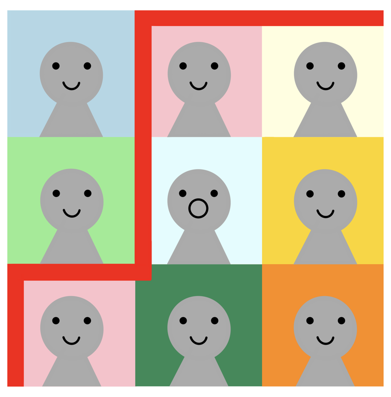

Alternating sign matrices and totally symmetric plane partitions

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1) Alternating sign matrix (ASM)

An alternating sign matrix (ASM) of size n is an $n \times n$ matrix with entries $1, 0, -1$, such that

- all row- and column-sums are equal to 1,
- in each row and column, the non-zero entries alternate.

Example An ASM of size 6.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Theorem(Zeilberger, 1996) The number of ASMs of size n is given by

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

2) Refined enumeration of ASMs

Given an ASM $A = (a_{i,j})_{1 \le i,j \le n}$ of size n, we regard the four refinements

- the position of the unique 1 in the top row $\rho(A)$,
- the inversion number $\mathrm{inv}(A)$,
- the complementary inversion number $\mathrm{inv}'(A)$,
- and the number of -1's denoted by $\mathcal{N}(A)$.

$$\mathrm{inv}(A) := \sum_{\substack{1 \le i' < i \le n \\ 1 \le j' \le j \le n}} a_{i',j} a_{i,j'} \quad \text{and} \quad \mathrm{inv}'(A) := \sum_{\substack{1 \le i' < i \le n \\ 1 \le j \le j' \le n}} a_{i',j} a_{i,j'},$$

The refinements are connected by $\mathrm{inv}(A) + \mathrm{inv}'(A) + \mathcal{N}(A) = \binom{n}{2}$.

Example All ASMs of size 3 and their weight $u^{\mathrm{inv}(A)} v^{\mathrm{inv}'(A)} z^{\rho(A)-1}$.

$$\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \quad \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix} \quad \begin{pmatrix} 0&1&0\\1&-1&1\\0&1&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}$$

$$v^3 \qquad uv^2 \qquad uv^2z \qquad uvz \qquad u^2vz \qquad u^2vz^2 \qquad u^3z^2$$

3) An antisymmetrizer formula

For a positive integer n we define

$$\mathcal{A}_n(\alpha,\beta,\gamma;\mathbf{x}) := \frac{\mathbf{ASym}_{x_1,\ldots,x_n}\left[\prod_{i=1}^{n} x_i^{i-1} \prod_{1 \le i < j \le n} (\alpha + \beta x_i + \gamma x_i x_j)\right]}{\prod_{1 \le i < j \le n} (x_j - x_i)},$$

where **ASym** denotes the antisymmetrizer, i.e.,

$$\mathbf{ASym}_{x_1,\ldots,x_n} f(\mathbf{x}) = \sum_{\sigma \in \mathcal{S}_n} \mathrm{sgn}(\sigma) \cdot f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

and $\mathbf{x} = (x_1, \ldots, x_n)$.

Theorem The number of $n \times n$ ASMs that have the unique 1 in the top row in column i and with inversion number a and complementary inversion number b is the coefficient of $u^a v^b z^{i-1}$ in $\mathcal{A}_n(v, 1-u-v, u; z, 1, \ldots, 1)$.

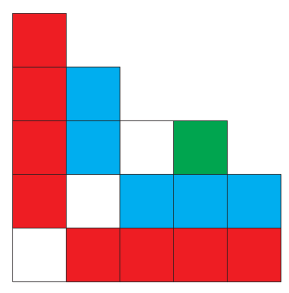
4) Modified balanced partitions

Let λ be a partition with non-negative parts and denote by l the side length of its Durfee square, i.e., $l = \max_i \{\lambda_i \ge i\}$. The Frobenius notation of λ is

$$(\lambda_1 - 1, \lambda_2 - 2, \ldots, \lambda_l - l | \lambda'_1 - 1, \lambda'_2 - 2, \ldots, \lambda'_l - l),$$

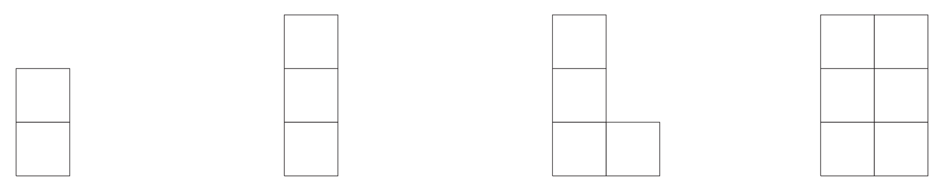
where $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ denotes the conjugate partition.

Example The Frobenius notation of the partition $(5,5,4,2,1)$ is $(4,3,1|4,2,0)$



A modified balanced partition of size n is a partition $\lambda = (a_1, \ldots, a_l | b_1, \ldots, b_l)$ with parts at most $n-1$ such that $a_i < b_i$ for $1 \le i \le l$.

Example All modified balanced partitions of size 3.



$\emptyset$				
$\emptyset$	$(1,1)$	$(1,1,1)$	$(2,1,1)$	$(2,2,2)$
$(\vert)$	$(0\vert 1)$	$(0\vert 2)$	$(1\vert 2)$	$(1,0\vert 2,1)$

5) Totally symmetric plane partitions

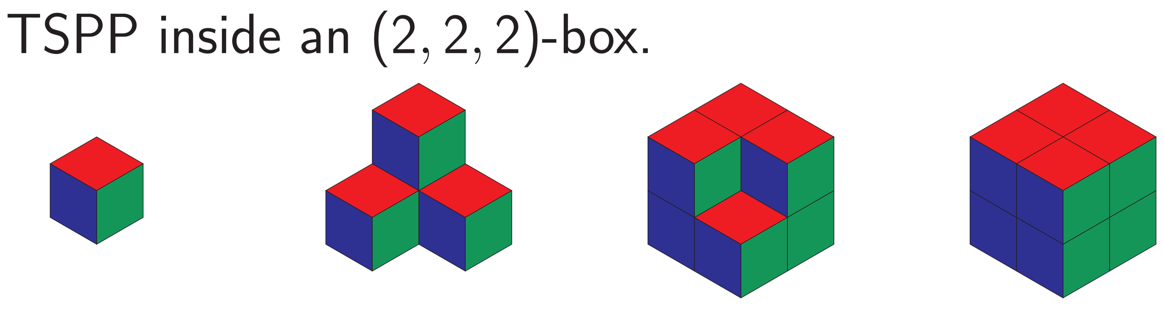
A Totally symmetric plane partition (or short TSPP) T inside an (n,n,n)-box is a subset of $\{1,\ldots,n\}^3$ such that

- if $(i,j,k) \in T$ then every $(i',j',k') \in T$ for $i' \le i, j' \le j, k' \le k$,
- if $(i,j,k) \in T$, all permutations of the coordinates (i,j,k) are in T.

We denote by TSPP_n the set of all TSPP inside an (n,n,n)-box.

Example All TSPP inside an $(2,2,2)$-box.

$\emptyset$



6) Refined TSPPs

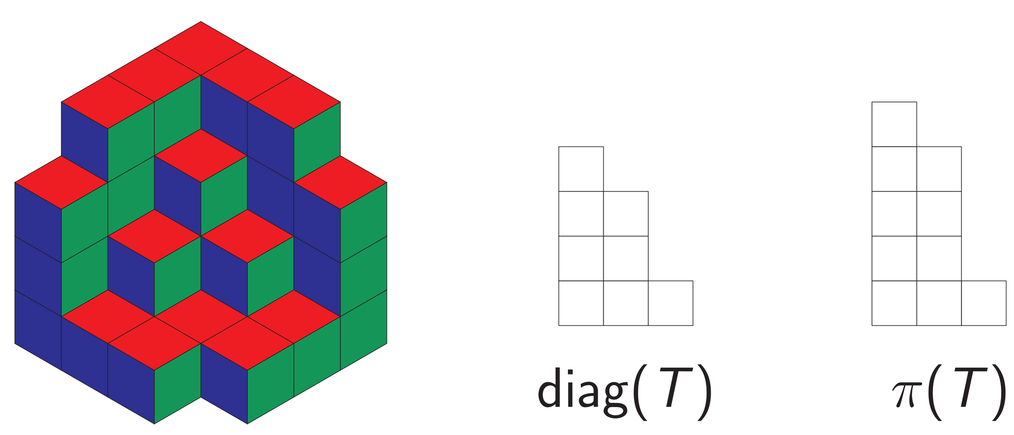
For $T \in \mathrm{TSPP}_{n-1}$ we define

$$\begin{aligned} \mathrm{diag}(T) &= (|\{(k,k,1) \in T\}|, \ldots, |\{(k,k,n-1) \in T\}|) \\ &= (a_1, \ldots, a_l | b_1, \ldots, b_l), \end{aligned}$$

and

$$\pi(T) = (a_1, \ldots, a_l | b_1 + 1, \ldots, b_l + 1).$$

Example



diag(T) $\qquad$ $\pi(T)$

7) The main Theorem

Theorem For all positive integers n, we have

$$\mathcal{A}_n(\alpha,\beta,\gamma;\mathbf{x}) = \sum_{T \in \mathrm{TSPP}_{n-1}} \omega_{\pi(T)}(\alpha,\beta,\gamma) s_{\pi(T)}(\mathbf{x}),$$

where

$$\omega_{\pi(T)}(\alpha,\beta,\gamma) = \alpha^{\binom{n}{2} - \sum_{i=1}^{l} b_i} \beta^{\sum_{i=1}^{l}(b_i - a_i)} \gamma^{\sum_{i=1}^{l}(a_i+1)},$$

for $\pi(T) = (a_1, \ldots, a_l | b_1 + 1, \ldots, b_l + 1)$.

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ASMs and TSPPs - A complement

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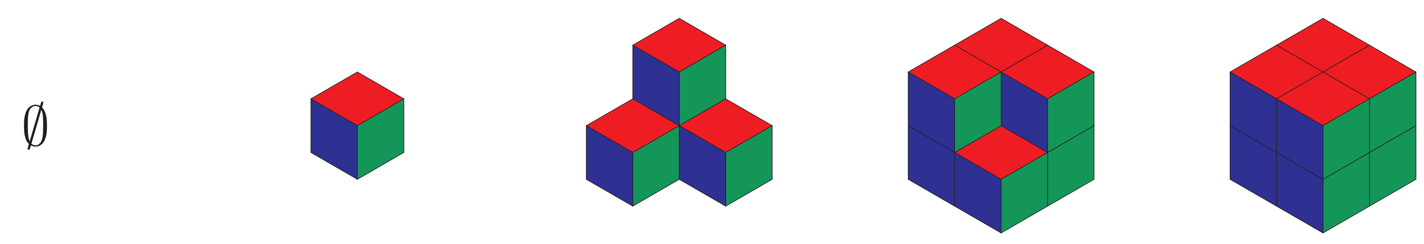
$$\mathcal{A}_1(\alpha,\beta,\gamma;\mathbf{x}) = 1$$

$$\mathcal{A}_2(\alpha,\beta,\gamma;\mathbf{x}) = \alpha + \gamma\, s_{(1,1)}$$

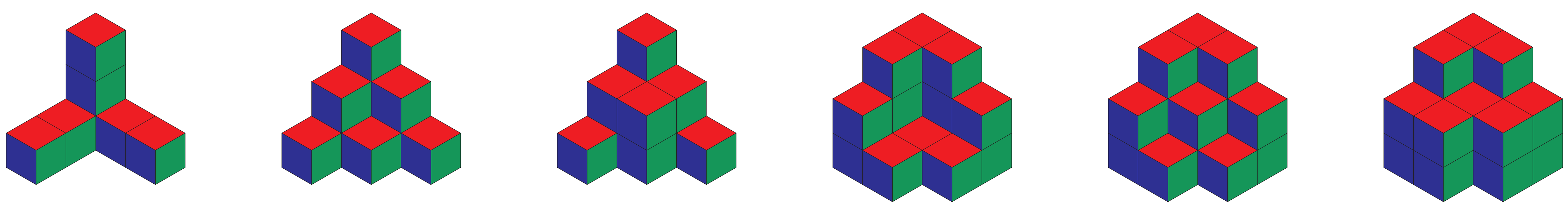
$$\mathcal{A}_3(\alpha,\beta,\gamma;\mathbf{x}) = \alpha^3 + \alpha^2\gamma\, s_{(1,1)} + \alpha\beta\gamma\, s_{(1,1,1)} + \alpha\gamma^2\, s_{(2,1,1)} + \gamma^3\, s_{(2,2,2)}$$

$$\begin{aligned}\mathcal{A}_4(\alpha,\beta,\gamma;\mathbf{x}) = {}& \alpha^6 + \alpha^5\gamma\, s_{(1,1)} + \alpha^4\beta\gamma\, s_{(1,1,1)} + \alpha^4\gamma^2\, s_{(2,1,1)} + \alpha^3\gamma^3\, s_{(2,2,2)} \\ & + \alpha^3\beta^2\gamma\, s_{(1,1,1,1)} + 2\alpha^3\beta\gamma^2\, s_{(2,1,1,1)} + \alpha^3\gamma^3\, s_{(3,1,1,1)} + 2\alpha^2\beta\gamma^3\, s_{(2,2,2,1)} \\ & + \alpha^2\gamma^4\, s_{(3,2,2,1)} + \alpha\beta^2\gamma^3\, s_{(2,2,2,2)} + \alpha\beta\gamma^4\, s_{(3,2,2,2)} + \alpha\gamma^5\, s_{(3,3,2,2)} + \gamma^6\, s_{(3,3,3,3)}\end{aligned}$$

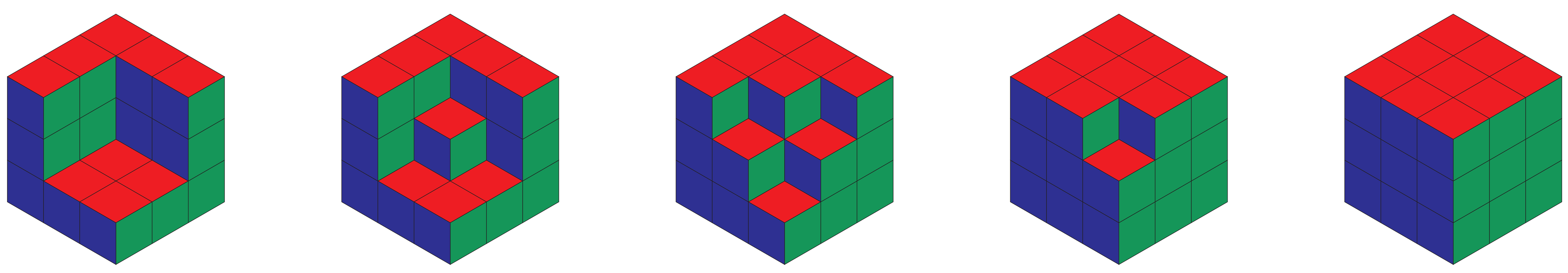
n	#(different Schur polynomials)	sum of coefficients
1	1	1
2	2	2
3	5	5
4	14	16
5	42	66
6	132	352
	Catalan numbers	Totally symmetric plane partitions



1 $s_{(1,1)}$ $s_{(1,1,1)}$ $s_{(2,1,1)}$ $s_{(2,2,2)}$



$s_{(1,1,1,1)}$ $s_{(2,1,1,1)}$ $s_{(2,2,2,1)}$ $s_{(2,1,1,1)}$ $s_{(2,2,2,1)}$ $s_{(2,2,2,1)}$



$s_{(3,1,1,1)}$ $s_{(3,2,2,1)}$ $s_{(3,2,2,2)}$ $s_{(3,3,2,2)}$ $s_{(3,3,3,3)}$