcal{H}^\alpha_\infty(X) = 0\}.$$

Moreover the *α-dimensional ε-Hausdorff content* of the set X is defined as

$$\mathcal{H}^\alpha_\varepsilon(X) = \inf\left\{\sum_{i} |U_i|^\alpha \,\middle|\, X \subseteq \bigcup_{i} U_i, |U_i| < \varepsilon\right\},$$

so now the covering sets have diameter less than ε. Finally the *Hausdorff measure* of X is defined to be

$$\mathcal{H}^\alpha(X) = \lim_{\varepsilon \searrow 0} \mathcal{H}^\alpha_\varepsilon(X).$$

First we discuss the fundamental properties of the Hausdorff measure.

Proposition 4.5. *Let $\alpha > 0$. The Hausdorff measure $\mathcal{H}^\alpha : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ is a metric outer measure. Therefore it is a Borel measure.*

Proof. It is obvious that $\mathcal{H}^\alpha(\emptyset) = 0$. Let $X \subseteq Y \subseteq \mathbb{R}^n$. Every cover of Y is a cover of X as well and therefore $\mathcal{H}^\alpha(X) \leq \mathcal{H}^\alpha(Y)$. Now let $(X_i)_{i \in \mathbb{N}}$ be a sequence of subsets of $\mathbb{R}^n$. Then

$$\mathcal{H}^\alpha\left(\bigcup_{i\in\mathbb{N}} X_i\right) \leq \sum_{i\in\mathbb{N}} \mathcal{H}^\alpha\left(X_i\right),$$

because a union of covers for each X_i forms a cover for the union of the sets X_i. Finally let $X, Y \subseteq \mathbb{R}^n$, such that $D = \inf_{x\in X, y\in Y} |x-y| > 0$. For all ε with $\varepsilon < \frac{D}{2}$ it holds that $\mathcal{H}^\alpha_\varepsilon(X \cup Y) = \mathcal{H}^\alpha_\varepsilon(X) + \mathcal{H}^\alpha_\varepsilon(Y)$, as no set of diameter less than ε can intersect both X and Y. Now letting $\varepsilon \searrow 0$ gives $\mathcal{H}^\alpha(X \cup Y) = \mathcal{H}^\alpha(X) + \mathcal{H}^\alpha(Y)$. Using theorem 3.7 we get that every Borel set is measurable, i.e. $\mathcal{H}^\alpha$ is a Borel measure. □

One can show, that if $\alpha \in \mathbb{N}$, then the α-dimensional Hausdorff measure is a constant multiple of the α-dimensional Lebesgue measure. For a proof see Theorem 1.12 in Falconers book [Fal85].

We will now show, that for a fixed set the α-dimensional Hausdorff measure is infinite for all α, up to one critical value, the Hausdorff dimension, and then drops to 0 for all larger α. At the Hausdorff dimension itself the Hausdorff measure of our set can be anything from 0 to ∞. Moreover it is irrelevant whether we use the Hausdorff content or the Hausdorff measure to define the Hausdorff dimension.

Proposition 4.6. *Let $X \subset \mathbb{R}^n$ and let $\alpha < \beta$. Then the following statements are true.*

(1) If $\mathcal{H}^\alpha(X) < \infty$ then $\mathcal{H}^\beta(X) = 0$.

(2) The α-Hausdorff content of x vanishes if and only if the α-Hausdorff measure vanishes, i.e.

$$\mathcal{H}^\alpha_\infty(X) = 0 \quad \Leftrightarrow \quad \mathcal{H}^\alpha(X) = 0.$$

(3) For the Hausdorff dimension we have

$$\dim(X) = \inf\{\alpha | \mathcal{H}^\alpha(X) = 0\} = \sup\{\alpha | \mathcal{H}^\alpha(X) = \infty\} = \sup\{\alpha | \mathcal{H}^\alpha_\infty(X) = \infty\}.$$

Proof.

(1) We have $\mathcal{H}^\beta_\varepsilon(X) \leq \varepsilon^{\beta-\alpha}\mathcal{H}^\alpha_\varepsilon(X)$, if we use the same cover and use the estimate $|U_i|^\beta \leq |U_i|^\alpha \varepsilon^{\beta-\alpha}$ for a set U_i. Now letting $\varepsilon \searrow 0$ yields the desired result.

(2) As $\mathcal{H}^\alpha_\infty(X) \leq \mathcal{H}^\alpha(X)$ the first implication is trivial. Now assume conversely that $\mathcal{H}^\alpha_\infty(X) = 0$. For every ε there is a cover $\{U_i\}_{i\in I}$ of X, such that $\sum_{i\in I} |U_i|^\alpha \leq \varepsilon$. But this implies that $|U_i|^\alpha \leq \varepsilon$ for all i, so $|U_i| \leq \varepsilon^{\frac{1}{\alpha}}$. Using this coverings we get $\mathcal{H}^\alpha_{\varepsilon^{\frac{1}{\alpha}}}(X) \leq \varepsilon$. Now letting$\varepsilon \searrow 0$ yields $\mathcal{H}^\alpha(X) = 0$.

(3) This is an immediate consequence of (1) and (2).

□

Now we compare the Hausdorff dimension to the Minkowski dimension.

Proposition 4.7. *Let $X \subseteq \mathbb{R}^n$ be bounded. Then* $\dim_H(X) \leq \underline{\dim}_M(X)$.

Proof. Let $\alpha = \underline{\dim}_M(X)$ and $\varepsilon > 0$. By the definition of the lower Minkowski dimension, there exists a sequence of radii, $r_i \searrow 0$, such that $\frac{\log N_{r_i}(X)}{-\log r_i} \leq \alpha + \varepsilon$. Hence $N_{r_i}(X) \leq r_i^{-(\alpha+\varepsilon)}$. So there exists a cover of X with sets of radius smaller than r_i containing at most $r_i^{-(\alpha+\varepsilon)}$ sets. This gives an upper bound for the $(\alpha + 2\varepsilon)$)-Hausdorff content:

$$\mathcal{H}_\infty^{\alpha+2\varepsilon}(X) \leq N_{r_i} r_i^{\alpha+2\varepsilon} \leq r_i^{\varepsilon}.$$

As the r_i tend to zero, $\mathcal{H}_\infty^{\alpha+2\varepsilon}(X) = 0$ and as this is true for all ε we get $\dim_H(X) \leq \alpha$. $\square$

So we have an upper bound by the Minkowski dimension. Note that is often easy to find an upper bound for the Hausdorff dimension by finding a suitable cover. To find a lower bound for the Hausdorff dimension, which is normally harder to achieve, the following fundamental Lemma is usefull. It connects dimension theory to measure theory.

Lemma 4.8 (Mass distribution principle)**.** *Let $X \subseteq \mathbb{R}^n$ be bounded and $C, \alpha > 0$. Suppose there is a non-zero Borel measure μ on X, such that*

$$\mu(B_r(x)) \leq C r^\alpha$$

for any ball of any radius r with center x in X. Then

$$\mathcal{H}_\infty^\alpha(X) \geq \frac{\mu(X)}{C}.$$

In particular

$$\dim_H(X) \geq \alpha.$$

Proof. Let $\{U_i\}_{i \in I}$ be a finite cover of X. W.l.o.g we may assume that each U_i is bounded, as X is assumed bounded. Chose $x_i \in U_i$ for all i and let $r_i = |U_i|$. This ensures that $B_{r_i}(x_i) \supseteq |U_i|$. Thus we get

$$\sum_{i \in I} |U_i|^r \geq \sum_{i \in I} \frac{\mu\left(B_{r_i}(x_i)\right)}{C} \geq \sum_{i \in I} \frac{\mu(U_i)}{C} \geq \frac{\mu(X)}{C}.$$

Thus the α-dimensional Hausdorff content is greater or equal than $\frac{\mu(X)}{C}$, hence positive and the Hausdorff dimension of X must be greater or equal to α. $\square$

Now we have everything to prove basic properties of the Hausdorff dimension, analogous to the ones of the Minkowski dimension in Lemma 4.3.

Proposition 4.9. *Let $X, Y, Y_i \subseteq \mathbb{R}^n$ for $i \in \mathbb{N}$. The Hausdorff dimension has the following properties:*

(1) If $X \subseteq Y$ then $\dim_H(X) \leq \dim_H(Y)$.

(2) $0 \leq \dim_H(X) \leq n$.

(3) The Hausdorff dimension is countably stable, i.e.

$$\dim_H\left(\bigcup_{i \in \mathbb{N}} Y_i\right) = \sup_{i \in \mathbb{N}}\{\dim_H(Y_i)\}.$$

(4) If X is open and non-empty then $\dim_H(X) = n$.

(5) If X is countable, then $\dim_H(X) = 0$.

(6) If $f : X \to \mathbb{R}^n$ is bi-Lipschitz, then

$$\dim_H(f(X)) = \dim_H(X) \quad .$$

Proof.

(1) This follows from the fact, that every covering of Y is a covering of X as well.

(2) The lower bound is trivial by the definition. For the upper bound let $\alpha > n$. We show that $\mathcal{H}^\alpha_\infty(\mathbb{R}^n) = 0$ and by (1) the result follows. To this end we chose $\delta > 0$ and cover $\mathbb{R}^n$ by hypercubes Q_k with side length δ for $k \in \mathbb{N}$. Then we divide Q_k in 2^{dk} of side length $\delta \cdot 2^{-k}$ and use this as a covering for $\mathbb{R}^d$. The diameter of such a cube is given by $\delta\sqrt{d}2^{-k}$. This yields

$$\mathcal{H}^\alpha_\infty(\mathbb{R}^d) \leq \sum_{k\in\mathbb{N}} 2^{kd}\left(\delta \cdot 2^{-k}\sqrt{d}\right)^\alpha = \delta^\alpha d^{\alpha/2} \sum_{k\in\mathbb{N}} \left(2^{d-\alpha}\right)^k = C\delta^\alpha,$$

where C is independent of δ. As δ was chosen arbitrarily, it follows $\mathcal{H}^\alpha_\infty(\mathbb{R}^d) = 0$ and taking the infinmum yields $\dim_H(\mathbb{R}^d) \leq d$.

(3) By (1) it is clear, that $\dim_H(Y_i) \leq \dim_H\left(\bigcup_i Y_i\right)$. Let $\alpha > \sup_i \dim_H(Y_i)$. Then $\mathcal{H}^\alpha_\infty(Y_i) = 0$ for all i and we can chose a cover $\{U_{i,j}\}_{j\in\mathbb{N}}$ of Y_i, such that $\sum_{j\in\mathbb{N}} |U_j|^\alpha \leq \varepsilon \cdot 2^{-i}$. So $\{U_{i,j}\}_{i,j\in\mathbb{N}}$ is a cover of $\bigcup_i Y_i$ and $\sum_{i,j\in\mathbb{N}} |U_{i,j}|^\alpha = \varepsilon$. As ε was chosen arbitrarily we get $\mathcal{H}^\alpha_\infty(\bigcup_i Y_i) = 0$ and the claim follows.

(4) As X is open, it contains an open ball B. We restrict the n-dimensional Lebesgue measure to B and use the mass distribution principle to get that $\dim_H(B) \geq n$. By (1) and (2) we may conclude that $\dim_H(X) = n$.

(5) The Hausdorff dimension of a single point is 0 and using (3) we get that $\dim_H(X) = 0$.

(6) As is the analogous claim for the Minkowski dimension we prove that if $|f(x)-f(y)| \leq c|x-y|$, then $\dim_H(f(X) \leq \dim_H(X)$. Let $\alpha > \dim_H(X)$. Then $\mathcal{H}^\alpha_\infty(X) = 0$, so there is a cover $\{U_i\}_{i\in Iy}$ of X, with $\sum_i |U_i|^\alpha \leq \frac{\varepsilon}{c^\alpha}$. But then $f(U_i \cap X)$ is a cover of $f(X)$, moreover $\sum_i |f(U_i)|^\alpha \leq \varepsilon$. As ε was arbitrary, we get $\mathcal{H}^\alpha_\infty(f(X)) = 0$ and therefore $\dim_H(f(X)) \leq \dim_H(X)$. Using the inverse of f we obtain equality.

□

So now we got rid of the "flaw" of countable sets with positive dimension, in comparison to the Minkowski dimension. Moreover we have the powerful tool of measure theory at our hands when computing the dimension of a set.

To close this chapter we compute the dimension of a well-known fractal.

Example 4.10 (Cantor set)**.** Let $A_0 = [0,1]$ be the unit line segment. Further let $A_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, $A_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$ and in general let A_{n+1} be the set in which we have removed the middle third of every interval of A_n. Finally let

$$A = \bigcap_{n\in\mathbb{N}} A_n$$

be the Cantor set. The first construction steps are illustrated in figure 3.

Figure 3: The sets A_0 to A_5 in the construction of the Cantor set

We now want to compute the dimension of A. Let $3^{-n} \geq r > 3^{-n-1}$. Then the 2^{n+1} Intervals of which A_{n+1} consist, form a cover of A with diameter less than r. Thus we get

$$\overline{\dim}_M(A) = \limsup_{r \searrow 0} \frac{\log N_r}{-\log r} \leq \limsup_{n \to \infty} \frac{\log 2^{n+1}}{-\log 3^{-n}} = \limsup_{n \to \infty} \frac{n+1}{n} \frac{\log 2}{\log 3} = \frac{\log 2}{\log 3}.$$

Next we define a measure μ on A in the following way. We set $\mu(I_{n_k} \cap A) = 2^{-n}$ for each of the 2^n intervals I_{n_k} of which A_n consists. Note that the sets $I_{n_k} \cap A$ form a semi ring on A which generates all Borel sets and we can extend μ to a measure on the Borel σ-Algebra. Now let $3^{-n-1} < r \leq 3^{-n}$ and let B_r be a ball of radius r. It can intersect at maximum two of the Intervals of length 3^{-n}. Thus we get

$$\mu(B_r) \leq 2\mu(I_n) = 2^{-n+1} \leq 4 \cdot 3^{(\log 2/\log 3)\cdot(-n-1)} \leq 4r^{\log 2/\log 3}.$$

By the mass distribution principle and the first computation we now may conclude that

$$\frac{\log 2}{\log 3} \leq \dim_H(A) \leq \underline{\dim}_M(A) \leq \overline{\dim}_M(A) \leq \frac{\log 2}{\log 3}$$

and therefore both Minkowski dimensions and the Hausdorf dimension of the Cantor set all agree and are equal to $\frac{\log 2}{\log 3}$.

5 Kakeya Sets and their Applications

5.1 Construction of Kakeya and Besikovitch Sets

In this section we want to construct sets of arbitrarily small area, such that you can rotate a needle within and sets of Lebesgue measure zero, which contain a unit line segment in every direction. We will follow the construction in Falconer's book [Fal85].

Definition 5.1. A set $X \subseteq \mathbb{R}^2$ is called a *Kakeya set*, if there are continuous functions $x, y, \omega : [0,1] \to \mathbb{R}$ and $k \in \mathbb{Z}$, such that for all t in

$$\left\{ x(t) + s\cos(\omega(t)), y(t) + s\sin(\omega(t)) \middle| s \in \left[-\frac{1}{2}, \frac{1}{2}\right] \right\} \subseteq X,$$

$x(0) = x(1)$, $y(0) = y(1)$ and $\omega(0) = \omega(1) + (2k+1)\pi$.

Definition 5.2. A set $X \subseteq \mathbb{R}^n$ is called *Besikovitch set* if for all $x \in S^{n-1}$ there is a $y \in \mathbb{R}^n$, such that

$$\{y + sx | s \in [0,1]\} \subseteq X$$

and if $\mathcal{L}^n(X) = 0$.

These two definitions formalize the concept of "a set, in which one can rotate a needle continuously" and "a set of zero area which contains a unit line segment in every direction." However we will not stick to this formal definition most of the times.

Now we want to show, that there are arbitrarily small Kakeya sets as well as Besikovitch sets. The sets we are going to construct are called Perron trees and are the result of an improved construction of Besikovitch sets, compared to the original one from 1921. First we start with a Lemma.

Lemma 5.3. *Let T be a triangle and M be the midpoint of its base line of length $2b$. Let $\frac{1}{2} < \alpha < 1$. Split T along the line from M to the opposite vertex of T and slide one of the resulting triangles along the base line, such that the base lines overlap for a distance of $2(1-\alpha)b$ and let S be the resulting figure. Then S consists of a triangle T', similar to T with $\mathcal{L}^2(T') = \alpha^2\mathcal{L}^2(T)$ and two further triangles. Furthermore*

$$\mathcal{L}^2(T) - \mathcal{L}^2(S) = \mathcal{L}^2(T)(1-\alpha)(3\alpha - 1).$$

Proof. The construction is illustrated in figure 4. The triangle T' is similar to T, as all of the sides are pairwise parallel. Its base line has length $2\alpha b$, hence we have $\mathcal{L}^2(T') = \alpha^2\mathcal{L}^2(T)$. Note that $\mathcal{L}^2(T) - \mathcal{L}^2(S)$ is the area of the overlap of the two triangles of which S consists. We observe that the two triangles next to the overlapping area at the bottom form together a third triangle T'', similar to T as well. It's base length is $2(2\alpha - 1)b$. We compute the area of the overlap:

$$\mathcal{L}^2(T) - \mathcal{L}^2(S) = \mathcal{L}^2(T') - \mathcal{L}^2(T'') = (\alpha^2 - (2\alpha-1)^2)\mathcal{L}^2(T) = (1-\alpha)(3\alpha-1)\mathcal{L}^2(T)$$

□

Starting from an equilateral triangle of height 1, where the needle can be rotated by $\frac{\pi}{3}$ at the top vertex, this Lemma allows us to rotate a needle the same angle in a smaller area. We can let the needle rotate for half of the angle at the top of the first triangle and then the rest at the top of the second one. There is only the problem that it has to "jump" in between. We can use this Lemma repeatedly to create a figure of arbitrarily small area in which the needle may be rotated provided we can take care of the "jumps."

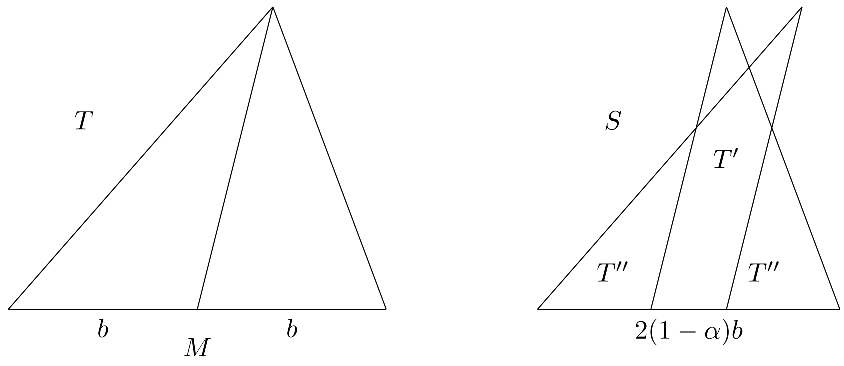

Figure 4: Sliding Triangles

Theorem 5.4. *Let T be a triangle and $\varepsilon > 0$. It is possible to divide the base line of T, join the dividing points to the opposite vertex of T and translate the resulting triangles along the base line, such that the resulting figure S has $\mathcal{L}^2(S) < \varepsilon$. Moreover let U be an open set containing T. Then this may be done, such that $S \subseteq U$.*

Proof. Let b be half the length of the base line of T. First chose $\alpha \in \left(\frac{1}{2}, 1\right)$ and $k \in \mathbb{N}$. Divide the base line into 2^k equal parts and join the points of division to the opposite vertex of T. We label the resulting triangles $T_1^0, T_2^0, \ldots, T_{2^k}^0$. Now we slide T_{2l}^0 to the right, such that the overlap of the base line with T_{2l-1}^0 has length $2(\alpha - 1)b$. So we can apply the previous lemma to the triangle $T_{2l-1}^0 \cup T_{2l}^0$. Let T_l^1 be the triangle called T' in Lemma 5.3. Doing this with every pair of consecutive triangles, we denote the resulting figure by S^1. We have achieved a total reduction of area equal to $\mathcal{L}^2(T) - \mathcal{L}^2(S^1) = (3\alpha - 1)(1 - \alpha)\mathcal{L}^2(T)$ in this first step. Now assume we have already constructed S_l^r and T_l^r for $l = 1, \ldots, 2^{k-r}$. We slide the S_{2l}^r to the right, such that the base lines of the triangles T_{2l-1}^r and T_{2l}^r overlap for $2(1 - \alpha)$ times their base length. This leads to a resulting figure S_l^{r+1} with a triangle T_l^{r+1} inside, similar to $T_{2l-1}^r \cup T_{2l}^r$, with proportionality factor α. The construction is illustrated in figure 5. As the overlap of S_{2l-1}^r and S_{2l}^r after the translation is at least as big as the overlap of the triangles T_{2l-1}^r and T_{2l}^r we have

$$\mathcal{L}^2(S_{2l-1}^r) + \mathcal{L}^2(S_{2l}^r) - \mathcal{L}^2(S_l^{r+1}) \geq (1 - \alpha)(3\alpha - 1)\mathcal{L}^2(T_{2l-1}^r \cup T_{2l}^r).$$

Summing up these inequalities for all l from 1 to 2^{k-r-1} and all r from 0 to $k - 1$ we get

$$\mathcal{L}^2(T) - \mathcal{L}^2(S_1^k) \geq (1 - \alpha)(3\alpha - 1)(1 + \alpha^2 + \ldots + \alpha^{2k-2})\mathcal{L}^2(T) = \mathcal{L}^2(T)\frac{(3\alpha - 1)(1 - \alpha^{2k})}{(1 + \alpha)}.$$

Chose α to be close to one, such that $\frac{(3\alpha-1)}{1+\alpha}$ is sufficiently close to 1 and let k then be large enough, such that $(1 - \alpha^{2k})$ is small enough. Then $\mathcal{L}^2(S_1^k)$ will be smaller than ε and we may set $S = S_1^k$.

Now let U be an open set containing T. There is $\delta > 0$, such that a δ-neighbourhood of T is contained in U. In the construction above we have moved each of the triangles we started with for a distance less than the base of the starting triangle. Now we divide the base line of T in equal parts, such that each part has length smaller than δ. Then we apply the construction from above to each of the triangles obtained by joining the division points to the opposite vertex of T. So we have moved each of the starting triangles by a maximum of δ, thus the resulting figure is still contained in U. $\square$

Next we need to take care of the jumps to obtain Kakeya sets by introducing Pál joints.

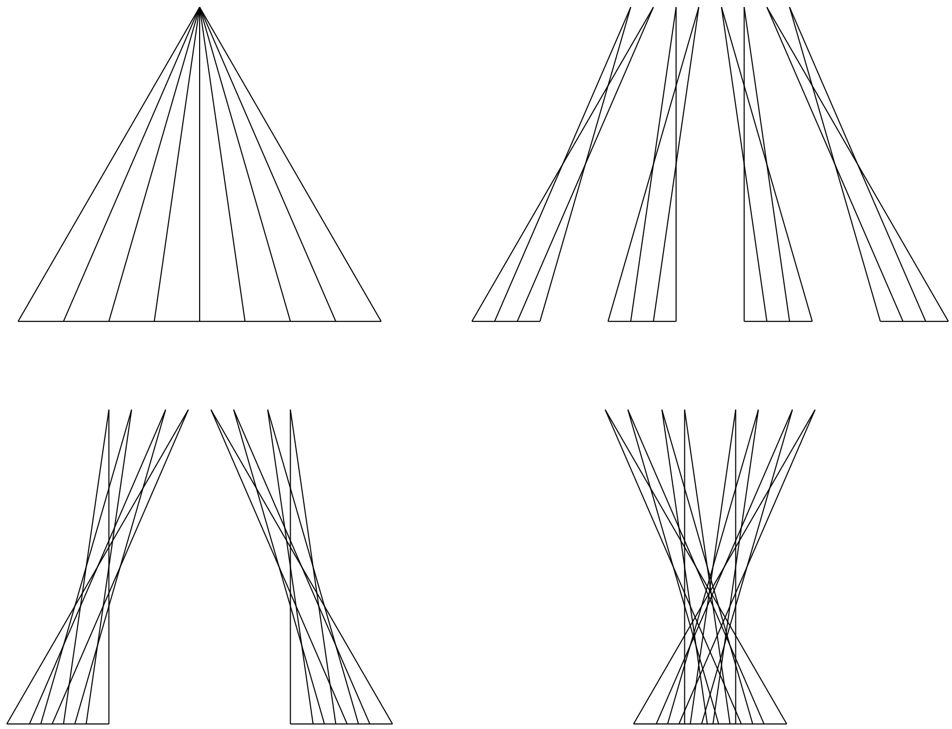

Figure 5: The construction of a Perron tree

Lemma 5.5. *Let G_1, G_2 be two parallel unit line segments and let $\varepsilon > 0$. There is a set J such that $\mathcal{L}^2(J) < \varepsilon$ and a needle can be moved continuously from G_1 to G_2 within J.*

Proof. Let g_1, h_1 and g_2, h_2 be the endpoints of G_1 and G_2 respectively. Chose x on the line k defined by G_1 and let J_x be the set consisting of the line segment joining h_1 and x, the line segment l joining x and h_2 as well as G_2 and two rectangular triangles: one with one edge of length 1 starting at x on k and the hypotenuse on l, the other one with one edge equal to G_2 and the hypotenuse on l as well. The set is visualized in figure 6. If we chose x sufficiently far apart from h the length h of the third edges of the triangles get smaller than ε. Thus the area of J_x, which is just the area of the two triangles, gets sufficiently small, i.e. $\mathcal{L}^2(J_x) < \varepsilon$. We can move the needle from G_1 to G_2 by moving it along k, turning it inside the first triangle, pushing it along l and finally rotate it within the second triangle. □

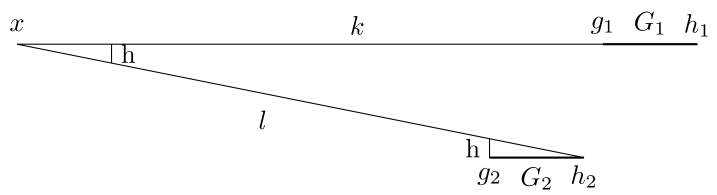

Figure 6: Connecting two parallel unit line segments

Now we have everything we need to prove that a needle can be rotated inside a set with arbitrarily small measure, hence prove Besikovitch's original result.

Theorem 5.6. *Let $\varepsilon > 0$. There exists a Kakeya set K, such that $\mathcal{L}^2(K) < \varepsilon$.*

Proof. We will show, that there is a set L, with $\mathcal{L}^2(L) < \frac{\varepsilon}{3}$ such that we can rotate a needle for an angle $\frac{\pi}{3}$. By taking three copies of such sets we obtain the desired set K. Let T be a equilateral triangle of height one. It is possible to rotate a needle by an angle of $\frac{\pi}{3}$ starting at one edge and rotating around the top vertex. Now we apply Theorem 5.4 to T. We split T into 2^k triangles in each of which the needle can be rotated for an angle of $\frac{\pi}{3 \cdot 2^k}$. Two neighbouring triangles have a pair of parallel edges. We rearrange those triangles, such that the area of the resulting figure S is less than $\frac{\varepsilon}{6}$. Now we construct for each pair of consecutive triangles a joint as in Lemma 5.5 of area smaller that $\frac{\varepsilon}{3 \cdot 2^{k+1}}$ to connect their parallel edges. Then we take L to be the union of S and all those joints and we get

$$\mathcal{L}^2(L) \leq \frac{\varepsilon}{6} + (2^k - 1)\frac{\varepsilon}{3 \cdot 2^{k+1}} < \frac{\varepsilon}{3},$$

so we are done. □

Having solved the original Kakeya Problem we proceed by showing, that Besikovitch sets exist.

Theorem 5.7. *There is a two dimensional Besikovitch set.*

Proof. We start again with an equilateral triangle T of unit height to construct a set of measure zero such that it contains a unit line segment in every direction which encloses an angle of more than $\frac{\pi}{3}$ with the base line of T. Let U_0 be an open ball containing T. In the first step we apply Theorem 5.4 to split T into triangles and rearrange them, such that the area of the resulting figure S_1, constructed in a way to be contained in U_0, is smaller than $\frac{1}{2}$. As S_1 is a finite union of triangles we find an open set U_1 with $S_1 \subseteq U_1 \subseteq U_0$ such that $\mathcal{L}^2\left(\overline{U_1}\right) < 1$. Assume we have already constructed sets U_r, S_r. We apply Theorem 5.4 to every triangle of S_r such that the resulting figure S_{r+1} has area less than $\frac{1}{2^{r+1}}$ and is contained within U_r. Then, as S_{r+1} still consists of a finite union of triangles, we can find U_{r+1}, such that $S_{r+1} \subseteq U_{r+1} \subseteq U_r$ and

$$\mathcal{L}^2\left(\overline{U^{r+1}}\right) < \frac{1}{2^r}.$$

Now we want to show, that

$$B = \bigcap_{r=1}^{\infty} \overline{U_r}$$

is a Besikovitch set. As B is a subset of sets with arbitrarily small area, we have $\mathcal{L}^2(B) = 0$. Let $x \in S^1$ be a unit vector, such that it encloses an angle of more than $\frac{\pi}{3}$ with the base line of T. When rotating the needle at the top of T, the needle points in direction x at some point. This line segment gets translated when we construct the sets S_r, but in each of the S_r there is a unit line segment x_r parallel to x. As all the S_r are contained in the compact set $\overline{U_0}$, there is a subsequence x_{r_k} converging to a unit line segment x_0. For a fixed r we have for all $q \geq r$, that $S_q \subseteq \overline{U_r}$. Hence $x_q \subseteq \overline{U_r}$ and as $\overline{U_r}$ is compact we get $x_0 \subseteq \overline{U_r}$ for all r. Thus we get $x_0 \subseteq B$ and we have shown that B is a Besikovitch set. □

Note that the theorems above proved the existence of Besicovitch sets of Lebesgue-measure zero as well as the existence of a Kakeya set of arbitrarily small area. However, it is clear that no Kakeya set of area zero may exist, as rotating a needle inevitably leads to a positive area covered. For a rigorous proof see Question 1.22.1 in Terence Tao's collection of blog entries [Tao09].

From here on it is easy to construct higher dimensional Besicovitch sets. Let $d > 2$ and consider $B = B_2 \times [0,1]^{d-2}$ for a two-dimensional Besikovitch set B_2. By Fubini's theorem we know that $\mathcal{L}^d(B) = 0$. Let $x \in S^{d-1}$ be a d-dimensional direction and let x_2 be the projection of x onto $\mathbb{R}^2$. Then there is a unit line G segment in the direction of x_2 in B_2. The hyperplane $E = G \times [0,1]^{d-2}$ is therefore contained in B and E contains a unit line segment in the direction of x. Thus B is a Besikovitch set.

5.2 An application of Besicovitch sets – Fefferman's Disk Multiplier Theorem

In 1971 Fefferman provided a disproof of a long-standing conjecture in Fourier analysis using a slightly modified version of the construction of Besikovitch sets.

Recall that the Fourier transform of a function f is given by

$$\mathcal{F}f(x) = \hat{f}(y) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot y} f(x) dx$$

It can be shown that the Fourier transform extends to an isometry of $L^2(\mathbb{R}^n)$. Its inverse is given by

$$f(x) = \mathcal{F}^{-1}\hat{f}(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} \hat{f}(y) dy.$$

Now consider the following class of operators: Let m be a bounded function. We define $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ by

$$Tf = \mathcal{F}^{-1}\left(m\mathcal{F}f\right).$$

One of the easiest cases occurs, when m is the characteristic function of some set X. Let now X be the unit ball B and consider the disk multiplier T_B defined by $T_B f = \mathcal{F}^{-1}(\chi_B \mathcal{F}f)$. We want to investigate, for which values of p this operator is bounded. As mentioned above, for $p = 2$ the Fourier transform is an isometry, therefore we get that $\|T_B\| = 1$. It was believed, that there is actually a range of values for p, such that T_B is bounded as operator from L^p to itself. However Fefferman proved the following result:

Theorem 5.8. *The operator $T_B : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is unbounded for all $n \geq 2$ and $p \neq 2$.*

The proof is a bit involved and not presented here. See Fefferman's original paper [Fef71] or chapter 9.3 in [BP17].

6 The Kakeya Conjecture

As mentioned above we want to analyse the size of Besikovitch sets even further, by means of fractal geometry.

Conjecture 6.1. *Let $B \subseteq \mathbb{R}^d$ be a compact Besikovitch set. Then*

$$\dim_H(B) = \underline{\dim}_M(B) = \overline{\dim}_M(B) = d,$$

i.e. the Minkowski dimension of B exists, is equal to the Hausdorff dimension and is maximal.

The conjecture remains unproven for $d > 2$, but there is a proof for $d = 2$ which will be given in the following section. Moreover there are already some bounds on the minimal dimension for higher dimensional Besikovitch sets, some of the results are summed up in the second part of this chapter.

6.1 The two dimensional Case

Let $B \subseteq \mathbb{R}^2$ be a Besikovitch set. Indeed it is enough to prove that $\dim_H(B) = d$, as the Minkowski dimension is bounded from below by the Hausdorff dimension and from above by 2 . We need two Theorems, that are due to John Marstrand, one of Besikovitch's students. For one of them there is a fairly straightforward proof, the other one relies on Frostman's lemma and capacity theory, which we will discuss here. We follow the arguments of Bishop and Peres [BP17].

Theorem 6.2 (Marstrand Slicing Theorem)**.** *Let $E \subseteq \mathbb{R}^2$ be a bounded set. Let $E_x = \{y|(x, y) \in E\}$ be the vertical slice of E at x and suppose* $\dim_H(E) > 1$. *Then*

$$\dim_H(E_x) \le \dim_H(E) - 1$$

for almost all x.

Proof. First we want to prove, that

$$\int_{\mathbb{R}} \mathcal{H}^{\alpha-1}_{\varepsilon}(E_x)dx \le \mathcal{H}^{\alpha}(E)$$

for all $1 \le \alpha \le 2$. Let $\varepsilon, \delta > 0$ and $\{A_j\}_{j\in J}$ be a cover of E with sets of diameter less than ε, such that $\sum_j |A_j|^\alpha < \mathcal{H}^\alpha_\varepsilon(E) + \delta$. For each A_j chose a square Q_j, parallel to the axis, with side length $|A_j|$, such that $A_j \subseteq Q_j$. Let R_j be the projection of Q_j onto the vertical axis and let

$$J_x = \{j \in J | \exists y \ (x, y) \in Q_j\}.$$

Then $\{R_j\}_{j\in J_x}$ is a cover of E_x. Hence $\mathcal{H}^{\alpha-1}_\varepsilon(E_x) \le \sum_{j\in J_x} |R_j|^{\alpha-1}$. Now integrating with respect to x yields

$$\int_{\mathbb{R}} \mathcal{H}^{\alpha-1}_\varepsilon(E_x)dx = \sum_{j\in J} \int_{\{x|j\in J_x\}} |R_j|^{\alpha-1}dx = \sum_{j\in J} |R_j|^\alpha = \sum_{j\in J} |A_j|^\alpha < \mathcal{H}^\alpha_\varepsilon(E) + \delta$$

Now with $\delta \searrow 0$ we get

$$\int_{\mathbb{R}} \mathcal{H}^{\alpha-1}_\varepsilon(E_x)dx \le \mathcal{H}^\alpha_\varepsilon(E)$$

and as the Hausdorff content does not decrease when ε tends to 0, by the monotone convergence theorem we get

$$\int_{\mathbb{R}} \mathcal{H}^{\alpha-1}(E_x)dx \le \mathcal{H}^\alpha(E).$$

Now let $\alpha > \dim_H(E)$. Then we have

$$0 = \mathcal{H}^\alpha(E) \geq \int_{\mathbb{R}} \mathcal{H}^{\alpha-1}(E_x)dx,$$

so the non-negative integrand must vanish almost everywhere. $\square$

Next we need to prove a converse statement to the mass distribution principle, Frostman's Lemma.

Lemma 6.3 (Frostman's Lemma)**.** *Let $\alpha > 0$ and $E \subseteq \mathbb{R}^d$, be a compact set with $\mathcal{H}^\alpha_\infty(E) > 0$. Then there is a positive Borel measure μ on E and a constant A, such that*

$$\mu(B_r) \leq Ar^\alpha$$

for all balls B_r of radius r and

$$\mu(E) \geq \mathcal{H}^\alpha_\infty(E).$$

Proof. By rescaling and applying a translation we may assume without loss of generality that E is contained in the unit cube. We now split the unit cube in 2^d cubes of side length $\frac{1}{2}$ and those even further. Then we construct a tree Γ in the following way: The vertices correspond to those dyadic cubes, which intersect E and each of them is connected to its "parent" cube, the cube of double side length in which it is contained. We define the following capacity C on the edges of Γ:

$$C(v'v) = \left(\sqrt{d} \cdot 2^{-|v|}\right)^\alpha.$$

Let f be a maximal legal flow on Γ, which exists in view of Theorem 3.9.

We now consider the space of infinite paths in Γ starting at the root V_0. First we define a metric d on it: For two paths p, q, let v be the last common vertex and we set $d(p,q) = 2^{-|v|}$. Open balls are then given by all paths passing through a given edge. One can show that the space of infinite paths is compact with respect to this metric (see [BP17], section 3.1). Let v be a vertex and

$$S_v = \{\text{all paths passing through } v'v\},$$

then the collection R of all the S_v form a semi ring. We define a pre-measure $\tilde{\mu}$ on it by setting

$$\tilde{\mu}(\{\text{all paths passing through } v'v\}) = f(v'v).$$

This is finitely additive by the property of a flow. Moreover compactness implies that the set of all paths passing through $v'v$ is compact as well and thus it can be split up in only finitely many disjoint elements of R. This yields countable additivity. So $\tilde{\mu}$ can be extended by Carathéodory's extension theorem (Theorem 3.6) to a measure μ on the σ-algebra generated by R. Now we may interpret μ as a Borel measure on E by setting the measure of a dyadic cube corresponding to v as $\mu(v'v)$ and the dyadic cubes generate all Borel sets. Let $x \notin E$. Then there is an open neighbourhood of x in the complement of E and a dyadic cube contained within. This ensures that the support of μ is a subset of E. Now let D be a dyadic cube associated to the vertex v. Then $\mu(D) = f(v'v) \leq C(v'v) = |D|^\alpha$ and every ball B can be covered by a finite number of dyadic cubes of side length smaller than the radius. So there exists a constant A, only dependant on the dimension d, such that $\mu(B_r) \leq Ar^\alpha$ for all balls of radius r.

It remains to prove that $\mu(E) \geq \mathcal{H}^\alpha_\infty(E)$. Note that each cover of dyadic cubes of E corresponds to a cut-set of the tree. Thus we get by the Max-Flow-Min-Cut theorem (Theorem 3.9) that

$$\mu(E) = \|f\| = \inf_{D \text{ cut-set}} C(D) \geq \mathcal{H}^\alpha_\infty(E),$$

where we have used that the capacity of an edge is equal to the area of the corresponding dyadic cube. $\square$

Now we need to make a detour to capacity theory to establish the second of Marstrand's theorems we need.

Definition 6.4. Let μ be a Borel measure on $\mathbb{R}^d$ and let $\alpha > 0$. We define the *α-dimensional energy $\mathcal{E}_\alpha(\mu)$ of μ* by

$$\mathcal{E}_\alpha(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\alpha} d\mu(x) d\mu(y)$$

Moreover for a set $E \subset \mathbb{R}^d$ we define the *α-dimensional capacity* $\operatorname{Cap}_\alpha(E)$ as

$$\operatorname{Cap}_\alpha(E) = \frac{1}{\inf_\mu \mathcal{E}_\alpha(\mu)},$$

where the infimum is taken over all Borel probability measures μ, which are supported on E. Furthermore if $\mathcal{E}_\alpha(\mu) = \infty$ for all such measures μ, we set $\operatorname{Cap}_\alpha(E) = 0$.

For a given set E we now want to find the critical value α_0, such that $\operatorname{Cap}_\alpha(E) = 0$ for all $\alpha > \alpha_0$. It turns out, that this value is equal to the Hausdorff dimension of E.

Proposition 6.5. *Let $E \subset \mathbb{R}^d$ be a compact set and suppose $\alpha > \beta$. Then the following statements are true:*

(1) If $\mathcal{H}^\alpha(E) > 0$, then $\operatorname{Cap}_\beta(E) > 0$.

(2) If $\operatorname{Cap}_\alpha(E) > 0$, then $\mathcal{H}^\alpha(E) = \infty$.

(3) $\dim_H(E) = \inf\{\alpha | \operatorname{Cap}_\alpha(E) = 0\}$.

Proof.

(1) Suppose $\mathcal{H}^\alpha(E) > 0$. Then by Frostman's lemma (Lemma 6.3), there is a non vanishing measure μ, such that $\mu(B) < C|B|^\alpha$ for all balls. As E is compact, μ is finite and by rescaling we may assume that μ is a probability measure. Now we choose r such that $|E| < 2^r$ and we fix $x \in E$ to compute

$$\begin{aligned}
\int_E \frac{1}{|x-y|^\beta} d\mu(y) &= \sum_{n=-\infty}^{r} \int_{B_{2^n}(x) \setminus B_{2^{n-1}}(x)} \frac{1}{|x-y|^\beta} d\mu(y) \\
&\le \sum_{n=-\infty}^{r} \mu(B_{2^n}(x)) 2^{\beta(-n+1)} \\
&\le \sum_{n=-\infty}^{r} C 2^{\alpha n} 2^{\beta(-n+1)} \\
&\le 2^\beta C \sum_{n=-\infty}^{r} 2^{(\alpha-\beta)n} \\
&\le M < \infty,
\end{aligned}$$

where M is a constant, independent of x. Thus we may integrate once more and get

$$\int_{E \times E} \frac{1}{|x-y|^\beta} d\mu(y) d\mu(x) \le \mu(E) M < \infty,$$

which proves (i).

(2) Because $\operatorname{Cap}_\alpha(E) > 0$, there is a positive Borel measure μ supported on E, such that $\mathcal{E}_\alpha(\mu) < \infty$. Let M be large enough, such that

$$F = \left\{ x \in E \middle| \int_E \frac{1}{|x-y|^\alpha} \le M \right\}$$

has positive measure. This exists, as the α-dimensional energy of μ is finite. Now we may compute

$$\int_E \frac{1}{|x-y|^\alpha} d\mu(y) \geq \sum_{n=-\infty}^{r} \mu(B_{2^n}(x) \setminus B_{2^{n-1}}(x)) 2^{-\alpha n}$$
$$\geq \mu(B_{2^r}(x)) + (1-2^{-\alpha}) \sum_{n=-\infty}^{r-1} \mu(B_{2^n}(x)) 2^{-\alpha n},$$

where we have used the additivity of μ for disjoint sets. As for all $x \in F$ the integral is finite, we know that the sum on the right hand side of the equation has to be finite as well. So we get

$$\lim_{n\to\infty} \frac{\mu(B_{2^{-n}}(x))}{2^{-\alpha n}} = 0,$$

and $\mu(B_r(x)) \leq Cr^\alpha$ for all α and some constant C. A variant of the mass distribution principle ensures that $\mathcal{H}^\alpha(F) = \infty$ and therefore $\mathcal{H}^\alpha(E) = \infty$. To see this, replace $\mathcal{H}^\alpha_\infty$ by $\mathcal{H}^\alpha_\delta$ in the proof of the mass distribution principle and let $\delta \searrow 0$.

(3) This is a direct consequence of (1) and (2). Let $\alpha > \dim_H(E)$. Then $\mathcal{H}^\alpha(E) = 0$ and because of (2) $\mathrm{Cap}_\alpha(E) = 0$. If $\alpha < \dim_H(E)$, let $\gamma = (\alpha + \dim_H(E))/2$. Because $\gamma < \dim_H(E)$ we get that $\mathcal{H}^\gamma(E) > 0$ and with (1) and $\alpha < \gamma$ it follows that $\mathrm{Cap}_\alpha(E) > 0$.

$\square$

Now we return to $\mathbb{R}^2$. In the following let p_θ be the orthogonal projection in the direction of the angle θ onto a line through the origin.

Theorem 6.6 (Marstrand's Projection Theorem)**.** *Let* $0 < \alpha < 1$ *and* $E \subseteq \mathbb{R}^2$ *compact. Assume* $\mathrm{Cap}_\alpha(E) > 0$*. Then for almost all* θ *we have that* $\mathrm{Cap}_\alpha(p_\theta(E)) > 0$*. Moreover, if* $\dim_H(E) < 1$*, then* $\dim_H(p_\theta(E)) = \dim_H(E)$ *for almost all* θ*.*

Proof. As $\mathrm{Cap}_\alpha(E) > 0$, there is a measure μ supported on E, such that $\mathcal{E}_\alpha(\mu) < \infty$. We consider the pushforward measures μ_θ on $\mathbb{R}$ defined by $\mu_\theta(X) = \mu(p_\theta^{-1}(X))$, where we identify $\mathbb{R}$ with the line normal to the direction θ passing through the origin. We want to show that $\mathcal{E}_\alpha(\mu_\theta) < \infty$ for almost all θ, which we will achieve by proving that

$$\int_0^\pi \mathcal{E}_\alpha(\mu_\theta) d\theta < \infty.$$

We compute by using Fubini's theorem to swap integrals:

$$\int_0^\pi \mathcal{E}_\alpha(\mu_\theta) d\theta = \int_0^\pi \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{|\hat{x}-\hat{y}|^\alpha} d\mu_\theta(\hat{x}) d\mu_\theta(\hat{y}) d\theta$$
$$= \int_0^\pi \int_E \int_E \frac{1}{|p_\theta(x-y)|^\alpha} d\mu(x) d\mu(y) d\theta$$
$$= \int_K \int_K \frac{1}{|x-y|^\alpha} \int_0^\pi \frac{1}{|p_\theta(\frac{x-y}{|x-y|})|^\alpha} d\theta d\mu(x) d\mu(y)$$

Now note, that the inner integral is independent of x and y, as we project a unit vector in every direction, regardless of the "starting direction" of $\frac{x-y}{|x-y|}$. Thus it is a constant C depending only on α and we get

$$\int_0^\pi \mathcal{E}_\alpha(\mu_\theta) d\theta = C \int_K \int_K \frac{1}{|x-y|^\alpha} d\mu(x) d\mu(y) = C\mathcal{E}_\alpha(\mu) < \infty$$

Using Proposition 6.5(3) we get the second result. $\square$

Now we can finally prove the Kakeya conjecture in the two dimensional case.

Theorem 6.7. *Let $B \subseteq \mathbb{R}^2$ be a compact Besikovitch set. Then* $\dim_H(B) = 2$.

Proof. Let I be an interval. We consider the set

$$S_I = \{(a, b) \subseteq \mathbb{R}^2 | \forall x \in I \quad (x, b - ax) \in B\}.$$

Let $p : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first coordinate. As B is a Besikovitch set, for all $a \in \mathbb{R}$ there has to be an Interval I (without loss of generality with rational endpoints), such that a is the first coordinate of an element of S_I. So $\bigcup_I p(S_I) = \mathbb{R}$, where the union is over those Intervals with rational endpoints. As this is a countable union and the Hausdorff dimension is countable stable, for all ε we get an I_ε, such that $\dim_H(p(S_{I_\varepsilon})) = 1 - \varepsilon$, and, because p is Lipschitz, $\dim_H(S_{I_\varepsilon}) > 1 - \varepsilon$. Therefore by the Marstrand projection theorem (Theorem 6.6), we may conclude, that $\dim_H(p_\theta(S_{I_\varepsilon}) > 1 - \varepsilon$ for almost all θ. Let q_θ be the projection in the direction of lines with slope $t = \tan\theta$ onto the vertical axis. As the map, that maps $p_\theta(E)$ to $q_\theta(E)$ is bi-Lipschitz, the two sets have the same Hausdorff dimension. But the set $q_\theta(S_{I_\varepsilon})$ is the same as $\{(b - at|(a, b) \in S_{I_\varepsilon}\}$. So for almost all t we get that $\dim_H(\{b - at|(a, b) \in S_{I_\varepsilon}\}) > 1 - \varepsilon$. From $(a, b) \in S_{I_\varepsilon}$ it follows that $(t, b - at) \in B$ for all $t \in I_\varepsilon$. So we have

$$\{b - at|(a, b) \in S_{I_\varepsilon}\} \subseteq \{b - at|(t, b - at) \in B\}$$

and therefore

$$\dim_H(\{b - at|(t, b - at) \in B\}) > 1 - \varepsilon$$

for almost all t. These are just slices of the set B and the Marstrand slicing theorem (Theorem 6.2) gives us that $\dim_H > 1 + 1 - \varepsilon$ for all ε and therefore $\dim_H(B) \geq 2$. As it is a subset of $\mathbb{R}^2$ we may conclude $\dim_H(B) = 2$. □

6.2 Progress in higher Dimensions

At the moment, there seem to be two promising approaches, how to tackle the Kakeya conjecture in higher dimensions. One is from a geometric point of view and uses δ-neighbourhoods of unit line segments, so called δ-tubes and the goal is to bound the area of their intersections. This leads to the so called bush-argument, published by Jean Bourgain, which proves that the dimension of a n-dimensional Besikovitch set is at least $\frac{(n+1)}{2}$. The other approach is more abstract, translates the problem to arithmetic and uses combinatorial arguments. It also yields the $\frac{(n+1)}{2}$ bound.

Good lower bounds for the Hausdorff dimension of Besikovitch sets found up until now, are $\frac{5}{2}$ for $n = 3$ and 3 for $n = 4$ by Thomas Wolff as well as $(2 - \sqrt{2})(n - 4) + 3$ by Nets Katz and Terence Tao, while bounds on the Minkowski dimension are slightly bigger. For more information, see the article of Katz and Tao [KT00] and the homepage of Izabella Laba [Lab02]. They both provide a good overview.

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The figures 2 and 3 are taken from [BP17], the rest was created using tikz [`https://github.com/pgf-tikz/pgf`] and/or Geogebra [`https://www.geogebra.org/`].