qRSt: A probabilistic Robinson-Schensted correspondence for Macdonald polynomials

Florian Aigner

joint work with Gabriel Frieden

Algebraic Combinatorics Seminar @ Waterloo, 29.10.2020

Outline

- The classical Robinson-Schensted correspondence
- Macdonald polynomials
- Probabilistic bijections
- A probabilistic Robinson-Schensted correspondence
- Properties of qRSt

Semistandard Young tableaux

Definition

Let λ be a partition. A semistandard Young tableau (SSYT) T of shape λ is a filling of the cells of λ with positive integers such that

- the rows are weakly increasing from left to right,
- the columns are strictly increasing from bottom to top (French notation).

Denote by $\mathrm{SSYT}(\lambda)$ the set of SSYTs of shape λ.

Example.

4	5			
2	3	4	4	
1	1	2	3	3

Semistandard Young tableaux

Definition

Let λ be a partition. A semistandard Young tableau (SSYT) T of shape λ is a filling of the cells of λ with positive integers such that

- the rows are weakly increasing from left to right,
- the columns are strictly increasing from bottom to top (French notation).

Denote by SSYT(λ) the set of SSYTs of shape λ.

Example.

4	5			
2	3	4	4	
1	1	2	3	3

$(2, 2, 3, 3, 1)$

The content of an SSYT T is $(\mu_1, \mu_2, \ldots)$ where μ_i is the number of entries i in T; denote by $\mathbf{x}^T = \prod_i x_i^{\mu_i}$.

Schur polynomials

Definition

Let λ be a partition. The Schur polynomial $s_\lambda(\mathbf{x})$ is defined as the sum

$$\sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{x}^T.$$

Theorem (Cauchy identity)

For two sequences of indeterminates $\mathbf{x} = (x_1, x_2 \ldots)$ *and* $\mathbf{y} = (y_1, y_2, \ldots)$*, we have*

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}).$$

Schur polynomials

Definition

Let λ be a partition. The Schur polynomial $s_\lambda(\mathbf{x})$ is defined as the sum

$$\sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{x}^T.$$

Theorem (Cauchy identity)

For two sequences of indeterminates $\mathbf{x} = (x_1, x_2 \ldots)$ *and* $\mathbf{y} = (y_1, y_2, \ldots)$, *we have*

$$\sum_{A=(a_{i,j})} \prod_{i,j} (x_i y_j)^{a_{i,j}} = \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}).$$

In this talk we are interested in the squarefree part, i.e., the coefficient of $x_1 \cdots x_n y_1 \cdots y_n$.

Young's lattice

For two partitions λ, μ we write

- $\mu \subseteq \lambda$ if the Young diagram of μ is contained in that of λ,
- $\mu \lessdot \lambda$, if λ covers μ, i.e., $\mu \subseteq \lambda$ and $|\lambda| = |\mu| + 1$.

We define the up operator U and down operator D on the $\mathbb{Q}$-vector space generated by all partitions as

$$U\lambda = \sum_{\nu \gtrdot \lambda} \nu, \qquad D\lambda = \sum_{\mu \lessdot \lambda} \mu.$$

Example. $U\left(\ydiagram{2,2}\right) = \ydiagram{3,2} + \ydiagram{2,2,1}, \qquad D\left(\ydiagram{2,2}\right) = \ydiagram{2,1}.$

Young's lattice

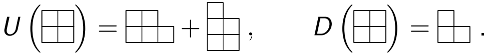
For two partitions λ, μ we write

- $\mu \subseteq \lambda$ if the Young diagram of μ is contained in that of λ,
- $\mu \lessdot \lambda$, if λ covers μ, i.e., $\mu \subseteq \lambda$ and $|\lambda| = |\mu| + 1$.

We define the up operator U and down operator D on the $\mathbb{Q}$-vector space generated by all partitions as

$$U\lambda = \sum_{\nu \gtrdot \lambda} \nu, \qquad D\lambda = \sum_{\mu \lessdot \lambda} \mu.$$

Example.



Theorem

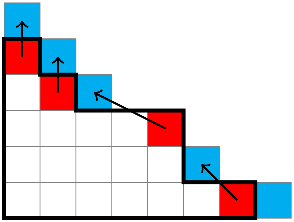
The two operator satisfy the commutation relation

$$DU - UD = I.$$

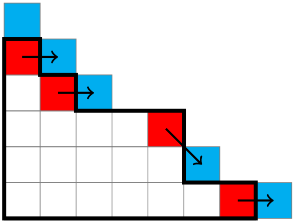
Bijective proof of the commutation relation

The commutation relation $DU = UD + I$ is equivalent to

$$|\{\nu \gtrdot \lambda\}| = |\{\mu \lessdot \lambda\} \cup \{\lambda\}|, \qquad \forall \lambda,$$
$$|\{\nu \gtrdot \lambda, \rho\}| = |\{\mu \lessdot \lambda, \rho\}|, \qquad \forall \lambda \neq \rho.$$



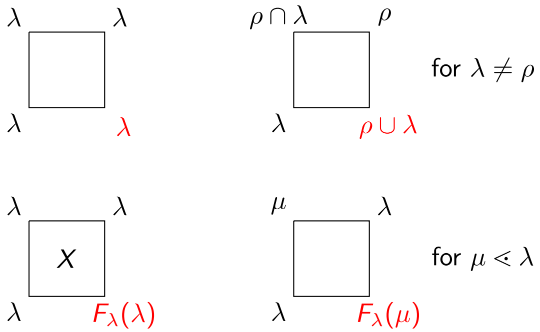
F_λ^{row}



F_λ^{col}

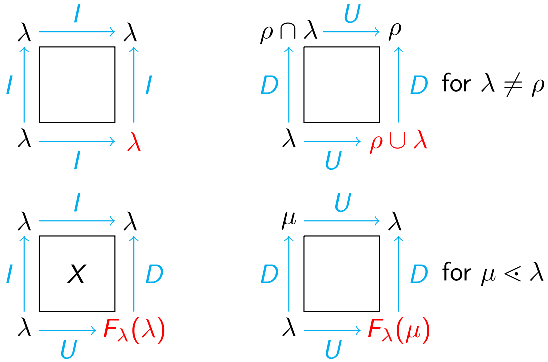
Local growth rules

Let $F_\lambda : \{\mu \lessdot \lambda\} \cup \{\lambda\} \to \{\nu \gtrdot \lambda\}$ is a bijection; in our case $F_\lambda = F_\lambda^{\text{row}}$. The local growth rules are an assignment of a partition to the bottom right corner of a square according to one of the four cases



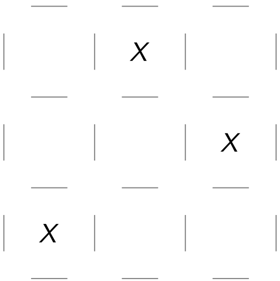
Local growth rules

Let $F_\lambda : \{\mu \lessdot \lambda\} \cup \{\lambda\} \to \{\nu \gtrdot \lambda\}$ is a bijection; in our case $F_\lambda = F_\lambda^{\text{row}}$.
The local growth rules are an assignment of a partition to the bottom right corner of a square according to one of the four cases



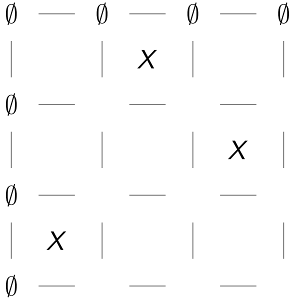
Fomin growth diagram

We consider a permutation matrix as an $n \times n$ grid of squares and associate permutations to the vertices recursively following the local growth rules.



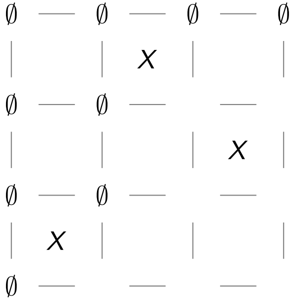
Fomin growth diagram

We consider a permutation matrix as an $n \times n$ grid of squares and associate permutations to the vertices recursively following the local growth rules.



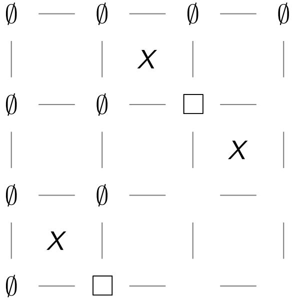
Fomin growth diagram

We consider a permutation matrix as an $n \times n$ grid of squares and associate permutations to the vertices recursively following the local growth rules.



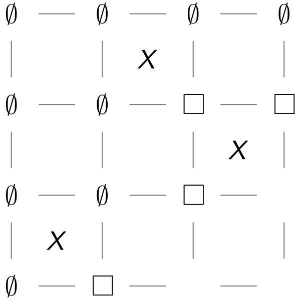
Fomin growth diagram

We consider a permutation matrix as an $n \times n$ grid of squares and associate permutations to the vertices recursively following the local growth rules.



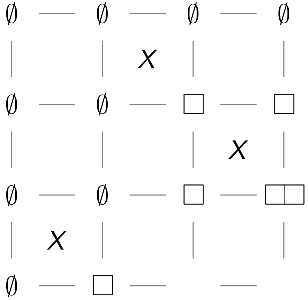
Fomin growth diagram

We consider a permutation matrix as an $n \times n$ grid of squares and associate permutations to the vertices recursively following the local growth rules.



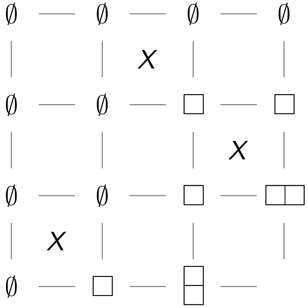
Fomin growth diagram

We consider a permutation matrix as an $n \times n$ grid of squares and associate permutations to the vertices recursively following the local growth rules.



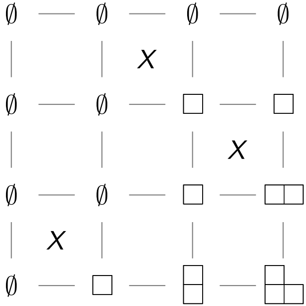
Fomin growth diagram

We consider a permutation matrix as an $n \times n$ grid of squares and associate permutations to the vertices recursively following the local growth rules.



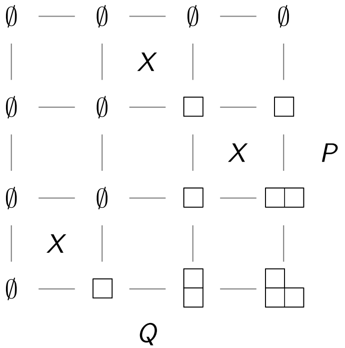
Fomin growth diagram

We consider a permutation matrix as an $n \times n$ grid of squares and associate permutations to the vertices recursively following the local growth rules.



Fomin growth diagram

We consider a permutation matrix as an $n \times n$ grid of squares and associate permutations to the vertices recursively following the local growth rules.



The ith partition along the right (bottom) boundary give the shape of the subtableau of P (Q) with entries at most i.

In our example we obtain

$$(P, Q) = \left(\begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \right)$$

Macdonald Polynomials

We define Macdonald polynomials by

$$P_\lambda(\mathbf{x}; q, t) = \sum_{T \in \mathrm{SSYT}(\lambda)} \psi_T(q,t)\mathbf{x}^T,$$

$$Q_\lambda(\mathbf{x}; q, t) = \sum_{T \in \mathrm{SSYT}(\lambda)} \varphi_T(q,t)\mathbf{x}^T,$$

where ψ, φ are certain rational functions in q, t.

Theorem (Macdonald)

Let $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2 \ldots)$ be two sets of variables. Then

$$\prod_{i,j \geq 1} \prod_{k=0}^{\infty} \frac{1 - t x_i y_j q^k}{1 - x_i y_j q^k} = \sum_\lambda P_\lambda(\mathbf{x}; q, t) Q_\lambda(\mathbf{y}; q, t).$$

Again we are interested in the squarefree part, i.e., the coefficient of $x_1 \cdots x_n y_1 \cdots y_n$.

Macdonald Polynomials

We define Macdonald polynomials by

$$P_\lambda(\mathbf{x};q,t) = \sum_{T\in \mathrm{SSYT}(\lambda)} \psi_T(q,t)\mathbf{x}^T,$$

$$Q_\lambda(\mathbf{x};q,t) = \sum_{T\in \mathrm{SSYT}(\lambda)} \varphi_T(q,t)\mathbf{x}^T,$$

where ψ, φ are certain rational functions in q, t.

Theorem (Macdonald)

Let $\mathbf{x} = (x_1, x_2, \ldots)$ *and* $\mathbf{y} = (y_1, y_2 \ldots)$ *be two sets of variables. Then*

$$\sum_{A=(a_{i,j})} \prod_{i,j\geq 1} (x_i y_j)^{a_{i,j}} \prod_{k=0}^{a_{i,j}-1} \frac{1-tq^k}{1-q^{k+1}} = \sum_\lambda P_\lambda(\mathbf{x};q,t) Q_\lambda(\mathbf{y};q,t).$$

Again we are interested in the squarefree part, i.e., the coefficient of $x_1 \cdots x_n y_1 \cdots y_n$.

The coefficient of $x_1x_2y_1y_2$ of the Cauchy identity

weight of A	A	(P, Q)	$\psi_P(q,t)\varphi_Q(q,t)$
$\frac{(1-t)^2}{(1-q)^2}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\boxed{1}\boxed{2}, \boxed{1}\boxed{2}$	$\frac{(1-t)^3(1-q^2)}{(1-q)^3(1-qt)}$
$\frac{(1-t)^2}{(1-q)^2}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}, \begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\frac{(1-t)(1-t^2)}{(1-q)(1-qt)}$

The Ups and Downs of Macdonald polynomials

The Macdonald weights are defined "recursively":

$$\psi_T(q,t) = \prod_i \psi_{T^{(i)}/T^{(i-1)}}(q,t), \qquad \varphi_T(q,t) = \prod_i \varphi_{T^{(i)}/T^{(i-1)}}(q,t),$$

where $T^{(i)}$ is the shape of the subtableau of an SSYT T of entries at most i. The ψ, φ are again rational functions in q, t.

We define the (q,t)-up and down operator as

$$U_{q,t}\lambda = \sum_{\nu \gtrdot \lambda} \psi_{\nu/\lambda}(q,t)\nu, \qquad D_{q,t}\lambda = \sum_{\mu \lessdot \lambda} \varphi_{\lambda/\mu}(q,t)\mu.$$

The Ups and Downs of Macdonald polynomials

The Macdonald weights are defined "recursively":

$$\psi_T(q,t) = \prod_i \psi_{T^{(i)}/T^{(i-1)}}(q,t), \qquad \varphi_T(q,t) = \prod_i \varphi_{T^{(i)}/T^{(i-1)}}(q,t),$$

where $T^{(i)}$ is the shape of the subtableau of an SSYT T of entries at most i. The ψ, φ are again rational functions in q, t.

We define the (q,t)-up and down operator as

$$U_{q,t}\lambda = \sum_{\nu \gtrdot \lambda} \psi_{\nu/\lambda}(q,t)\nu, \qquad D_{q,t}\lambda = \sum_{\mu \lessdot \lambda} \varphi_{\lambda/\mu}(q,t)\mu.$$

Theorem

The (q,t)-up and down operator satisfy the commutation relation

$$D_{q,t}U_{q,t} - U_{q,t}D_{q,t} = \frac{1-q}{1-t} I.$$

An equivalent formulation

The commutation relation

$$D_{q,t}U_{q,t} = U_{q,t}D_{q,t} + \frac{1-q}{1-t}I,$$

is equivalent to the two equations

$$\sum_{\nu \gtrdot \lambda, \rho} \psi_{\nu/\lambda}(q,t)\varphi_{\nu/\rho}(q,t) = \sum_{\mu \lessdot \lambda, \rho} \varphi_{\lambda/\mu}(q,t)\psi_{\rho/\mu}(q,t),$$

$$\sum_{\nu \gtrdot \lambda} \psi_{\nu/\lambda}(q,t)\varphi_{\nu/\lambda}(q,t) = \frac{1-q}{1-t} + \sum_{\mu \lessdot \lambda} \varphi_{\lambda/\mu}(q,t)\psi_{\lambda/\mu}(q,t),$$

for all $\lambda \neq \rho$.

Probabilistic bijections

Let X, Y be two sets equipped with weight functions $\omega : X \to k$, $\overline{\omega} : Y \to k$, where k is a field. A probabilistic bijection from (X, ω) to $(Y, \overline{\omega})$ is a pair of maps $\mathcal{P}, \overline{\mathcal{P}} : X \times Y \to k$ such that

$$\sum_{y \in Y} \mathcal{P}(x, y) = 1 \qquad \forall x \in X,$$

$$\sum_{x \in X} \overline{\mathcal{P}}(x, y) = 1 \qquad \forall y \in Y,$$

$$\omega(x)\mathcal{P}(x, y) = \overline{\omega}(y)\overline{\mathcal{P}}(x, y) \qquad \forall x \in X, y \in Y.$$

We usually write $\mathcal{P}(x \to y) := \mathcal{P}(x, y)$ and $\overline{\mathcal{P}}(x \leftarrow y) := \overline{\mathcal{P}}(x, y)$.

Probabilistic bijections

Let X, Y be two sets equipped with weight functions $\omega : X \to k$, $\overline{\omega} : Y \to k$, where k is a field. A probabilistic bijection from (X, ω) to $(Y, \overline{\omega})$ is a pair of maps $\mathcal{P}, \overline{\mathcal{P}} : X \times Y \to k$ such that

$$\sum_{y \in Y} \mathcal{P}(x \to y) = 1 \qquad \forall x \in X,$$

$$\sum_{x \in X} \overline{\mathcal{P}}(x \leftarrow y) = 1 \qquad \forall y \in Y,$$

$$\omega(x)\mathcal{P}(x \to y) = \overline{\omega}(y)\overline{\mathcal{P}}(x \leftarrow y) \qquad \forall x \in X, y \in Y.$$

We usually write $\mathcal{P}(x \to y) := \mathcal{P}(x, y)$ and $\overline{\mathcal{P}}(x \leftarrow y) := \overline{\mathcal{P}}(x, y)$.

Probabilistic bijections

Let X, Y be two sets equipped with weight functions $\omega : X \to k$, $\overline{\omega} : Y \to k$, where k is a field. A probabilistic bijection from (X, ω) to $(Y, \overline{\omega})$ is a pair of maps $\mathcal{P}, \overline{\mathcal{P}} : X \times Y \to k$ such that

$$\sum_{y \in Y} \mathcal{P}(x \to y) = 1 \qquad \forall x \in X,$$

$$\sum_{x \in X} \overline{\mathcal{P}}(x \leftarrow y) = 1 \qquad \forall y \in Y,$$

$$\omega(x)\mathcal{P}(x \to y) = \overline{\omega}(y)\overline{\mathcal{P}}(x \leftarrow y) \qquad \forall x \in X, y \in Y.$$

We usually write $\mathcal{P}(x \to y) := \mathcal{P}(x, y)$ and $\overline{\mathcal{P}}(x \leftarrow y) := \overline{\mathcal{P}}(x, y)$.

Lemma

If $\mathcal{P}, \overline{\mathcal{P}}$ is a probabilistic bijection from (X, ω) to $(Y, \overline{\omega})$, then

$$\sum_{x \in X} \omega(x) = \sum_{y \in Y} \overline{\omega}(y).$$

The weighted sets

We regard the sets $\{\mu \lessdot \lambda\} \cup \{\lambda\}$ and $\{\nu \gtrdot \lambda\}$ with weights

$$\omega(\mu) = \begin{cases} 1 & \mu = \lambda, \\ \frac{1-q}{1-t} \varphi_{\lambda/\mu}(q,t)\psi_{\lambda/\mu}(q,t) & \text{otherwise,} \end{cases}$$

$$\overline{\omega}(\nu) = \frac{1-q}{1-t} \psi_{\nu/\lambda}(q,t)\varphi_{\nu/\lambda}(q,t).$$

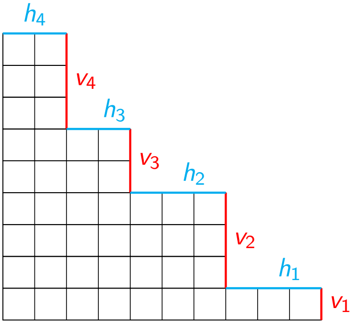
Hence, we need to show $\displaystyle\sum_{\mu \lessdot \lambda \text{ or } \mu = \lambda} \omega(\mu) = \sum_{\nu \gtrdot \lambda} \overline{\omega}(\nu).$

We prove this by finding a probabilistic bijection $\mathcal{P}_\lambda, \overline{\mathcal{P}}_\lambda$ from $(\{\mu \lessdot \lambda\} \cup \{\lambda\}, \omega)$ to $(\{\nu \gtrdot \lambda\}, \overline{\omega})$.

A few more notations

Denote by

- $(h_1, \ldots, h_d)$ the horizontal segment lengths on the boundary of λ,
- $(v_1, \ldots, v_d)$ the vertical segment lengths on the boundary of λ.



Let

$$x_i := h_1 + \ldots + h_i,$$
$$y_i := v_1 + \ldots + v_i.$$

Define for $0 \leq r, s \leq d$

- $\lambda^{(+s)}$ by adding a box to λ in row $y_s + 1$,
- $\lambda^{(-r)}$ by deleting a box of λ in row y_r.

The probabilities

Write $p_{r,s} := \mathcal{P}_\lambda\left(\lambda^{(-r)} \to \lambda^{(+s)}\right)$ and $\overline{p}_{r,s} := \overline{\mathcal{P}}_\lambda\left(\lambda^{(-r)} \leftarrow \lambda^{(+s)}\right)$. Then

$$p_{0,s} = \frac{\prod_{i=1}^{d}\left(q^{x_s}t^{y_s} - q^{x_{i-1}}t^{y_i}\right)}{\prod_{\substack{i=0\\ i\neq s}}^{d}\left(q^{x_s}t^{y_s} - q^{x_i}t^{y_i}\right)}, \qquad \overline{p}_{0,s} = \frac{\prod_{i=1}^{d}\left(q^{x_s-1}t^{y_s+1} - q^{x_{i-1}}t^{y_i}\right)}{\prod_{\substack{i=0\\ i\neq s}}^{d}\left(q^{x_s-1}t^{y_s+1} - q^{x_i}t^{y_i}\right)},$$

The probabilities

Write $p_{r,s} := \mathcal{P}_\lambda\left(\lambda^{(-r)} \to \lambda^{(+s)}\right)$ and $\overline{p}_{r,s} := \overline{\mathcal{P}}_\lambda\left(\lambda^{(-r)} \leftarrow \lambda^{(+s)}\right)$. Then

$$p_{0,s} = \frac{\prod_{i=1}^{d}\left(q^{x_s}t^{y_s} - q^{x_{i-1}}t^{y_i}\right)}{\prod_{\substack{i=0\\ i\neq s}}^{d}\left(q^{x_s}t^{y_s} - q^{x_i}t^{y_i}\right)}, \qquad \overline{p}_{0,s} = \frac{\prod_{i=1}^{d}\left(q^{x_s-1}t^{y_s+1} - q^{x_{i-1}}t^{y_i}\right)}{\prod_{\substack{i=0\\ i\neq s}}^{d}\left(q^{x_s-1}t^{y_s+1} - q^{x_i}t^{y_i}\right)},$$

and for $r > 0$,

$$p_{r,s} = \prod_{\substack{i=0\\ i\neq s}}^{d} \frac{q^{x_{r-1}+1}t^{y_r-1} - q^{x_i}t^{y_i}}{q^{x_s}t^{y_s} - q^{x_i}t^{y_i}} \prod_{\substack{i=1\\ i\neq r}}^{d} \frac{q^{x_s}t^{y_s} - q^{x_{i-1}}t^{y_i}}{q^{x_{r-1}+1}t^{y_r-1} - q^{x_{i-1}}t^{y_i}},$$

$$\overline{p}_{r,s} = \prod_{\substack{i=0\\ i\neq s}}^{d} \frac{q^{x_{r-1}}t^{y_r} - q^{x_i}t^{y_i}}{q^{x_s-1}t^{y_s+1} - q^{x_i}t^{y_i}} \prod_{\substack{i=1\\ i\neq r}}^{d} \frac{q^{x_s-1}t^{y_s+1} - q^{x_{i-1}}t^{y_i}}{q^{x_{r-1}}t^{y_r} - q^{x_{i-1}}t^{y_i}}.$$

Our main Theorem

Theorem (A.-Frieden)

The pair $\mathcal{P}_\lambda, \overline{\mathcal{P}}_\lambda$ are a probabilistic bijection from $(\{\mu \lessdot \lambda\} \cup \{\lambda\}, \omega)$ to $(\{\nu \gtrdot \lambda\}, \overline{\omega})$.

Our main Theorem

Theorem (A.-Frieden)

The pair $\mathcal{P}_\lambda, \overline{\mathcal{P}}_\lambda$ are a probabilistic bijection from $(\{\mu \lessdot \lambda\} \cup \{\lambda\}, \omega)$ to $(\{\nu \gtrdot \lambda\}, \overline{\omega})$.

The probabilities $\mathcal{P}_\lambda, \overline{\mathcal{P}}_\lambda$ are defined such that

$$\omega\left(\lambda^{(-r)}\right)\mathcal{P}_\lambda\left(\lambda^{(-r)} \to \lambda^{(+s)}\right) = \overline{\omega}\left(\lambda^{(+s)}\right)\overline{\mathcal{P}}_\lambda\left(\lambda^{(-r)} \leftarrow \lambda^{(+s)}\right),$$

holds for all $0 \leq r, s \leq d$. Therefore, it suffices to prove

$$\sum_{s=0}^{d} \mathcal{P}_\lambda\left(\lambda^{(-r)} \to \lambda^{(+s)}\right) = 1 \quad \forall\, 0 \leq r \leq d,$$

$$\sum_{r=0}^{d} \overline{\mathcal{P}}_\lambda\left(\lambda^{(-r)} \leftarrow \lambda^{(+s)}\right) = 1 \quad \forall\, 0 \leq s \leq d.$$

About the proof

We present the proof for $\sum_{s=0}^{d} \mathcal{P}_\lambda(\lambda \to \lambda^{(+s)}) = 1.$ By definition we have

$$\sum_{s=0}^{d} \mathcal{P}_\lambda(\lambda \to \lambda^{(+s)}) = \sum_{s=0}^{d} \frac{\prod_{i=1}^{d} (q^{x_s} t^{y_s} - q^{x_{i-1}} t^{y_i})}{\prod_{\substack{i=0 \\ i \neq s}}^{d} (q^{x_s} t^{y_s} - q^{x_i} t^{y_i})}.$$

About the proof

We present the proof for $\sum_{s=0}^{d} \mathcal{P}_\lambda(\lambda \to \lambda^{(+s)}) = 1$. By definition we have

$$\sum_{s=0}^{d} \mathcal{P}_\lambda(\lambda \to \lambda^{(+s)}) = \sum_{s=0}^{d} \frac{\prod_{i=1}^{d} (q^{x_s} t^{y_s} - q^{x_{i-1}} t^{y_i})}{\prod_{\substack{i=0 \\ i \neq s}}^{d} (q^{x_s} t^{y_s} - q^{x_i} t^{y_i})}.$$

Let us write $a_i = q^{x_i} t^{y_i}$ and $b_i = q^{x_{i-1}} t^{y_i}$ in the above expression.

About the proof

We present the proof for $\sum_{s=0}^{d} \mathcal{P}_\lambda(\lambda \to \lambda^{(+s)}) = 1$. By definition we have

$$\sum_{s=0}^{d} \mathcal{P}_\lambda(\lambda \to \lambda^{(+s)}) = \sum_{s=0}^{d} \frac{\prod_{i=1}^{d}(a_s - b_i)}{\prod_{\substack{i=0\\i\neq s}}^{d}(a_s - a_i)}.$$

Let us write $a_i = q^{x_i} t^{y_i}$ and $b_i = q^{x_i - 1} t^{y_i}$ in the above expression.

About the proof

We present the proof for $\sum_{s=0}^{d} \mathcal{P}_\lambda(\lambda \to \lambda^{(+s)}) = 1$. By definition we have

$$\sum_{s=0}^{d} \mathcal{P}_\lambda(\lambda \to \lambda^{(+s)}) = \sum_{s=0}^{d} \frac{\prod_{i=1}^{d}(a_s - b_i)}{\prod_{\substack{i=0\\ i\neq s}}^{d}(a_s - a_i)}.$$

Let us write $a_i = q^{x_i} t^{y_i}$ and $b_i = q^{x_{i-1}} t^{y_i}$ in the above expression.

The right hand side is actually the leading coefficient of the polynomial (in x)

$$\sum_{s=0}^{d} \prod_{i=1}^{d}(a_s - b_i) \prod_{\substack{i=0\\ i\neq s}}^{d} \frac{x - a_i}{a_s - a_i} \qquad ,$$

About the proof

We present the proof for $\sum_{s=0}^{d} \mathcal{P}_\lambda(\lambda \to \lambda^{(+s)}) = 1$. By definition we have

$$\sum_{s=0}^{d} \mathcal{P}_\lambda(\lambda \to \lambda^{(+s)}) = \sum_{s=0}^{d} \frac{\prod_{i=1}^{d}(a_s - b_i)}{\prod_{\substack{i=0\\ i\neq s}}^{d}(a_s - a_i)}.$$

Let us write $a_i = q^{x_i} t^{y_i}$ and $b_i = q^{x_i - 1} t^{y_i}$ in the above expression.

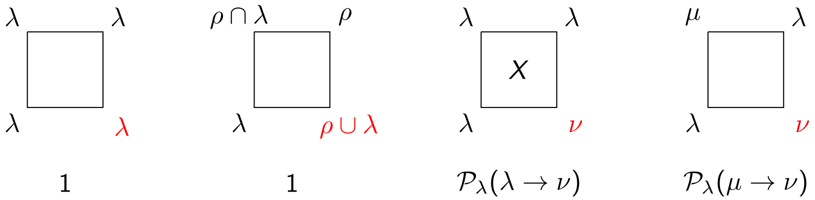
The right hand side is actually the leading coefficient of the polynomial (in x)

$$\sum_{s=0}^{d} \prod_{i=1}^{d}(a_s - b_i) \prod_{\substack{i=0\\ i\neq s}}^{d} \frac{x - a_i}{a_s - a_i} = \prod_{i=1}^{d}(x - b_i),$$

and hence equal to 1.

The probabilistic local growth rules

Let $\lambda \neq \rho$ be partitions and $\nu \gtrdot \lambda \gtrdot \mu$. We assign a partition to the bottom right corner of a square according to one of the four cases and their corresponding probabilities.



For the qRSt algorithm we use the probabilistic local growth rules instead of the deterministic ones.

Theorem (A.-Frieden)

The qRSt algorithm allows a probabilistic bijection proof of the square-free part of the Cauchy identity.

Inverting q and t

The Macdonald polynomials are invariant under inverting q and t,

$$P_\lambda(\mathbf{x}; q^{-1}, t^{-1}) = P_\lambda(\mathbf{x}; q, t), \qquad Q_\lambda(\mathbf{x}; q^{-1}, t^{-1}) = Q_\lambda(\mathbf{x}; q, t).$$

The weights $\omega, \overline{\omega}$ are also invariant, the probabilities $\mathcal{P}_\lambda, \overline{\mathcal{P}}_\lambda$ however not!

Define new probabilities

$$\mathcal{P}_\lambda^{col} = \mathcal{P}_\lambda\big|_{(q,t)\mapsto(q^{-1},t^{-1})},$$
$$\overline{\mathcal{P}}_\lambda^{col} = \overline{\mathcal{P}}_\lambda\big|_{(q,t)\mapsto(q^{-1},t^{-1})}.$$

Theorem (A.-Frieden)

The maps $\mathcal{P}_\lambda^{col}, \overline{\mathcal{P}}_\lambda^{col}$ are probabilistic bijections. The induced RS algorithm specialises for $q, t \to 0$ to the column insertion version of Robinson-Schensted.

Specialisations of qRSt

qRSt

Macdonald polynomials

Specialisations of qRSt

qRSt

$q \to 0$

t-RS
(row insertion)

Macdonald polynomials
Hall-Littlewood polynomials

Specialisations of qRSt

qRSt

$q \to 0$

$t \to 0$

t-RS
(row insertion)

q-RS
(row insertion)
Borodin-Petrov'16
Matveev-Petrov '17

Macdonald polynomials
Hall-Littlewood polynomials
q Whittaker polynomials

Specialisations of qRSt

qRSt

$q \to 0$: t-RS (row insertion)

$t \to 0$: q-RS (row insertion)
Borodin-Petrov'16
Matveev-Petrov '17

$t \to \infty, q \to q^{-1}$: q-RS (column insertion)
O'Connell-Pei'13
Pei'14

Macdonald polynomials
Hall-Littlewood polynomials
q Whittaker polynomials

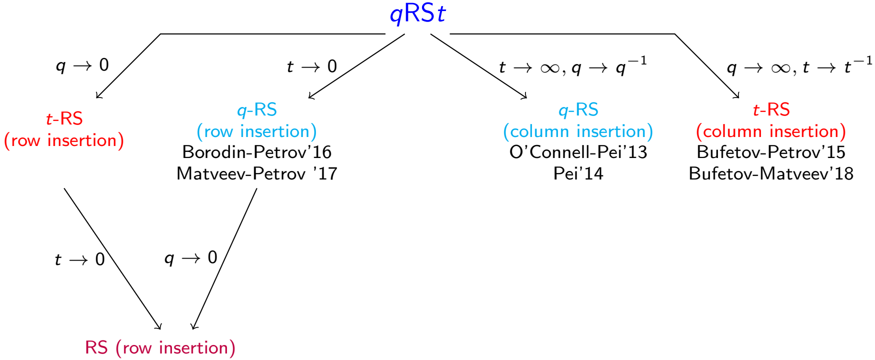
Specialisations of qRSt

qRSt

- $q \to 0$: t-RS (row insertion)
- $t \to 0$: q-RS (row insertion)
  Borodin-Petrov'16
  Matveev-Petrov '17
- $t \to \infty, q \to q^{-1}$: q-RS (column insertion)
  O'Connell-Pei'13
  Pei'14
- $q \to \infty, t \to t^{-1}$: t-RS (column insertion)
  Bufetov-Petrov'15
  Bufetov-Matveev'18

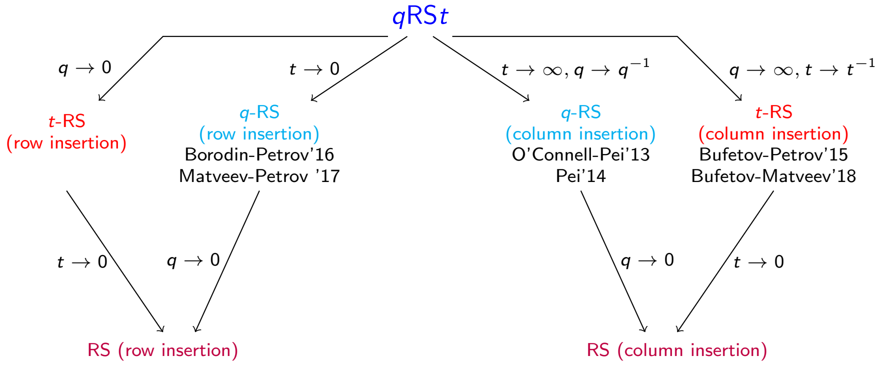
Macdonald polynomials
Hall-Littlewood polynomials
q Whittaker polynomials

Specialisations of qRSt



Macdonald polynomials
Hall-Littlewood polynomials
q Whittaker polynomials
Schur polynomials

Specialisations of qRSt



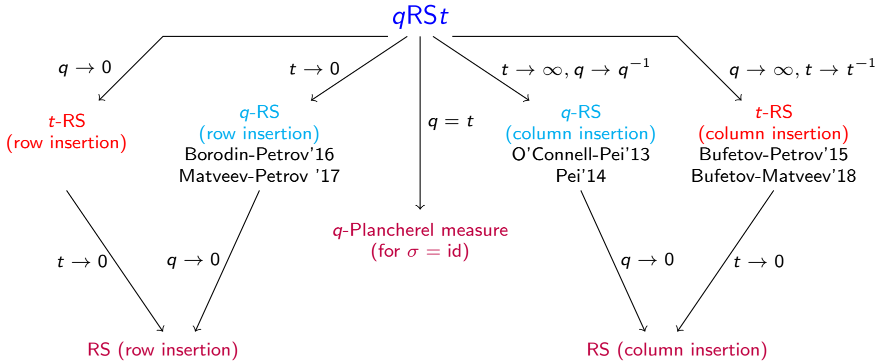
Macdonald polynomials

Hall-Littlewood polynomials

q Whittaker polynomials

Schur polynomials

Specialisations of qRSt



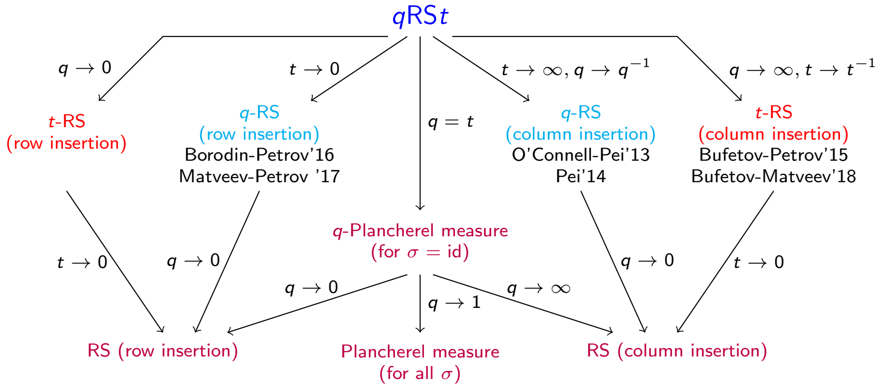
Macdonald polynomials

Hall-Littlewood polynomials

q Whittaker polynomials

Schur polynomials

Specialisations of $q\mathrm{RS}t$



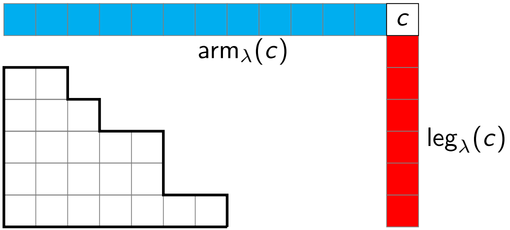
Macdonald polynomials
Hall-Littlewood polynomials
q Whittaker polynomials
Schur polynomials

exterior (q,t)-Hook walks I

1. Start with a cell $c=(x,y)$ "far away", i.e., $x>\lambda_1, y>\lambda'_1$.
2. Choose $c' \in \mathrm{arm}_\lambda(c) \cup \mathrm{leg}_\lambda(c)$ with

$$P(c \to c') = \begin{cases} q^{a(c)-i} \dfrac{t^{\ell(c)}(1-q)}{1-q^{a(c)}t^{\ell(c)}} & \text{if } c'=(x-i,y) \in \mathrm{arm}_\lambda(c) \\ t^{j-1} \dfrac{1-t}{1-q^{a(c)}t^{\ell(c)}} & \text{if } c'=(x,y-j) \in \mathrm{leg}_\lambda(c). \end{cases}$$

3. Repeat until we reach an exterior corner of λ.

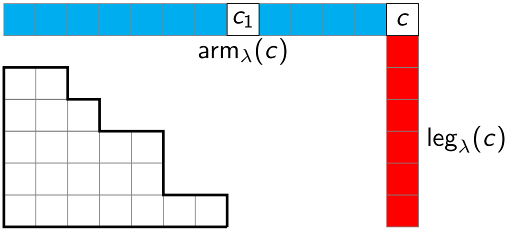


exterior (q, t)-Hook walks I

1. Start with a cell $c = (x, y)$ "far away", i.e., $x > \lambda_1, y > \lambda'_1$.
2. Choose $c' \in \text{arm}_\lambda(c) \cup \text{leg}_\lambda(c)$ with

$$P(c \to c') = \begin{cases} q^{a(c)-i} \dfrac{t^{\ell(c)}(1-q)}{1-q^{a(c)}t^{\ell(c)}} & \text{if } c' = (x-i, y) \in \text{arm}_\lambda(c) \\ \\ t^{j-1} \dfrac{1-t}{1-q^{a(c)}t^{\ell(c)}} & \text{if } c' = (x, y-j) \in \text{leg}_\lambda(c). \end{cases}$$

3. Repeat until we reach an exterior corner of λ.

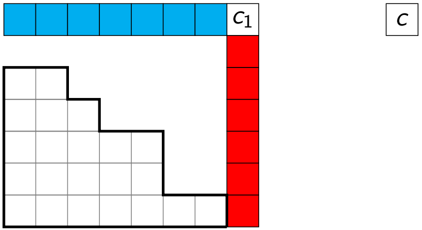


exterior (q,t)-Hook walks I

1. Start with a cell $c=(x,y)$ “far away”, i.e., $x>\lambda_1, y>\lambda'_1$.
2. Choose $c' \in \mathrm{arm}_\lambda(c) \cup \mathrm{leg}_\lambda(c)$ with

$$P(c \to c') = \begin{cases} q^{a(c)-i}\dfrac{t^{\ell(c)}(1-q)}{1-q^{a(c)}t^{\ell(c)}} & \text{if } c'=(x-i,y)\in \mathrm{arm}_\lambda(c) \\ t^{j-1}\dfrac{1-t}{1-q^{a(c)}t^{\ell(c)}} & \text{if } c'=(x,y-j)\in \mathrm{leg}_\lambda(c). \end{cases}$$

3. Repeat until we reach an exterior corner of λ.

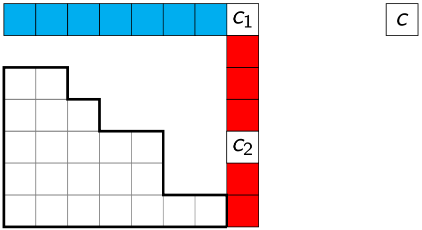


exterior (q,t)-Hook walks I

1. Start with a cell $c=(x,y)$ “far away”, i.e., $x>\lambda_1, y>\lambda'_1$.
2. Choose $c' \in \mathrm{arm}_\lambda(c) \cup \mathrm{leg}_\lambda(c)$ with

$$P(c\to c') = \begin{cases} q^{a(c)-i}\dfrac{t^{\ell(c)}(1-q)}{1-q^{a(c)}t^{\ell(c)}} & \text{if } c'=(x-i,y)\in \mathrm{arm}_\lambda(c) \\ t^{j-1}\dfrac{1-t}{1-q^{a(c)}t^{\ell(c)}} & \text{if } c'=(x,y-j)\in \mathrm{leg}_\lambda(c). \end{cases}$$

3. Repeat until we reach an exterior corner of λ.

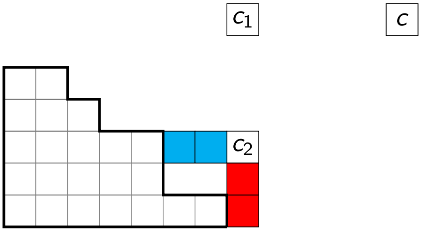


exterior (q, t)-Hook walks I

1. Start with a cell $c = (x, y)$ "far away", i.e., $x > \lambda_1, y > \lambda'_1$.
2. Choose $c' \in \mathrm{arm}_\lambda(c) \cup \mathrm{leg}_\lambda(c)$ with

$$P(c \to c') = \begin{cases} q^{a(c)-i} \dfrac{t^{\ell(c)}(1-q)}{1-q^{a(c)}t^{\ell(c)}} & \text{if } c' = (x-i, y) \in \mathrm{arm}_\lambda(c) \\[2em] t^{j-1} \dfrac{1-t}{1-q^{a(c)}t^{\ell(c)}} & \text{if } c' = (x, y-j) \in \mathrm{leg}_\lambda(c). \end{cases}$$

3. Repeat until we reach an exterior corner of λ.

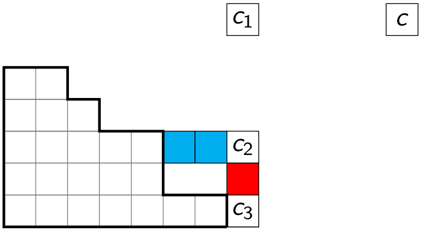


exterior (q, t)-Hook walks I

1. Start with a cell $c = (x, y)$ "far away", i.e., $x > \lambda_1, y > \lambda'_1$.
2. Choose $c' \in \text{arm}_\lambda(c) \cup \text{leg}_\lambda(c)$ with

$$P(c \to c') = \begin{cases} q^{a(c)-i} \dfrac{t^{\ell(c)}(1-q)}{1-q^{a(c)}t^{\ell(c)}} & \text{if } c' = (x-i, y) \in \text{arm}_\lambda(c) \\[2ex] t^{j-1} \dfrac{1-t}{1-q^{a(c)}t^{\ell(c)}} & \text{if } c' = (x, y-j) \in \text{leg}_\lambda(c). \end{cases}$$

3. Repeat until we reach an exterior corner of λ.



exterior (q, t)-Hook walks II

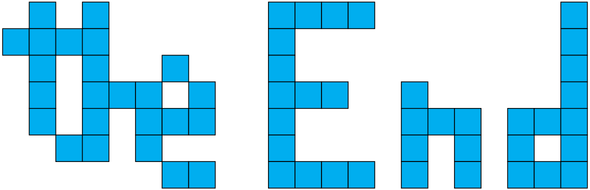
These walks are similar to the (q, t)-walks of Garsia and Haiman which generalise Greene–Nijenhuis–Wilf hook walks.

Let $P(\nu|c)$ be the probability that the *exterior* (q, t)*-Hook walk* ends at the exterior corner corresponding to $\nu \gtrdot \lambda$.

Theorem (A.-Frieden)

Let $c = (x, y)$ *with* $x > \lambda_1, y > \lambda'_1$*, then*

$$P(\nu|c) = \mathcal{P}_\lambda(\lambda \to \nu).$$


the End