An interaction of Combinatorics and Statistical Physics: Square Ice, the 6-vertex model and ASMs

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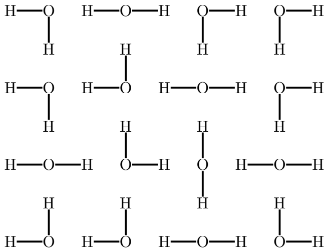
Student Retreat – Strobl, 25.4'17

Let's break the ice

A square ice (with domain wall boundary condition) of size n is an arrangement of n^2 water molecules, s.t.

- the oxygen atoms O are placed on an $n \times n$ square lattice,
- the hydrogen atoms H are placed in-between the oxygen atoms and to the left and right of the boundary oxygen atoms.



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$$S = k_B \log(Z(n)) = k_B n \log(W),$$

where S is the entropy, k_B the Boltzmann constant and $W = \left(\frac{4}{3}\right)^{\frac{3}{2}}$.

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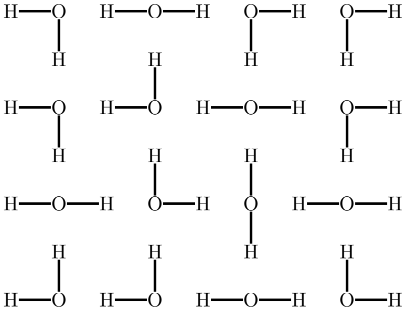
where S is the entropy, k_B the Boltzmann constant and $W = \left(\frac{4}{3}\right)^{\frac{3}{2}}$.
- The formula for the partition function for square ice with domain wall boundary condition was found by Korepin and Izergin .

Make the model more handy



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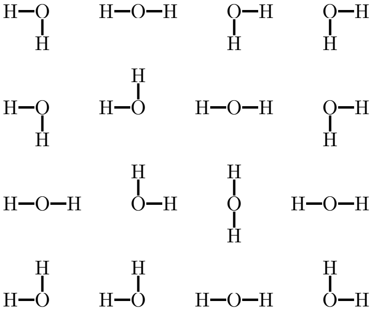

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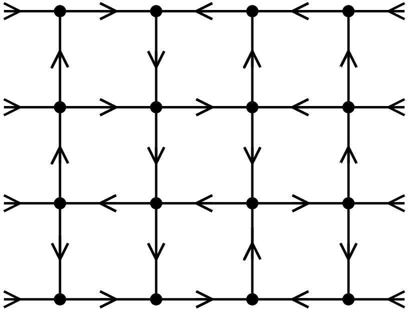

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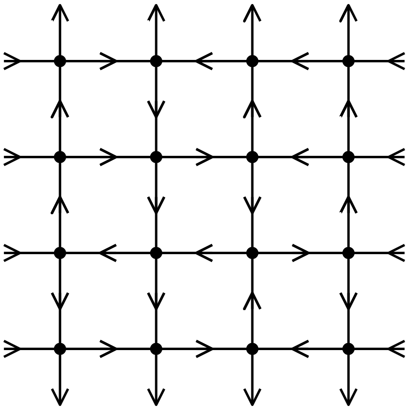
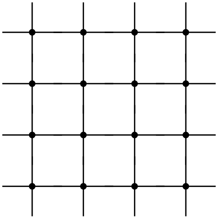


The 6-vertex model

A 6-vertex configuration of size n is an $n \times n$ grid with n external edges on every side and an edge orientation satisfying the following conditions.

- The external edges point inward at the left and right and outward at the top and bottom.
- Every vertex has two edges pointing towards it and two pointing away.



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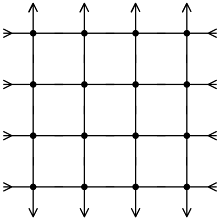

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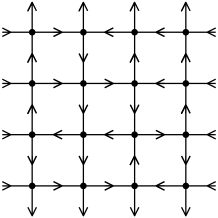

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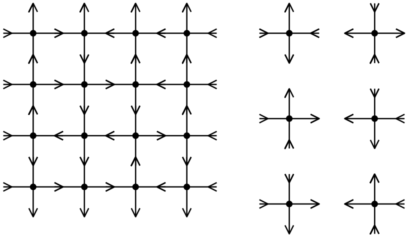

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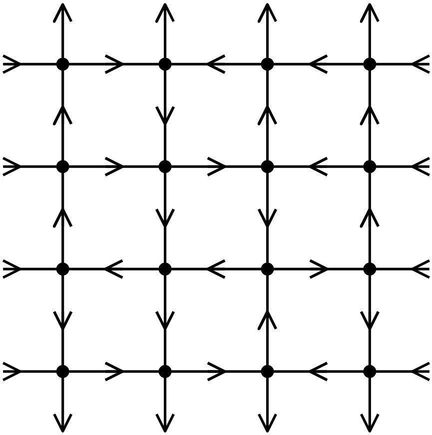

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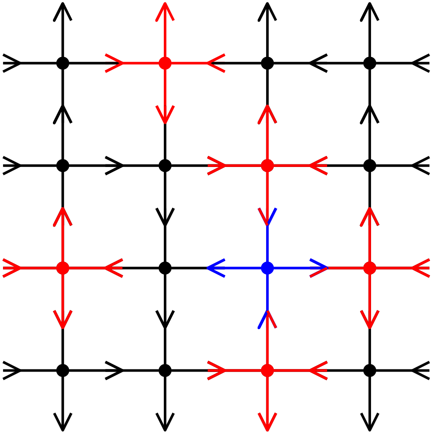

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$$\begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{matrix}$$

Alternating sign matrices

An alternating sign matrix (ASM) of size n is a $n \times n$ matrix with entries $-1, 0, 1$ such that

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- all column and row sums are equal to 1.

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Theorem (Folklore)

For an integer n, the following sets are in bijection

- *set of square ice of size n,*
- *set of 6-vertex configurations of size n,*
- *set of ASM of size n.*

Some facts

- ASMs were introduced by Robbins and Rumsey in the 1980s and arose from generalizing the determinant.

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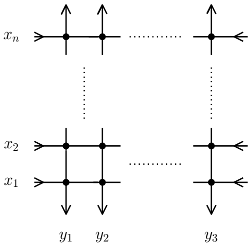
- Kuperberg discovered the relation between ASMs and the 6-vertex model.

The partition function

The partition function $Z(n; \mathbf{x}, \mathbf{y})$ is defined as $Z(n; \mathbf{x}, \mathbf{y}) = \sum_{\substack{A \text{ is a 6-vertex} \\ \text{conf. of size } n}} \prod_{v \in A} w_v$.



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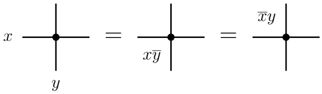
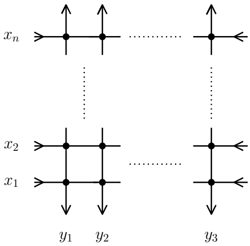
$\overline{x} := \frac{1}{x}$,
$\sigma(x) := x - \overline{x}$,
$q := e^{\frac{2\pi i}{3}}$.





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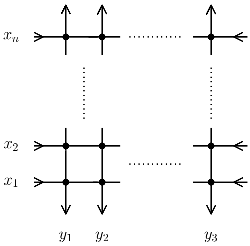

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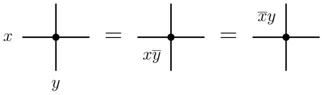
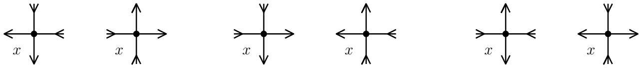


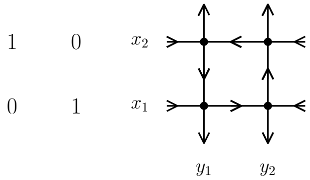
v

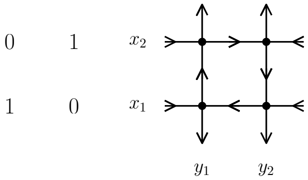


v			
$w_v(x)$	$\frac{\sigma(q\overline{x})}{\sigma(q^2)}$	$\frac{\sigma(qx)}{\sigma(q^2)}$	1

An example

Let $n = 2$.





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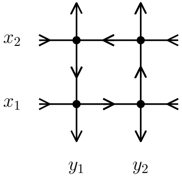

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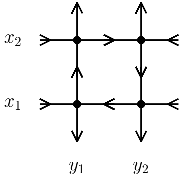

$$\begin{aligned} Z(2; \mathbf{x}, \mathbf{y}) &= \frac{\sigma(q x_2 \overline{y_2})}{\sigma(q^2)} \frac{\sigma(q x_1 \overline{y_1})}{\sigma(q^2)} + \frac{\sigma(q y_1 \overline{x_2})}{\sigma(q^2)} \frac{\sigma(q y_2 \overline{x_1})}{\sigma(q^2)} \\ &= -\frac{x_1^2 x_2^2 + y_1^2 y_2^2 + x_1^2 y_1^2 + x_2^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2}{x_1 x_2 y_1 y_2} \end{aligned}$$

A characterisation

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- $Z(n; \mathbf{x}, \mathbf{y})|_{x_1 = q y_1} = \prod_{i=2}^{n} \frac{\sigma(q x_1 \overline{y_i})}{\sigma(q^2)} \frac{\sigma(q y_1 \overline{x_i})}{\sigma(q^2)} \times Z(n-1; \mathbf{x} \backslash x_1, \mathbf{y} \backslash y_1)$.

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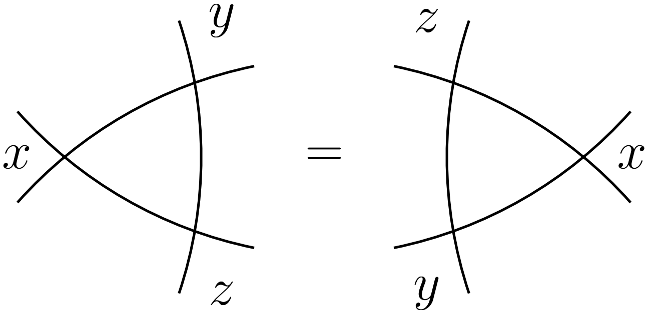
- The partition function $Z(n; \mathbf{x}, \mathbf{y})$ is symmetric in $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$.

Yang-Baxter equation

For $qxyz = 1$ holds



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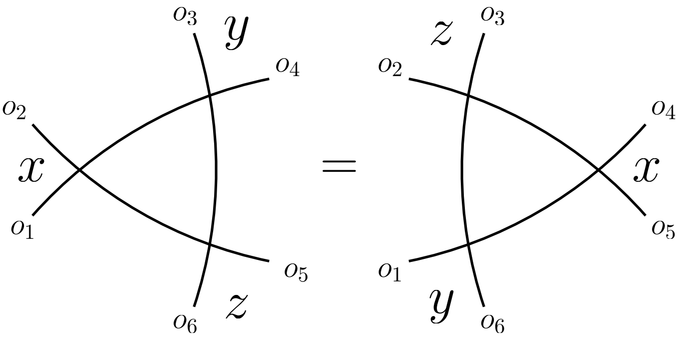

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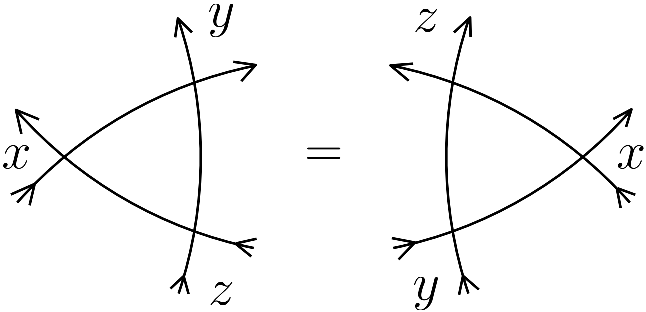

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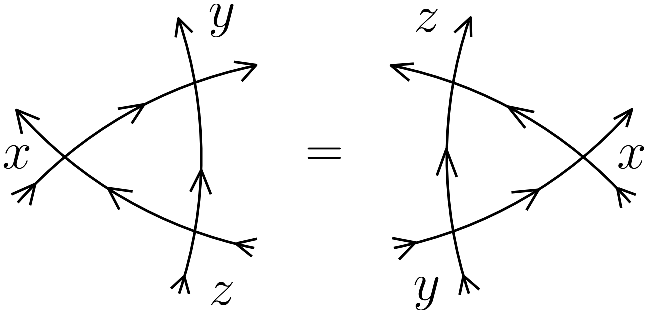

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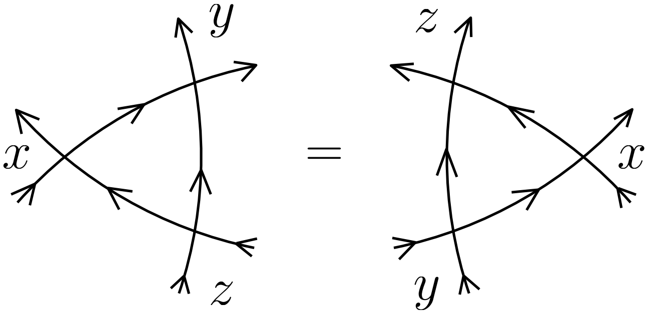
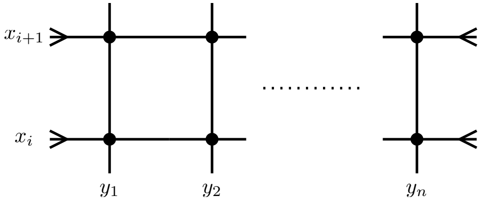


$$\frac{\sigma(qx)}{\sigma(q^2)}\frac{\sigma(q\bar{y})}{\sigma(q^2)}\frac{\sigma(q\bar{z})}{\sigma(q^2)} = \frac{\sigma(qx)}{\sigma(q^2)}\frac{\sigma(q\bar{y})}{\sigma(q^2)}\frac{\sigma(q\bar{z})}{\sigma(q^2)}$$

Symmetry of the partition function



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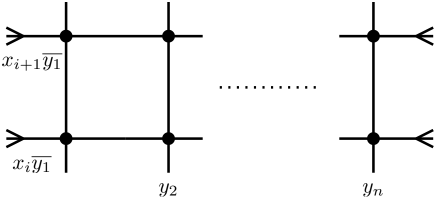

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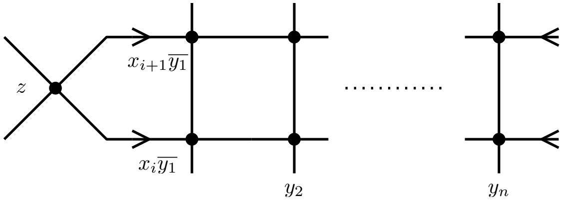

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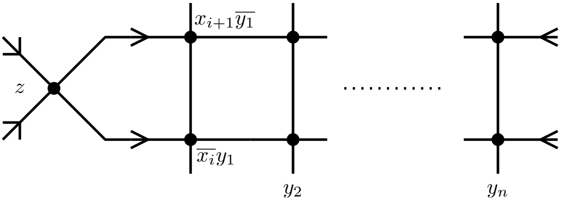

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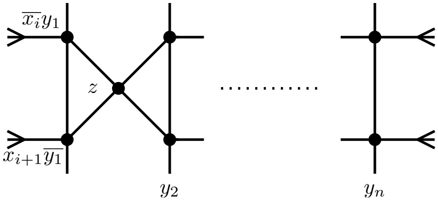

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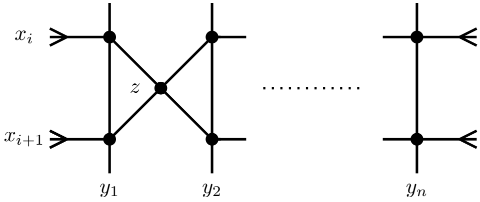

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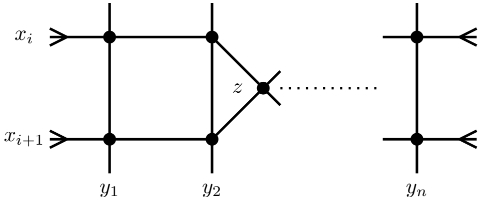

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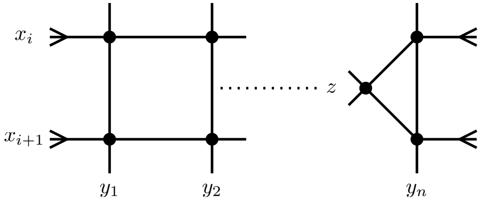

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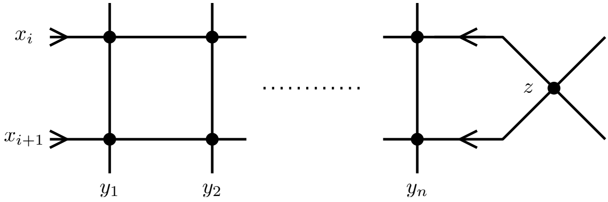

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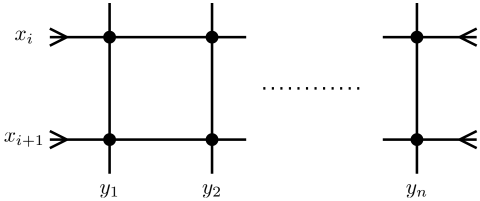

A closed formula

Theorem (Izergin, Korepin)

Define $M_{i,j} = \frac{1}{\sigma(qx_i\overline{y_j})\sigma(q\overline{x_i}y_j)}$, *then*

$$Z(n;\mathbf{x},\mathbf{y}) = \frac{\prod_{1\le i,j\le n}\sigma(qx_i\overline{y_j})\sigma(q\overline{x_i}y_j)}{\sigma(q^2)^{n^2-n}\prod_{1\le i<j\le n}\sigma(\overline{x_i}x_j)\sigma(y_i\overline{y_j})}\det(M).$$

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We can use the above formula together with $Z(n;(1,\ldots,1),(1,\ldots,1)) = \text{ASM}(n)$ to proof the ASM Theorem, which states $\text{ASM}(n) = \prod_{i=0}^{n-1}\frac{(3i+1)!}{(n+i)!}$.