Algebraicity of Hypergeometric Functions with Arbitrary Parameters

joint work with S. Yurkevich (arXiv:2308.12855)

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Overview

1. Introduction
2. A Little Bit of History
3. Some More History – Interlacing Criteria
4. Algebraicity for Arbitrary Parameters – A Complete Criterion
5. Examples

Definitions

Hypergeometric function:

$$ {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{x^n}{n!}, $$

where $(a)_n := a(a+1)\cdots(a+n-1)$ denotes the **rising factorial.**

Hypergeometric sequence:

$$ u_{n+1} = \frac{A(n)}{B(n)} u_n, $$

where $A(t), B(t) \in \mathbb{Q}[t]$ are polynomials.

Hypergeometric functions are generating functions of hypergeometric sequences.

Examples

- Logarithm:

$$ {}_2F_1\left[\begin{matrix}1,1\\2\end{matrix};x\right] = -\frac{\log(1-x)}{x} = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \ldots \in \mathbb{Q}[\![x]\!] $$

- **Catalan numbers:**

$$ C_n = \binom{2n}{n}\frac{1}{n+1} \in \mathbb{Z}, \quad \frac{C_{n+1}}{C_n} = \frac{(2n+1)(2n+2)(n+1)}{(n+1)(n+1)(n+2)}, \quad \sum_{n\geq 0} C_n x^n = {}_2F_1\left[\begin{matrix}\frac{1}{2},1\\2\end{matrix};4x\right] $$

- **Chebychev numbers:**

$$ \frac{(30n)!n!}{(15n)!(10n)!(6n)!} \in \mathbb{Z} $$

- Some other algebraic series, such as

$$ {}_3F_2\left[\begin{matrix}1/2, \sqrt{2}+1, -\sqrt{2}+1\\ \sqrt{2}, -\sqrt{2}\end{matrix};4x\right] = \frac{(7x-1)(2x-1)}{(1-4x)^{5/2}} = 1 + x - 6x^2 + \cdots \in \mathbb{Z}[\![x]\!] $$

Definitions

A power series $f(x) \in \mathbb{Q}[\![x]\!]$ is called **algebraic** (over $\mathbb{Q}(x)$) if there is $P(x,y) \in \mathbb{Q}[x,y]$, $P(x,y) \neq 0$, such that $P(x, f(x)) = 0$.

A power series $f(x) \in \mathbb{Q}[\![x]\!]$ is called **almost integral** if there are $\alpha, \beta \in \mathbb{Z} \setminus \{0\}$, such that $\beta f(\alpha x) \in \mathbb{Z}[\![x]\!]$.
In particular, only finitely many prime numbers appear in the denominators of the coefficients.

A power series $f(x) \in \mathbb{Q}[\![x]\!]$ is called **globally bounded** if it is almost integral and its convergence radius is nonzero and finite.

Theorem (Eisenstein 1852, Heine 1854)

Any algebraic $f(x) \in \mathbb{Q}[\![x]\!]$ is almost integral. More precisely, $f(x)$ is a polynomial or globally bounded.

Definitions

A power series $f(x) \in \mathbb{Q}[\![x]\!]$ is called **differentially finite** or **D-finite** if it satisfies a non-trivial linear ordinary differential equation with coefficients in $\mathbb{Q}[x]$ (ODE).

Theorem (Folklore, Abel 1827)

Any algebraic $f(x) \in \mathbb{Q}[\![x]\!]$ is D-finite.

Any hypergeometric function $F(x) \in \mathbb{Q}[\![x]\!]$ is D-finite. It satisfies the differential equation

$$x(\theta + a_1)\cdots(\theta + a_p)F(x) = \theta(\theta + b_1 - 1)\cdots(\theta + b_{p-1} - 1)F(x) \qquad (\theta = x\frac{\mathrm{d}}{\mathrm{d}x})$$

with coefficients in $\mathbb{Q}[x]$. All its solutions are linear combinations of hypergeometric functions.

Conjecture (weak form of p-curvature conjecture, Grothendieck 1969, Bézivin 1991)

An ODE of order n has n $\mathbb{Q}$-linearly independent algebraic (Puiseux series) solutions if and only if it is has n $\mathbb{Q}$-linearly independent globally bounded (Puiseux series) solutions.

Question

Which hypergeometric functions are algebraic?

The hypergeometric function

$$ {}_2F_1\left[\begin{matrix}1,1\\2\end{matrix};x\right] = -\frac{\log(1-x)}{x} = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \ldots \in \mathbb{Q}[\![x]\!] $$

clearly is **not algebraic.** It is not even globally bounded.

The function

$$ {}_3F_2\left[\begin{matrix}1/2, \sqrt{2}+1, -\sqrt{2}+1\\ \sqrt{2}, -\sqrt{2}\end{matrix};4x\right] = \frac{(7x-1)(2x-1)}{(1-4x)^{5/2}} $$

clearly is **algebraic.**

Schwarz' Classification

Schwarz 1873: Classification of all algebraic **Gaussian hypergeometric functions**, i.e., all $F(x) = {}_2F_1([a_1, a_2], [b_1]; x)$, with rational parameters $a_1, a_2, b_1 \in \mathbb{Q}$:

Define $(\lambda, \mu, \nu) = (1 - b_1, b_1 - a_1 - a_2, a_2 - a_1)$. Schwarz provided a list of triples, such that $F(x)$ is algebraic (assuming $\lambda, \mu, \nu \notin \mathbb{Z}$), if and only if (λ, μ, ν) appears in the list, up to permutations, sign changes and addition of triples of integers with even sum.

Example

$F(x) = {}_2F_1([-1/2, -1/6], [2/3]; x)$ is algebraic, as $(\lambda, \mu, \nu) = (1/3, 4/3, 1/3)$ and $(-(\nu - 1), \lambda, \mu - 1) = (2/3, 1/3, 1/3)$ is in the list.

No.	λ''	μ''	ν''	Inhalt / π	Polyeder
I.	$\frac{1}{2}$	$\frac{1}{2}$	ν	ν	Regelmässige Doppelpyramide
II.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6} = A$	Tetraeder
III.	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3} = 2A$	
IV.	[illegible]	[illegible]	[illegible]	[illegible]	

Landau-Errera Criterion

Landau 1904, 1911 exploited Eisenstein's Theorem, leading to a necessary condition for algebraicity of Gaussian hypergeometric functions with rational parameters:

Theorem (Landau)

Let $F(x) = {}_2F_1([a_1, a_2], [b_1]; x)$ *with* $a_1, a_2, b_1, a_1 - b_1, a_2 - b_1 \notin \mathbb{Z}$. *Then* $F(x)$ *is globally bounded iff for all* $1 \leq \lambda \leq N$ *coprime to the common denominator* N *of* a_1, a_2, b_1 *we have*

$$\langle \lambda a_1 \rangle < \langle \lambda b_1 \rangle < \langle \lambda a_2 \rangle \quad \text{or} \quad \langle \lambda a_2 \rangle < \langle \lambda b_1 \rangle < \langle \lambda a_1 \rangle, \tag{$\star$}$$

where $\langle \cdot \rangle$ *denotes the fractional part.*

Errera 1913 extended this to a criterion for algebraicity:

Theorem (Errera)

Condition $(\star)$ *for all* λ *is equivalent to Schwarz' classification.*

Christol's Interlacing Criterion

Christol 1986 studied a conjecture about **diagonals** of multivariate rational functions and globally bounded solutions of ODEs (still open and known as **Christol's conjecture**).

As a by-product of testing his conjecture he developed a classification of all globally bounded hypergeometric functions with rational parameters.

Global boundedness of hypergeometric function is only possible if $q = p - 1$ holds for the number of parameters, because of the restriction on the radius of convergence.

Idea: Counting multiplicities of prime numbers p in

$$\frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_{p-1})_n \; n!}.$$

Writing $a_i = r_i/d$, and $b_i = s_i/d$, this amounts to the p-adic evaluation of elements of the arithmetic progressions $r_i + k \cdot d$ and $s_i + k \cdot d$.

Christol's Interlacing Criterion

Define $\langle\cdot\rangle : \mathbb{R} \to (0,1]$ as the fractional part, where integers are assigned 1 instead of 0.
Define $\preceq$ on $\mathbb{R}^2$ via $a \preceq b$ if $\langle a\rangle < \langle b\rangle$ or $\langle a\rangle = \langle b\rangle$ and $a \geq b$.

Theorem (Christol, 1986)

Let

$$F(x) = {}_pF_{p-1}\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{p-1} \end{matrix}; x\right],$$

with rational parameters, $a_j, b_k \notin -\mathbb{N}$, denote by N the least common denominator of all parameters, and set $b_p = 1$. Then $F(x)$ is globally bounded if and only if for all $1 \leq \lambda \leq N$ with $\gcd(\lambda, N) = 1$ we have for all $1 \leq k \leq p$ that

$$|\{\lambda a_j \preceq \lambda b_k : 1 \leq j \leq p\}| - |\{\lambda b_j \preceq \lambda b_k : 1 \leq j \leq p\}| \geq 0.$$

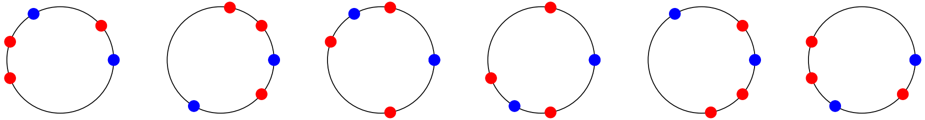
Christol's Interlacing Criterion

For $a_j - b_k \notin \mathbb{Z}$ the criterion can be interpreted graphically:

Draw the sets $\{\exp(2\pi i \lambda a_j)\}$ in red and $\{\exp(2\pi i \lambda b_k)\}$ in blue on the unit circle for all $1 \leq \lambda \leq N$ with $\gcd(\lambda, N) = 1$. Then F is globally bounded iff there are always at least as many red as blue points going counter-clockwise starting after 1 (count with multiplicity).

Example

${}_3F_2([1/9, 4/9, 5/9], [1/3, 1]; x)$ is globally bounded, as one can deduce from the pictures below. They correspond to $\lambda = 1, 2, 4, 5, 7, 8$ respectively.



Beukers–Heckman Interlacing Criterion

Theorem (Christol 1986, Beukers–Heckman 1989, Katz 1990)

Let

$$F(x) = {}_pF_{p-1}\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{p-1} \end{matrix}; x\right],$$

with rational parameters $a_j, b_k \notin -\mathbb{N}$ such that $a_j - b_k, a_j \notin \mathbb{Z}$, denote by N the least common denominator of all parameters, and set $b_p = 1$. Then $F(x)$ is algebraic if and only if for all $1 \leq \lambda \leq N$ with $\gcd(\lambda, N) = 1$ we have for all $1 \leq k \leq p$ that

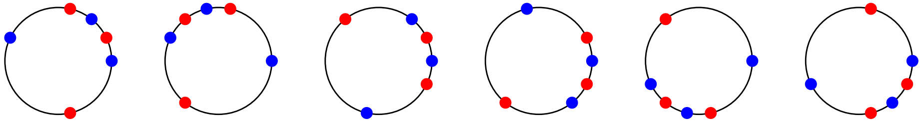
$$|\{\lambda a_j \preceq \lambda b_k : 1 \leq j \leq p\}| - |\{\lambda b_j \preceq \lambda b_k : 1 \leq j \leq p\}| = 0. \qquad \text{(IC)}$$

In other words, $F(x)$ is algebraic, if and only if the sets $\{2\pi i \lambda a_j\}$ and $\{2\pi i \lambda b_k\}$ **interlace** *on the unit circle for all λ.*

Beukers–Heckman Interlacing Criterion

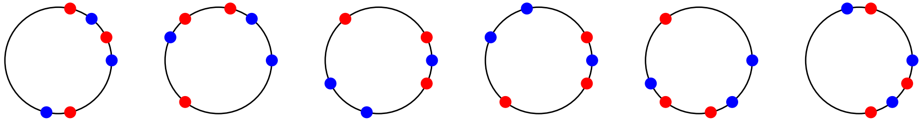
Example

$F(x) = {}_3F_2([1/14, 3/14, 11/14], [1/7, 3/7]; x)$ is algebraic:



Example

$F(x) = {}_3F_2([1/14, 3/14, 11/14], [1/7, 5/7]; x)$ is not algebraic:



Beukers–Heckman via Christol

Assuming that all parameters of $F(x)$ are pairwise disjoint modulo $\mathbb{Z}$, Christol's criterion for global boundedness is satisfied for $F(x)$ if and only if it is satisfied for **all** solutions of the hypergeometric differential equation and if and only if Beukers–Heckman interlacing (IC) holds.

> PROPOSITION 3 : *Toute fonction hypergéométrique F réduite et de hauteur 1 est globalement bornée si et seulement si, pour tout Δ tel que $(\Delta, N) = 1$, les nombres $\exp(2i\pi\Delta a_i)$ et $\exp(2i\pi\Delta b_i)$ sont entrelacés sur le cercle unité.*

Assuming the p-curvature conjecture, the interlacing criterion for algebraicity follows.

> COROLLAIRE (modulo la conjecture de GROTHENDIECK si $s>2$) : *Une fonction hypergéométrique de hauteur 1 est globalement bornée si et seulement si elle est algébrique.*

Hypergeometric equations are a special case of **factors of Picard-Fuchs differential equations**, a class of equations for which the conjecture was solved by Katz in 1972.

Ideas of Beukers–Heckman

Beukers and Heckman use a different approach:

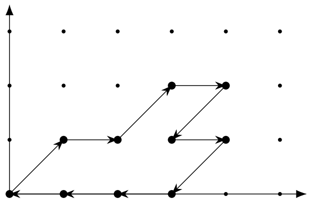
By analytic continuation of the solutions of the hypergeometric differential equation around its singularities $0, 1, \infty$ one obtains the **monodromy group** of the differential equation. It is finite if and only if all solutions are algebraic.

They construct an hermitian form, which is invariant under the monodromy group, which is positive definite if and only if the parameters interlace on the unit circle.

With this they show that the monodromy group is discrete and contained in the unitary group, which is compact, if and only if interlacing holds.

Example: Gessel Excursions

Lattice walks in the quaterplane with step set $\{\rightarrow, \leftarrow, \nearrow, \swarrow\}$: **Gessel walks**



Consider the generating function

$$G(x) = \sum_{n\geq 0} g_n x^n$$

of **excursions** of length n, i.e., walks with n steps that start and end at $(0,0)$.

Theorem (conjectured by Gessel 2001, Kauers–Koutschan–Zeilberger 2009, Bousquet-Mélou 2016, Bostan–Kurkova–Raschel 2017)

$$G(x) = \sum_{n\geq 0} \frac{(5/6)_n(1/2)_n}{(2)_n(5/3)_n} 16^n x^{2n} = {}_3F_2\left[\begin{matrix} \frac{5}{6}, \frac{1}{2}, 1 \\ 2, \frac{5}{3} \end{matrix}; 16x^2\right].$$

Example: Gessel Excursions

Is the generating function of Gessel excursions

$$G(x) = {}_3F_2\left[\begin{matrix}\frac{5}{6}, \frac{1}{2}, 1\\ 2, \frac{5}{3}\end{matrix}; 16x^2\right]$$

algebraic?

Direct application of the interlacing criterion is not possible, as $a_3 = 1 \in \mathbb{Z}$.
Trick: use identities for hypergeometric functions:

$$G(x) = \frac{1}{2x^2}\left({}_2F_1\left[\begin{matrix}-1/2, -1/6\\ 2/3\end{matrix}; 16x^2\right] - 1\right),$$

which is algebraic by Schwarz' classification (see earlier).

Algebraicity of $G(x)$ was overlooked until Bostan and Kauers proved the algebraicity of the trivariate generating function $Q(x, y, t)$ of Gessel walks ending at $(i, j) \in \mathbb{N}^2$ in 2010.

Irrational Parameters

Consider the recursion $(n+1)(n^2-2)u_{n+1} = 2(2n+1)(n^2+2n-1)u_n, \quad u_0 = 1.$

The generating function

$$\sum_{n\geq 0} u_n x^n = {}_3F_2\left[\begin{matrix}1/2, \sqrt{2}+1, -\sqrt{2}+1 \\ \sqrt{2}, -\sqrt{2}\end{matrix}; 4x\right] = \frac{(7x-1)(2x-1)}{(1-4x)^{5/2}}$$

is algebraic, although it has irrational parameters. The interlacing criterion is not applicable.

Recall: The interlacing criterion of Beukers and Heckman treats the case of $a_j, b_k \in \mathbb{Q} \setminus -\mathbb{N}$ with $a_j - b_k, a_j \notin \mathbb{Z}$.

Aim

An easy to use criterion to account for irrational parameters and integer differences.

Change of Setting

Define

$$\mathcal{F}\left[\begin{matrix} c_1, \dots, c_r \\ d_1, \dots, d_s \end{matrix}; x\right] := \sum_{n \geq 0} \frac{(c_1)_n \cdots (c_r)_n}{(d_1)_n \cdots (d_s)_n} x^n.$$

Note:

$$\begin{aligned} {}_pF_q\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; x\right] &= \mathcal{F}\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q, 1 \end{matrix}; x\right] \\ \mathcal{F}\left[\begin{matrix} c_1, \dots, c_r \\ d_1, \dots, d_s \end{matrix}; x\right] &= {}_{r+1}F_s\left[\begin{matrix} c_1, \dots, c_r, 1 \\ d_1, \dots, d_s \end{matrix}; x\right]. \end{aligned}$$

Definitions

$$F(x) = \mathcal{F}\left[\begin{matrix} c_1, \ldots, c_r \\ d_1, \ldots, d_s \end{matrix}; x\right] := \sum_{n \geq 0} \frac{(c_1)_n \cdots (c_r)_n}{(d_1)_n \cdots (d_s)_n} x^n.$$

$F(x)$ is **contracted** if $c_j - d_k \notin \mathbb{N}$. $F(x)$ is **reduced** if $c_j - d_k \notin \mathbb{Z}$.

The **contraction** $F^c(x)$ of $F(x)$ is obtained from $F(x)$ by removing pairs of parameters (c_j, d_k) with minimal difference $c_j - d_k \in \mathbb{N}$. It is contracted by definition.

If $F(x)$ is given as ${}_pF_q$, convert to $\mathcal{F}$ first.

Example

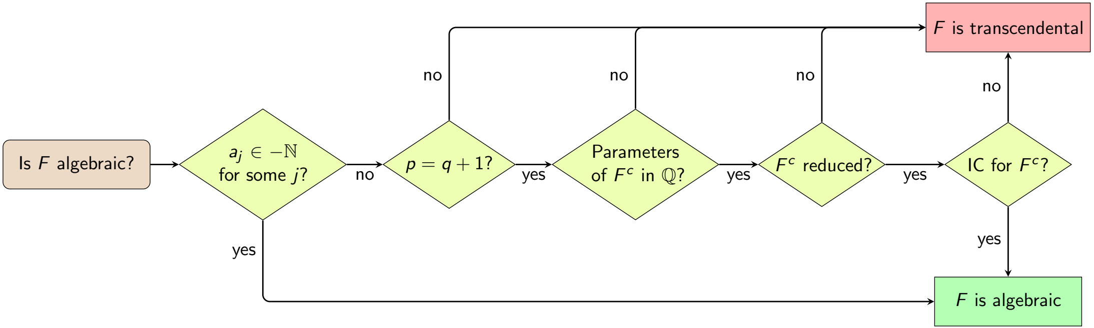
$${}_4F_3\left[\begin{matrix} \frac{1}{3}, \frac{1}{2}, 2, 4 \\ \frac{3}{2}, 3, 1 \end{matrix}; x\right]^c = \mathcal{F}\left[\begin{matrix} \frac{1}{3}, \frac{1}{2}, 2, 4 \\ \frac{3}{2}, 3, 1, 1 \end{matrix}; x\right]^c = \mathcal{F}\left[\begin{matrix} \frac{1}{3}, \frac{1}{2} \\ \frac{3}{2}, 1 \end{matrix}; x\right].$$

This contraction is not reduced, as $1/2 - 3/2 \in \mathbb{Z}$.

The Criterion

Theorem (F.–Yurkevich 2023)

For any hypergeometric function $F(x) = {}_pF_q([a_1, \ldots, a_p], [b_1, \ldots, b_q]; x) \in \mathbb{Q}[\![x]\!]$ the following decision tree answers the question whether it is algebraic over $\mathbb{Q}(x)$.



Ideas of the Proof

Proposition

The hypergeometric function $F(x)$ is algebraic if and only if $F^c(x)$ is algebraic.

Proof. If $a_1 + 1 \neq b_k$ for all k, then F is a $\bar{\mathbb{Q}}[x]$-linear combination of the functions

$$F^{+i} := {}_pF_{p-1}\left[\begin{matrix} a_1+1, \dots, a_i+1, a_{i+1}, \dots, a_p \\ b_1, \dots, b_{p-1} \end{matrix}; x\right],$$

a so-called **contiguous relation**, following from the hypergeometric differential equation and

$$(\theta + a_1)F(x) = a_1 F^{+1}(x) \qquad \left(\theta = x\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

As derivatives of algebraic series are algebraic, this proves that $F(x)$ is algebraic if and only if $F^{+1}(x)$ is and iteratively, the proposition follows. □

Ideas of the Proof

Proposition

If $F(x)$ is contracted, the hypergeometric differential equation is the differential equation of minimal order satisfied by $F(x)$.

All other solutions of the hypergeometric equations are linear combinations of series of the form $x^\rho G(x)$, where $G(x)$ is a hypergeometric function and $\rho \in \mathbb{C}$.

If one of the parameters of $F(x)$ is not rational, then one of the values of ρ is not rational, thus one solution is not algebraic. This contradicts the fact that all solutions of the minimal differential equation of an algebraic series are algebraic. From this we obtain:

Proposition

If $F(x)$ is contracted and has at least one irrational parameter, it is transcendental.

Ideas of the Proof

Proposition

If $F(x)$ is algebraic and contracted, but not reduced, then $F(x)$ is not algebraic.

Sketch of proof. Consider the hypergeometric functions

- $G(x)$ arising from $F(x)$ by removing all but one pair of integer differences between top and bottom parameters and
- $H(x)$ arising from $F(x)$ by removing all such pairs.

As $F(x)$ is algebraic and in particular globally bounded, so are $G(x)$ and $H(x)$.

$H(x)$ satisfies the interlacing criterion for algebraicity and $G(x)$ the interlacing criterion for global boundedness as both of these properties are preserved under removal of pairs of integer parameters. But these two facts contradict each other. □

Example 1

$$u_{n+1} = \frac{(14n+1)(14n+3)(14n+11)\left(n^2+2n+4\right)}{56(7n+1)(7n+3)(n+3)\left(n^2+3\right)} u_n, \quad u_0 = 1$$

Generating function:

$$f(x) = \mathcal{F}\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1+i\sqrt{3}, 1-i\sqrt{3} \\ \frac{1}{7}, \frac{3}{7}, i\sqrt{3}, -i\sqrt{3}, 3 \end{matrix}; x\right] = {}_6F_5\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1+i\sqrt{3}, 1-i\sqrt{3}, 1 \\ \frac{1}{7}, \frac{3}{7}, i\sqrt{3}, -i\sqrt{3}, 3 \end{matrix}; x\right].$$

Contraction has rational parameters and is reduced:

$$f^c(x) = \mathcal{F}\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14} \\ \frac{1}{7}, \frac{3}{7}, 3 \end{matrix}; x\right] = {}_4F_3\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1 \\ \frac{1}{7}, \frac{3}{7}, 3 \end{matrix}; x\right].$$

We have already seen that $f^c(x)$ is **algebraic** by the interlacing criterion, thus so is $f(x)$.

Example 2

$$u_n = \frac{3}{2}\binom{4n}{n}\frac{n+2}{(n+1)(n+3)}.$$

Generating function:

$$f(x) = {}_6F_5\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 3, 3, 1\\ \frac{1}{3}, \frac{2}{3}, 4, 2, 2\end{matrix}; \frac{256}{27}x\right]$$

Contraction:

$$f^c(x) = {}_4F_3\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\\ \frac{1}{3}, \frac{2}{3}, 4\end{matrix}; \frac{256}{27}x\right]$$

Interlacing criterion: $f(x)$ **algebraic**.

$$v_n = \frac{3}{2}\binom{4n}{n}\frac{n+2}{(n+1)^2}.$$

Generating function:

$$g(x) = {}_6F_5\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 3, 1, 1\\ \frac{1}{3}, \frac{2}{3}, 2, 2, 2\end{matrix}; \frac{256}{27}x\right]$$

Contraction:

$$g^c(x) = {}_5F_4\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1\\ \frac{1}{3}, \frac{2}{3}, 2, 2\end{matrix}; \frac{256}{27}x\right]$$

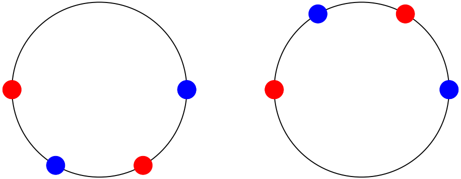
Not reduced: $g(x)$ **not algebraic**.

Example 3 – Gessel Revisited

Recall the generating function of Gessel excursions

$$G(x) = {}_3F_2\left[\begin{matrix}\frac{5}{6}, \frac{1}{2}, 1\\ 2, \frac{5}{3}\end{matrix}; 16x^2\right] = \mathcal{F}\left[\begin{matrix}\frac{5}{6}, \frac{1}{2}\\ 2, \frac{5}{3}\end{matrix}; x\right].$$

$G(x)$ is contracted, reduced, has only rational parameters and satisfies the interlacing criterion:



The End

Thank you for your attention!