Algebraicity of Hypergeometric Functions with Arbitrary Parameters

joint work with S. Yurkevich (arXiv:2308.12855)

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Definitions

Hypergeometric function:

$$
{}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{x^n}{n!},
$$

where $(a)_n := a(a+1)\cdots(a+n-1)$ denotes the **rising factorial.**

Hypergeometric sequence:

$$
u_{n+1} = \frac{A(n)}{B(n)} u_n,
$$

where $A(t), B(t) \in \mathbb{Q}[t]$ are polynomials.

Hypergeometric functions are generating functions of hypergeometric sequences.

Examples

- Logarithm:

$$ {}_2F_1\left[\begin{matrix}1,1\\2\end{matrix};x\right] = -\frac{\log(1-x)}{x} = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \ldots \in \mathbb{Q}[\![x]\!] $$

- **Catalan numbers:**

$$ C_n = \binom{2n}{n}\frac{1}{n+1} \in \mathbb{Z}, \quad \frac{C_{n+1}}{C_n} = \frac{(2n+1)(2n+2)(n+1)}{(n+1)(n+1)(n+2)}, \quad \sum_{n\geq 0} C_n x^n = {}_2F_1\left[\begin{matrix}\frac{1}{2},1\\2\end{matrix};4x\right] $$

- **Chebychev numbers:**

$$ \frac{(30n)!n!}{(15n)!(10n)!(6n)!} \in \mathbb{Z} $$

- Some other algebraic series, such as

$$ {}_3F_2\left[\begin{matrix}1/2,\sqrt{2}+1,-\sqrt{2}+1\\\sqrt{2},-\sqrt{2}\end{matrix};4x\right] = \frac{(7x-1)(2x-1)}{(1-4x)^{5/2}} = 1 + x - 6x^2 + \cdots \in \mathbb{Z}[\![x]\!] $$

Definitions

A power series $f(x) \in \mathbb{Q}[\![x]\!]$ is called **algebraic** (over $\mathbb{Q}(x)$) if there is $P(x, y) \in \mathbb{Q}[x, y]$, $P(x, y) \neq 0$, such that $P(x, f(x)) = 0$.

A power series $f(x) \in \mathbb{Q}[\![x]\!]$ is called **almost integral** if there are $\alpha, \beta \in \mathbb{Z} \setminus \{0\}$, such that $\beta f(\alpha x) \in \mathbb{Z}[\![x]\!]$.
In particular, only finitely many prime numbers appear in the denominators of the coefficients.

A power series $f(x) \in \mathbb{Q}[\![x]\!]$ is called **globally bounded** if it is almost integral and its convergence radius is nonzero and finite.

Theorem (Eisenstein 1852, Heine 1854)

Any algebraic $f(x) \in \mathbb{Q}[\![x]\!]$ is almost integral. More precisely, $f(x)$ is a polynomial or globally bounded.

Question

Which hypergeometric functions are algebraic?

The hypergeometric function

$$ {}_2F_1\left[\begin{matrix}1,1\\2\end{matrix};x\right] = -\frac{\log(1-x)}{x} = 1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\ldots \in \mathbb{Q}[\![x]\!] $$

clearly is **not algebraic.** It is not even globally bounded.

The function

$$ {}_3F_2\left[\begin{matrix}1/2, \sqrt{2}+1, -\sqrt{2}+1\\ \sqrt{2}, -\sqrt{2}\end{matrix}; 4x\right] = \frac{(7x-1)(2x-1)}{(1-4x)^{5/2}} $$

clearly is **algebraic.**

Schwarz' Classification

Schwarz 1873: Classification of all algebraic **Gaussian hypergeometric functions**, i.e., all $F(x) = {}_2F_1([a_1, a_2], [b_1]; x)$, with rational parameters $a_1, a_2, b_1 \in \mathbb{Q}$:

Define $(\lambda, \mu, \nu) = (1 - b_1, b_1 - a_1 - a_2, a_2 - a_1)$. Schwarz provided a list of triples, such that $F(x)$ is algebraic (assuming $\lambda, \mu, \nu \notin \mathbb{Z}$), iff (λ, μ, ν) appears in the list, up to permutations, sign changes and addition of triples of integers with even sum.

Example

$F(x) = {}_2F_1([-1/2, -1/6], [2/3]; x)$ is algebraic, as $(\lambda, \mu, \nu) = (1/3, 4/3, 1/3)$ and $(-(\nu - 1), \lambda, \mu - 1) = (2/3, 1/3, 1/3)$ is in the list.

No.	λ''	μ''	ν''	Inhalt / π	Polyeder
I.	$\frac{1}{2}$	$\frac{1}{2}$	ν	ν	Regelmässige Doppelpyramide
II.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6} = A$	Tetraeder
III.	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3} = 2A$	
IV.	[illegible]	[illegible]	[illegible]	[illegible]	

Landau-Errera Criterion

Landau 1904, 1911 exploited Eisenstein's Theorem, leading to a necessary condition for algebraicity of Gaussian hypergeometric functions with rational parameters:

Theorem (Landau)

Let $F(x) = {}_2F_1([a_1, a_2], [b_1]; x)$ with $a_1, a_2, b_1, a_1 - b_1, a_2 - b_1 \notin \mathbb{Z}$. Then $F(x)$ is globally bounded iff for all $1 \leq \lambda \leq N$ coprime to the common denominator N of a_1, a_2, b_1 we have

$$\langle \lambda a_1 \rangle < \langle \lambda b_1 \rangle < \langle \lambda a_2 \rangle \quad \text{or} \quad \langle \lambda a_2 \rangle < \langle \lambda b_1 \rangle < \langle \lambda a_1 \rangle, \tag{$\star$}$$

where $\langle \cdot \rangle$ denotes the fractional part.

Errera 1913 extended this to a criterion for algebraicity:

Theorem (Errera)

Condition $(\star)$ for all λ is equivalent to Schwarz' classification.

Christol's Interlacing Criterion

Define $\langle\cdot\rangle : \mathbb{R} \to (0,1]$ as the fractional part, where integers are assigned 1 instead of 0.
Define $\preceq$ on $\mathbb{R}^2$ via $a \preceq b$ if $\langle a\rangle < \langle b\rangle$ or $\langle a\rangle = \langle b\rangle$ and $a \geq b$.

Theorem (Christol, 1986)

Let

$$F(x) = {}_pF_{p-1}\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{p-1} \end{matrix}; x\right],$$

with rational parameters, $a_j, b_k \notin -\mathbb{N}$, denote by N the least common denominator of all parameters, and set $b_p = 1$. Then $F(x)$ is globally bounded if and only if for all $1 \leq \lambda \leq N$ with $\gcd(\lambda, N) = 1$ we have for all $1 \leq k \leq p$ that

$$|\{\lambda a_j \preceq \lambda b_k : 1 \leq j \leq p\}| - |\{\lambda b_j \preceq \lambda b_k : 1 \leq j \leq p\}| \geq 0.$$

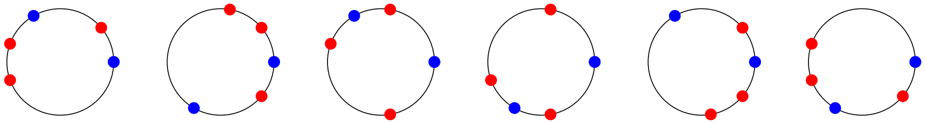
Christol's Interlacing Criterion

For $a_j - b_k \notin \mathbb{Z}$ the criterion can be interpreted graphically:

Draw the sets $\{\exp(2\pi i \lambda a_j)\}$ in red and $\{\exp(2\pi i \lambda b_k)\}$ in blue on the unit circle for all $1 \le \lambda \le N$ with $\gcd(\lambda, N) = 1$. Then F is globally bounded iff there are always at least as many red as blue points going counter-clockwise starting after 1 (count with multiplicity).

Example

${}_3F_2([1/9, 4/9, 5/9], [1/3, 1]; x)$ is globally bounded, as one can deduce from the pictures below. They correspond to $\lambda = 1, 2, 4, 5, 7, 8$ respectively.



Beukers–Heckman Interlacing Criterion

Theorem (Christol 1986, Beukers–Heckman 1989, Katz 1990)

Let

$$F(x) = {}_pF_{p-1}\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{p-1} \end{matrix}; x\right],$$

with rational parameters $a_j, b_k \notin -\mathbb{N}$ *such that* $a_j - b_k, a_j \notin \mathbb{Z}$, *denote by* N *the least common denominator of all parameters, and set* $b_p = 1$. *Then* $F(x)$ *is algebraic if and only if for all* $1 \le \lambda \le N$ *with* $\gcd(\lambda, N) = 1$ *we have for all* $1 \le k \le p$ *that*

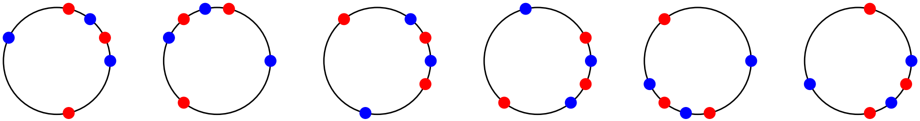
$$|\{\lambda a_j \preceq \lambda b_k : 1 \le j \le p\}| - |\{\lambda b_j \preceq \lambda b_k : 1 \le j \le p\}| = 0. \tag{IC}$$

In other words, $F(x)$ *is algebraic, if and only if the sets* $\{2\pi i \lambda a_j\}$ *and* $\{2\pi i \lambda b_k\}$ **interlace** *on the unit circle for all* λ.

Beukers–Heckman Interlacing Criterion

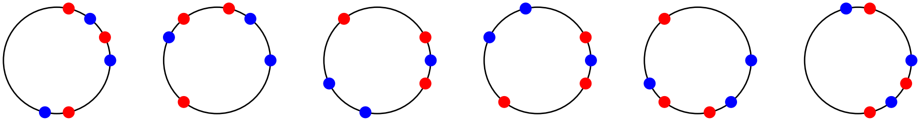
Example

$F(x) = {}_3F_2([1/14, 3/14, 11/14], [1/7, 3/7]; x)$ is algebraic:



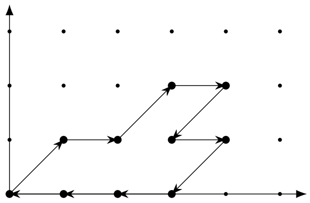
Example

$F(x) = {}_3F_2([1/14, 3/14, 11/14], [1/7, 5/7]; x)$ is not algebraic:



Example: Gessel Excursions

Lattice walks in the quaterplane with step set $\{\rightarrow, \leftarrow, \nearrow, \swarrow\}$: **Gessel walks**



Consider the generating function

$$G(x) = \sum_{n\geq 0} g_n x^n$$

of **excursions** of length n, i.e., walks with n steps that start and end at $(0,0)$.

Theorem (conjectured by Gessel 2001, Kauers–Koutschan–Zeilberger 2009, Bousquet-Mélou 2016, Bostan–Kurkova–Raschel 2017)

$$G(x) = \sum_{n\geq 0} \frac{(5/6)_n (1/2)_n}{(2)_n (5/3)_n} 16^n x^{2n} = {}_3F_2\left[\begin{matrix} \frac{5}{6}, \frac{1}{2}, 1 \\ 2, \frac{5}{3} \end{matrix}; 16x^2\right].$$

Example: Gessel Excursions

Is the generating function of Gessel excursions

$$G(x) = {}_3F_2\left[\begin{matrix}\frac{5}{6}, \frac{1}{2}, 1\\ 2, \frac{5}{3}\end{matrix}; 16x^2\right]$$

algebraic?

Direct application of the interlacing criterion is not possible, as $a_3 = 1 \in \mathbb{Z}$.
Trick: use identities for hypergeometric functions:

$$G(x) = \frac{1}{2x^2}\left({}_2F_1\left[\begin{matrix}-1/2, -1/6\\ 2/3\end{matrix}; 16x^2\right] - 1\right),$$

which is algebraic by Schwarz' classification (see earlier).

Algebraicity of $G(x)$ was overlooked until Bostan and Kauers proved the algebraicity of the trivariate generating function $Q(x, y, t)$ of Gessel walks ending at $(i, j) \in \mathbb{N}^2$ in 2010.

Irrational Parameters

Consider the recursion $(n+1)(n^2-2)u_{n+1} = 2(2n+1)(n^2+2n-1)u_n, \quad u_0 = 1.$

The generating function

$$\sum_{n\geq 0} u_n x^n = {}_3F_2\left[\begin{matrix}1/2, \sqrt{2}+1, -\sqrt{2}+1 \\ \sqrt{2}, -\sqrt{2}\end{matrix}; 4x\right] = \frac{(7x-1)(2x-1)}{(1-4x)^{5/2}}$$

is algebraic, although it has irrational parameters. The interlacing criterion is not applicable.

Recall: The interlacing criterion of Beukers and Heckman treats the case of $a_j, b_k \in \mathbb{Q} \setminus -\mathbb{N}$ with $a_j - b_k, a_j \notin \mathbb{Z}$.

Aim

An easy to use criterion to account for irrational parameters and integer differences.

Change of Setting

Define

$$\mathcal{F}\left[\begin{matrix} c_1, \ldots, c_r \\ d_1, \ldots, d_s \end{matrix}; x\right] := \sum_{n\geq 0} \frac{(c_1)_n \cdots (c_r)_n}{(d_1)_n \cdots (d_s)_n} x^n.$$

Note:

$$\begin{aligned} {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right] &= \mathcal{F}\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q, 1 \end{matrix}; x\right] \\ \mathcal{F}\left[\begin{matrix} c_1, \ldots, c_r \\ d_1, \ldots, d_s \end{matrix}; x\right] &= {}_{r+1}F_s\left[\begin{matrix} c_1, \ldots, c_r, 1 \\ d_1, \ldots, d_s \end{matrix}; x\right]. \end{aligned}$$

Definitions

$$F(x) = \mathcal{F}\left[\begin{matrix} c_1, \dots, c_r \\ d_1, \dots, d_s \end{matrix}; x\right] := \sum_{n\geq 0} \frac{(c_1)_n \cdots (c_r)_n}{(d_1)_n \cdots (d_s)_n} x^n.$$

$F(x)$ is **contracted** if $c_j - d_k \notin \mathbb{N}$. $F(x)$ is **reduced** if $c_j - d_k \notin \mathbb{Z}$.

The **contraction** $F^c(x)$ of $F(x)$ is obtained from $F(x)$ by removing pairs of parameters (c_j, d_k) with minimal difference $c_j - d_k \in \mathbb{N}$. It is contracted by definition.

If $F(x)$ is given as ${}_pF_q$, convert to $\mathcal{F}$ first.

Example

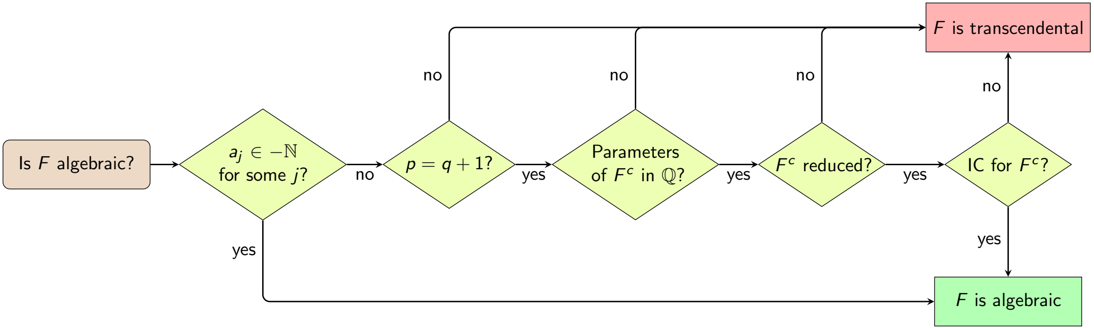
$${}_4F_3\left[\begin{matrix} \frac{1}{3}, \frac{1}{2}, 2, 4 \\ \frac{3}{2}, 3, 1 \end{matrix}; x\right]^c = \mathcal{F}\left[\begin{matrix} \frac{1}{3}, \frac{1}{2}, 2, 4 \\ \frac{3}{2}, 3, 1, 1 \end{matrix}; x\right]^c = \mathcal{F}\left[\begin{matrix} \frac{1}{3}, \frac{1}{2} \\ \frac{3}{2}, 1 \end{matrix}; x\right].$$

This contraction is not reduced, as $1/2 - 3/2 \in \mathbb{Z}$.

The Criterion

Theorem (F.–Yurkevich 2023)

For any hypergeometric function $F(x) = {}_pF_q([a_1, \dots, a_p], [b_1, \dots, b_q]; x) \in \mathbb{Q}[\![x]\!]$ *the following decision tree answers the question whether it is algebraic over* $\mathbb{Q}(x)$.



Example 1

$$u_{n+1} = \frac{(14n+1)(14n+3)(14n+11)\left(n^2+2n+4\right)}{56(7n+1)(7n+3)(n+3)\left(n^2+3\right)} u_n, \quad u_0 = 1$$

Generating function:

$$f(x) = \mathcal{F}\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1+i\sqrt{3}, 1-i\sqrt{3} \\ \frac{1}{7}, \frac{3}{7}, i\sqrt{3}, -i\sqrt{3}, 3 \end{matrix} ; x\right] = {}_6F_5\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1+i\sqrt{3}, 1-i\sqrt{3}, 1 \\ \frac{1}{7}, \frac{3}{7}, i\sqrt{3}, -i\sqrt{3}, 3 \end{matrix} ; x\right].$$

Contraction has rational parameters and is reduced:

$$f^c(x) = \mathcal{F}\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14} \\ \frac{1}{7}, \frac{3}{7}, 3 \end{matrix} ; x\right] = {}_4F_3\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1 \\ \frac{1}{7}, \frac{3}{7}, 3 \end{matrix} ; x\right].$$

We have already seen that $f^c(x)$ is **algebraic** by the interlacing criterion, thus so is $f(x)$.

Example 2

$$u_n = \frac{3}{2}\binom{4n}{n}\frac{n+2}{(n+1)(n+3)}.$$

Generating function:

$$f(x) = {}_6F_5\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 3, 3, 1\\ \frac{1}{3}, \frac{2}{3}, 4, 2, 2\end{matrix}; \frac{256}{27}x\right]$$

Contraction:

$$f^c(x) = {}_4F_3\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\\ \frac{1}{3}, \frac{2}{3}, 4\end{matrix}; \frac{256}{27}x\right]$$

Interlacing criterion: $f(x)$ **algebraic**.

$$v_n = \frac{3}{2}\binom{4n}{n}\frac{n+2}{(n+1)^2}.$$

Generating function:

$$g(x) = {}_6F_5\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 3, 1, 1\\ \frac{1}{3}, \frac{2}{3}, 2, 2, 2\end{matrix}; \frac{256}{27}x\right]$$

Contraction:

$$g^c(x) = {}_5F_4\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1\\ \frac{1}{3}, \frac{2}{3}, 2, 2\end{matrix}; \frac{256}{27}x\right]$$

Not reduced: $g(x)$ **not algebraic**.

The End

Thank you for your attention!