Refined enumerations of alternating sign triangles

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Alternating sign triangle (AST)

An AST of order n is a configuration of n centred rows where the i-th row, counted from the bottom, has $2i-1$ elements with entries $-1, 0, 1$ such that

- the non-zero entries alternate in all rows and columns,
- all row-sums are 1,
- the topmost non-zero entry is 1 for all columns.

Example The following is an AST of order 7.

$A =$

−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0	0	1	0	0	0
	0	0	0	0	0	0	0	1	−1	0	1	
		1	0	0	0	0	0	−1	0	1		
			0	0	0	1	0	0	0			
				1	0	−1	1	0				
					1	0	0					
						1						

Theorem (Ayyer, Behrend, Fischer, 2016) The number of AST of order n is given by

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!},$$

i. e. order n ASTs and $n \times n$ ASMs are equinumerous.

Centred Catalan set (CCS)

A centred Catalan set of size n is a n-subset of $\{-(n-1), \ldots, n-1\}$ such that $|S \cap \{-i, -i+1, \ldots, i\}| \geq i+1$ for all $0 \leq i \leq n-1$.

Proposition Label the columns of an AST A of order n from left to right with $-(n-1), \cdots, n-1$. The set $S(A)$ of columns with positive column-sum is a centred Catalan set.

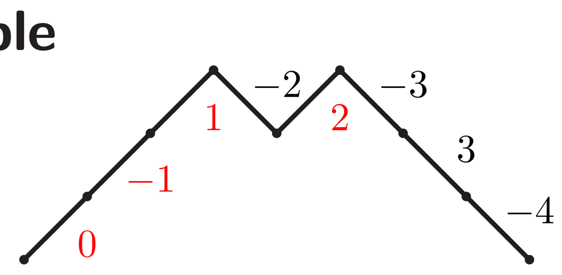
Example $S(A) = \{-4, -2, -1, 0, 1, 4, 5\}$, where A is the above AST.

Let S be a centred Catalan set of size n. We define the weight $w(S)$ as the number of ASTs with associated centred Catalan set equal to S.

CCS and Dyck paths

Given a Dyck path D of length $2n$, we label the steps from left to right by $0, -1, 1, -2, 2, \ldots, -n$. The set $S(D)$ of labels of the North-East steps is a centred Catalan set. The map $D \mapsto S(D)$ is a bijection between Dyck paths of length $2n$ and centred Catalan sets of size n.

Example



$S(D) = \{-1, 0, 1, 2\}$

Concatenating centred Catalan sets

We define for a positive integer l

$$\mathfrak{s}_l(x) = \begin{cases} x+l & x > 0, \\ 0 & x = 0, \\ x-l & x < 0. \end{cases}$$

Let S_1, S_2 be two centred Catalan sets of size n_1 or n_2 respectively. The concatenation of S_1 and S_2 is $(S_1, S_2) := S_1 \cup \mathfrak{s}_{n_1-1}(S_2)$. We call a centred Catalan set irreducible iff it can not be written as a non-trivial concatenation.

Example $(\{-2, -1, 0, 1\}, \{-1, 0, 1, 2\}) = \{-4, -2, -1, 0, 1, 4, 5\}$.

Alternating sign (AS)-trapezoids

An (n, l)-AS-trapezoid is a configuration of n centred rows, where the i-th row from bottom has $2(i+l)-1$ elements, with entries $-1, 0, 1$ such that

- the non-zero entries alternate in all rows and columns,
- all row-sums are 1,
- the topmost non-zero entry is 1 for all columns,
- the central $(2l-1)$ columns have column-sum 0.

Example The following is a (3,4)-AS-trapezoid.

$A =$

−3	−2	−1								1	2	3
0	0	0	0	0	0	0	0	0	1	0	0	0
	0	0	0	0	0	0	0	1	−1	0	1	
		1	0	0	0	0	0	−1	0	1		

Associating CCSs to AS-trapezoids

We label the non-central columns of an (n, l)-AS-trapezoid A form left to right with $-n, \ldots, -1, 1, \cdots, n$. Denote by $S(A)$ the centred Catalan set of size $n+1$ such that $S(A) \setminus \{0\}$ is the set of columns with column-sum equal to 1. The weight $w_l(S)$ is the number of (n, l)-AS-trapezoids with $S(A) = S$.

Example $S(A) = \{-1, 0, 1, 2\}$, where A is the above AS-trapezoid.

Remark ASTs of order $n+1$ and $(n, 1)$-AS-trapezoids are in bijection. This bijection preserves the assignment of centred Catalan sets, i. e., $w(S) = w_1(S)$ for every centred Catalan set S.

A splitting theorem

Theorem (A., 2016) Let S_1, S_2 be centred Catalan sets of size n_1, n_2 respectively, then $w_l((S_1, S_2)) = w_l(S_1) w_{n_1+l-1}(S_2)$.

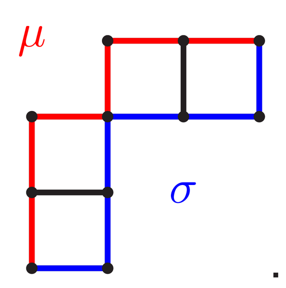
CCSs and skew-shaped Young diagrams

We assign to an centred Catalan set S a skew shaped Young diagram $Y(S)$. First we construct a pair $(\sigma(S), \mu(S))$ of paths of length $n-1$ where for $1 \leq i \leq n$ the i-th step σ_i, μ_i is given as in the table to the right. The skew-shaped Young diagram $Y(S)$ is defined as the boxes between the paths $\sigma(S)$ and $\mu(S)$.

	σ_i	μ_i
$\{-i, i\} \subseteq S$	E	N
$-i \in S, i \notin S$	N	N
$i \in S, -i \notin S$	E	E
$-i, i \notin S$	N	E

Example

The skew-shaped Young diagram of $\{-4, -2, -1, 0, 1, 4, 5\}$ is



A polynomiality theorem for ASTs

Theorem (A., 2016) For a centred Catalan set S the weight $w_l(S)$ is a polynomial function in l of degree $|Y(S)|$ with leading coefficient

$$\frac{2^{|Y(S)|} \#(\text{SYT of shape } Y(S))}{|Y(S)|!}.$$

Conjecture (A.) Let S be a centred Catalan set and k a positive integer. Then $\{-k, -k+1, \cdots, k\} \subseteq S$ if and only if

$$\prod_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} (2l+1+3i)_{k-2i} \,\Big|\, w_l(S),$$

where $(x)_j = (x)(x+1)\cdots(x+j-1)$.

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