Lattice Path Conference
21-25 June 2021
Presentation times for this poster:

Tuesday	6-7 pm
Thursday	1:30-2:30 pm

qRSt: A probabilistic Robinson-Schensted correspondence for Macdonald polynomials

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Semistandard Young tableaux

A semistandard Young tableau of shape λ is a filling of λ with positive integers, such that

- rows are weakly increasing,
- columns are strictly increasing.

4	5			
2	3	4	4	
1	1	2	3	3

Denote by $\mathbf{x}^T = \prod_i x_i^{\#\, i\text{'s in } T}$.

$\mathbf{x}^T = x_1^2 x_2^2 x_3^3 x_4^3 x_5$

A standard Young tableau is an SSYT whose entries are exactly $1, \dots, |\lambda|$.

Macdonald polynomials

The Macdonald polynomials P_λ, Q_λ are defined as

$$P_\lambda(q,t;\mathbf{x}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \psi_T(q,t)\mathbf{x}^T,$$

$$Q_\lambda(q,t;\mathbf{x}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \varphi_T(q,t)\mathbf{x}^T,$$

where $\psi_T(q,t), \varphi_T(q,t)$ are certain rational functions in q, t.

Theorem (Cauchy identity)

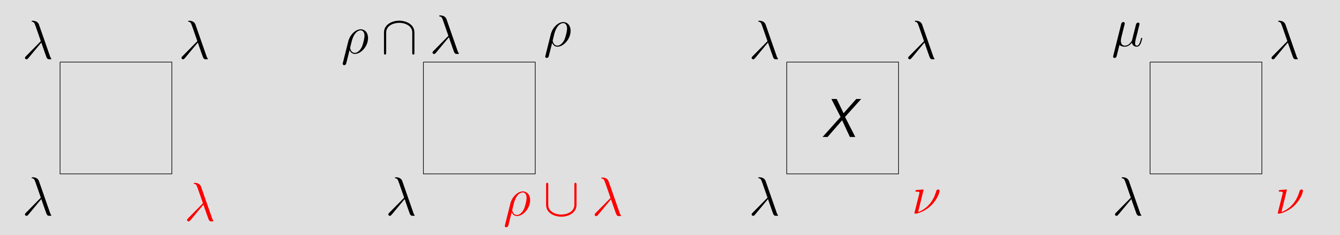
Let $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ be two sets of variables. Then

$$\sum_\lambda P_\lambda(q,t;\mathbf{x}) Q_\lambda(q,t;\mathbf{y}) = \sum_{A=(a_{i,j})} \prod_{i,j \geq 1} (x_i y_j)^{a_{i,j}} \prod_{k=0}^{a_{i,j}-1} \frac{1-tq^k}{1-q^{k+1}}.$$

The qRSt correspondence

We restrict ourselves to the coefficient of $x_1 \dots x_n y_1 \dots y_n$ in the Cauchy identity, i.e., SYTs and permutation matrices.

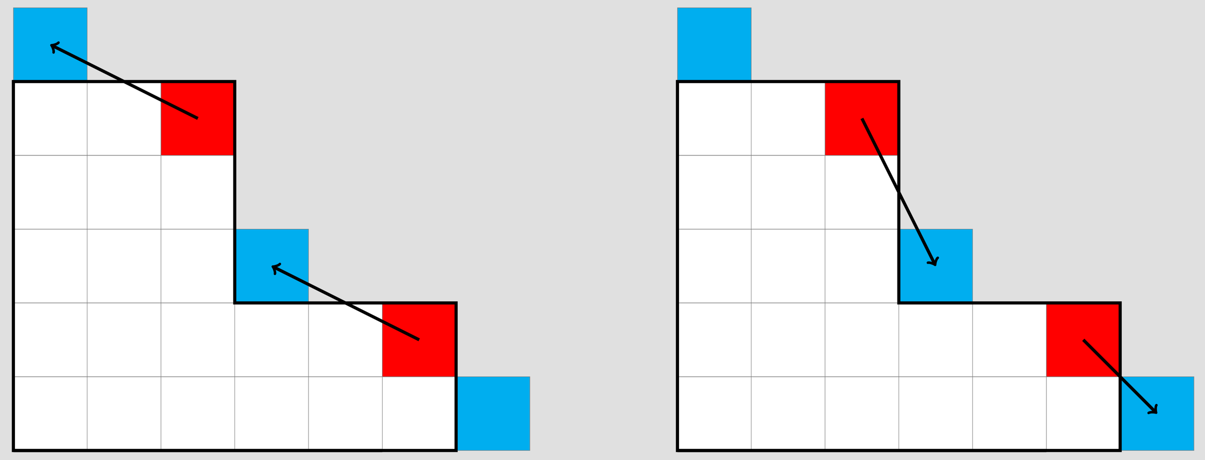
Use Fomin growth diagrams to construct pairs of SYTs of the same shape. The probabilistic local growth rules are



with prob.	1	1	$\mathcal{P}_\lambda(\lambda \to \nu)$	$\mathcal{P}_\lambda(\mu \to \nu)$

where $\lambda \neq \rho$ and $\nu > \lambda > \mu$.

Classical RS



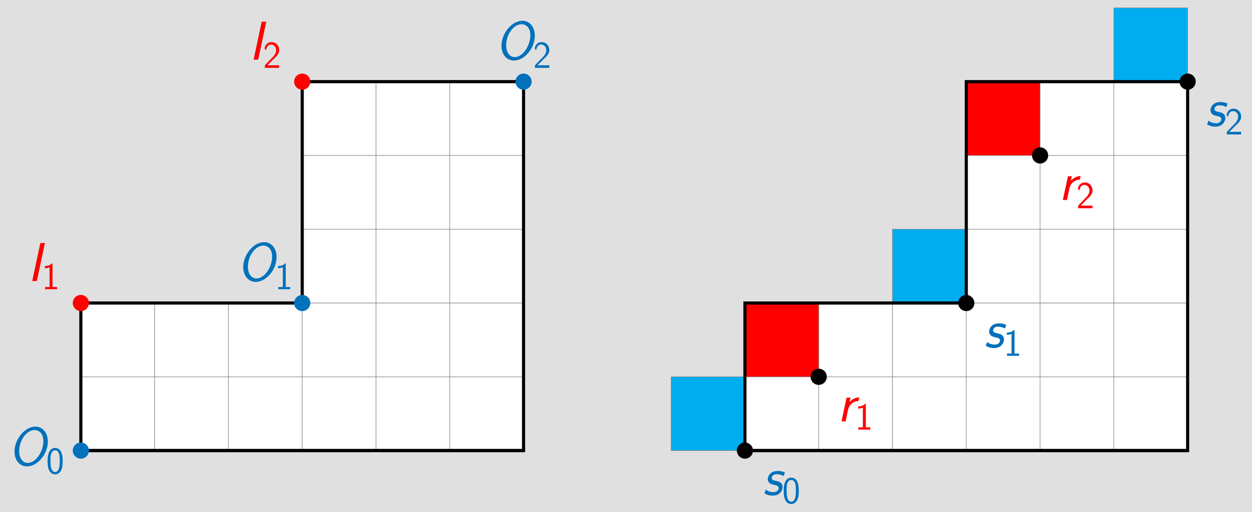
row insertion column insertion

The probabilities (via Quebecois notation)

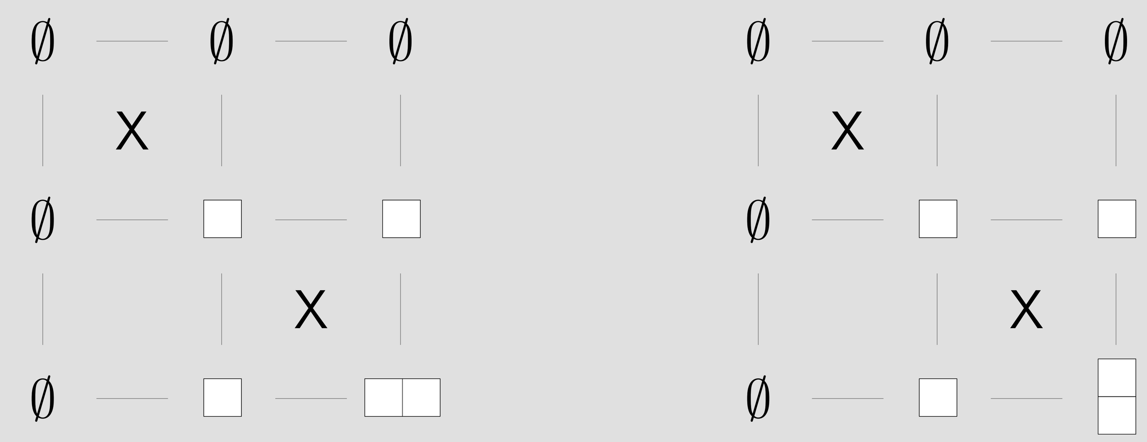
- Let $\lambda^{(\pm i)}$ denote λ with the i-th possible box supplemented or removed.
- Interpret a point (a, b) as $q^a t^b$.

$$\mathcal{P}_\lambda\left(\lambda \to \lambda^{(+j)}\right) = \prod_{k \neq j} \frac{1}{s_j - O_k} \prod_k (s_j - I_k),$$

$$\mathcal{P}_\lambda\left(\lambda^{(-i)} \to \lambda^{(+j)}\right) = \prod_{k \neq j} \frac{r_i - O_k}{s_j - O_k} \prod_{k \neq i} \frac{s_j - I_k}{r_i - I_k}.$$

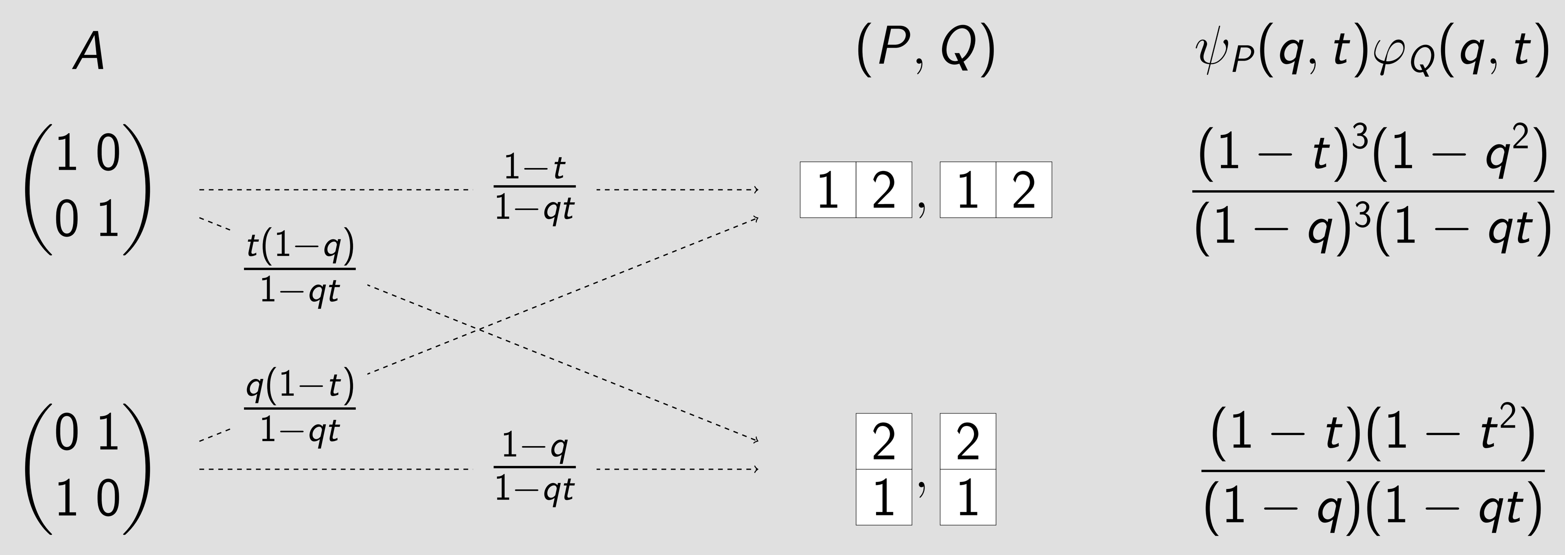


An Example



with prob.	$\dfrac{1-t}{1-qt}$	$\dfrac{t(1-q)}{1-qt}$

qRSt for $n = 2$



The weight of A is always $\dfrac{(1-t)^2}{(1-q)^2}$.

Probabilistic Bijections

Let X, Y be sets together with weights ω_X, ω_Y. A probabilistic bijection is a pair of functions $\mathcal{P}(x \to y), \overline{\mathcal{P}}(x \leftarrow y)$ such that

$$\sum_{y \in Y} \mathcal{P}(x \to y) = 1 \qquad \forall x \in X,$$

$$\sum_{x \in X} \overline{\mathcal{P}}(x \leftarrow y) = 1 \qquad \forall y \in Y,$$

$$\omega_X(x)\mathcal{P}(x \to y) = \omega_Y(y)\overline{\mathcal{P}}(x \leftarrow y) \qquad \forall x \in X, y \in Y.$$

A probabilistic bijection implies between (X, ω_X) and (Y, ω_Y) implies

$$\sum_{x \in X} \omega_X(x) = \sum_{y \in Y} \omega_Y(y).$$

Theorem (Aigner-Frieden)

The qRSt correspondence yields a probabilistic bijective proof of the square-free part of the Cauchy identity. Restricting to $q = t = 0$ ($q = t = \infty$ resp.) results in row (column resp.) insertion of RS.

An interesting identity

Let $\lambda > \mu$, and $f_\lambda = \#($ SYTs of shape $\lambda)$.
For $q = t = 1$ our Theorem implies

$$\sum_{\nu > \lambda} \frac{f_\mu f_\nu}{(h_\lambda(c_{\mu,\nu}))^2} = \frac{|\lambda| + 1}{|\lambda|}(f_\lambda)^2.$$