Fully packed loop configurations: polynomiality and nested arches

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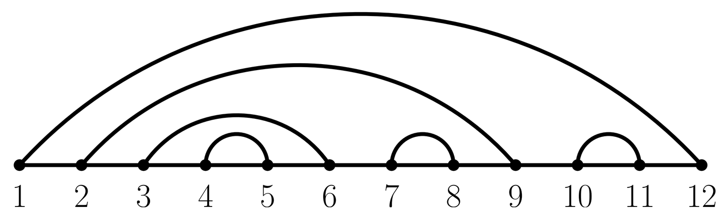
Noncrossing matchings

A noncrossing matching consists out of $2n$ aligned points which are connected by n noncrossing arches (lying above the points).

Example.

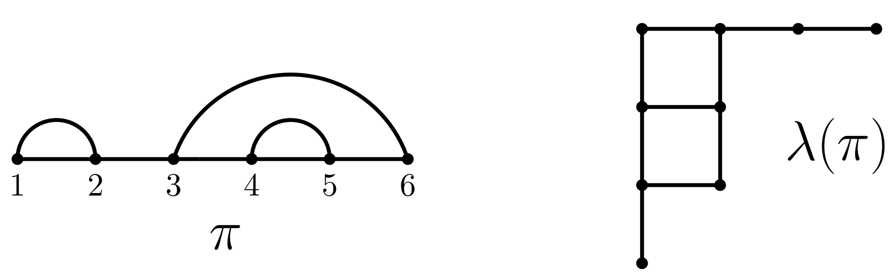


- Denote by $(\pi)_m$ the noncrossing matching π surrounded by m nested arches.
- Write $\pi\sigma$ for the concatenation of π and σ.
- Denote by NC_n the set of noncrossing matchings of size n.

From noncrossing matchings to Young diagrams

Noncrossing matchings of size n are in bijection to Young diagrams with at most $n-i$ boxes in the i-th row. Opening arches correspond to north-steps, closing arches to east-steps.

Example.

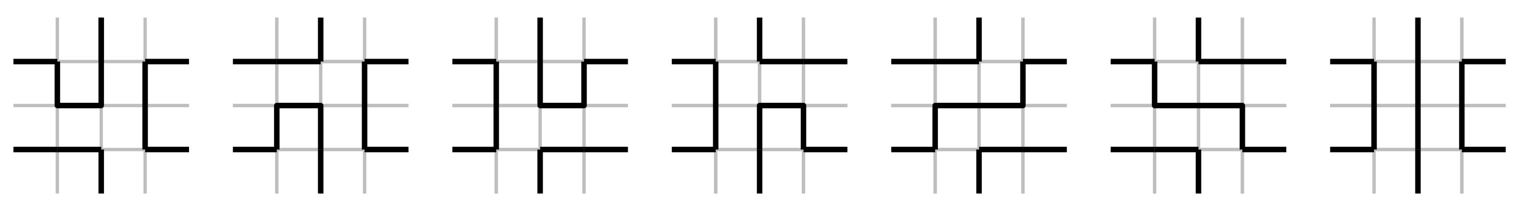


Fully packed loops

A FPL F of size n is a subgraph of the $n \times n$ grid with n external edges (they have only one vertex) such that:

- F contains all vertices of the $n \times n$ grid and every vertex has degree 2.
- F contains every other external edge, beginning with the topmost at the left side.

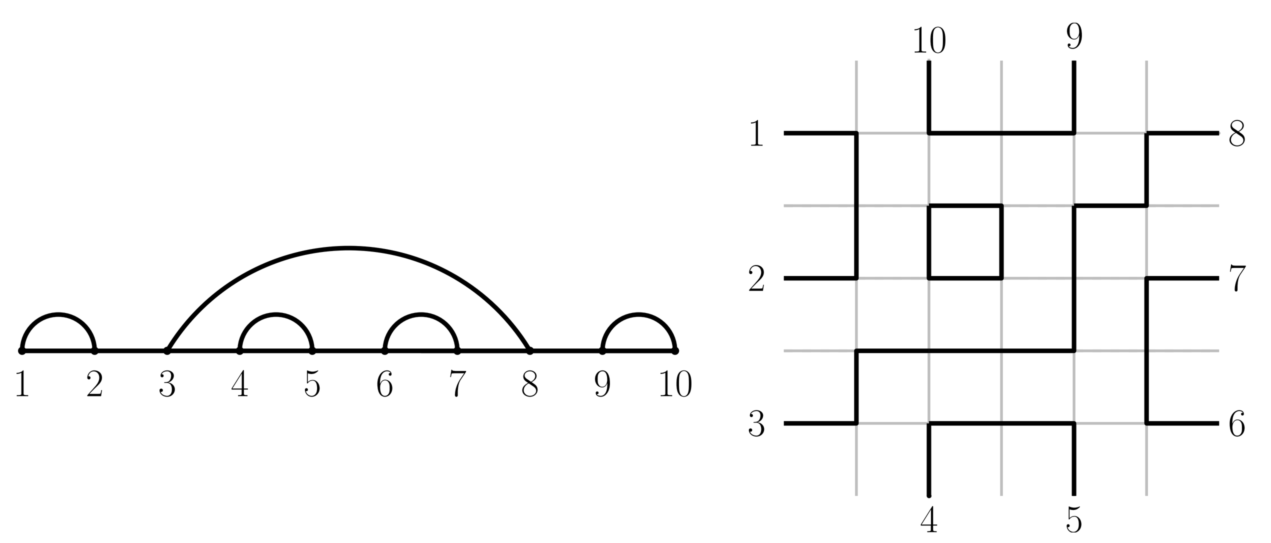
Example(All FPLs of size 3).



The link pattern

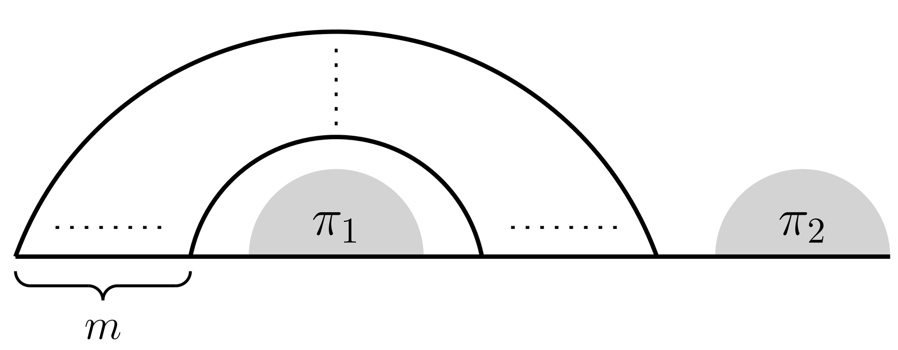
We assign to every FPL F a noncrossing matching $\pi(F)$, called its link pattern, by connecting the numbers i and j in $\pi(F)$ iff they are connected in F.

Example.



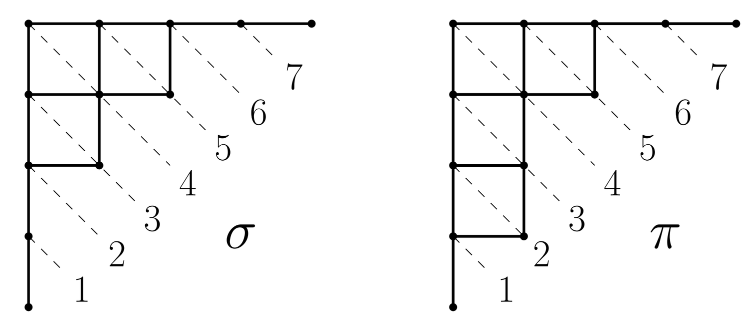
The main theorem

Theorem(Zuber; Caselli, Krattenthaler, Lass, Nadeau; A.). Let π_1, π_2 be noncrossing matchings of size n_1 or n_2 respectively and let m be an integer. The number of FPLs with link pattern $(\pi_1)_m \pi_2$ is a polynomial in m of degree $|\lambda(\pi_1)| + |\lambda(\pi_2)|$ with leading coefficient $\frac{\dim(\lambda(\pi_1)) \dim(\lambda(\pi_2))}{|\lambda(\pi_1)|!|\lambda(\pi_2)|!}$.



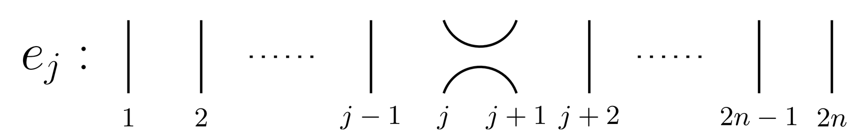
A partial order on noncrossing matchings

We write $\sigma \nearrow_j \pi$ if $\lambda(\pi)$ is obtained by adding a box to $\lambda(\sigma)$ on the j-th diagonal.

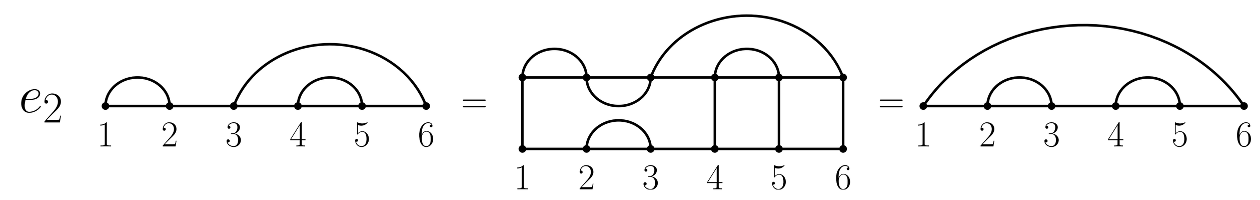


Temperley-Lieb operator

The Temperley-Lieb operator e_j is given for $1 \leq j \leq 2n$ by:



Example.



Wheel polynomials

A polynomial $p \in \mathbb{Q}(q)[z_1, \ldots, z_{2n}]$ is called wheel polynomial of order n if:

- p is homogeneous of degree $n(n-1)$.
- $p(z_1, \ldots, z_{2n})_{|q^4 z_i = q^2 z_j = z_k} = 0$ for all $1 \leq i < j < k \leq 2n$.

Denote by $W_n[z]$ the $\mathbb{Q}(q)$-vector space of wheel polynomials.

A family of operators

Define for $1 \leq k \leq 2n$

$$D_k(f) = \frac{q z_k - q^{-1} z_{k+1}}{z_{k+1} - z_k} (f(z_1, \ldots, z_{k+1}, z_k, \ldots, z_{2n}) - f(z_1, \ldots, z_{2n})).$$

The vector space $W_n[z]$

Theorem(Zinn-Justin; Cantini, Sportiello) There exists a $\mathbb{Q}(q)$-basis $\{\Psi_\pi | \pi \in NC_n\}$ of $W_n[z]$ such that:

- $\Psi_{()_n} = (q + q^{-1})^{-n(n-1)} \prod_{1 \leq i < j \leq n} (q z_i - q^{-1} z_j)(q z_{n+i} - q^{-1} z_{n+j})$.
- $\Psi_\pi(z) = D_j(\Psi_\sigma) - \sum_{\tau \in e_j^{-1}(\sigma) \setminus \{\sigma, \pi\}} \Psi_\tau$, if $\sigma \nearrow_j \pi$.
- $\Psi_\pi(1, \ldots, 1) = A_\pi$ for $q = e^{\frac{2\pi i}{3}}$.
- $\Psi_{\rho^{-1}(\pi)}(z_1, \ldots, z_{2n}) = \Psi_\pi(z_2, \ldots, z_{2n}, q^6 z_1)$

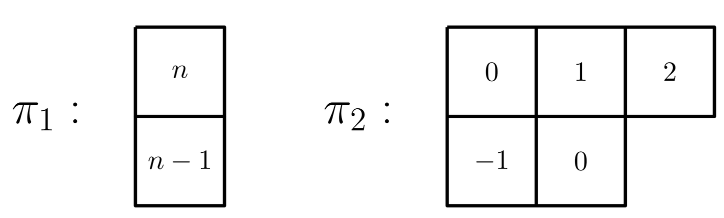
A new family of wheel polynomials

Let π_1, π_2 be noncrossing matchings of size n_1, n_2. We define D_{π_1,π_2} recursively:

- Write in the boxes of $\lambda(\pi_i)$ the number of the diagonal the boxes lie on, with the top-left box lying on the $(n_1 + n_2)$-th or 0-th diagonal respectively.
- Read in $\lambda(\pi_i)$ the rows from top to bottom, the boxes from left to right and apply $D_{\text{number in the box}}$ on the previous wheel polynomial, starting with $\Psi_{()_n}$.

Example.

The wheel polynomial D_{π_1,π_2} for π_1, π_2 as in the picture is



$$D_{\pi_1,\pi_2} = (D_0 \circ D_{-1} \circ D_2 \circ D_1 \circ D_0) \circ (D_{n-1} \circ D_n)(\Psi_{()_n}).$$

Theorem. $\Psi_{\rho^{n_2}(\pi_1\pi_2)}$ is a linear combination of D_{τ_1,τ_2} with $\tau_i \leq \pi_i$ for $i = 1, 2$ and the coefficient of D_{π_1,π_2} is 1.

A polynomiality theorem

Theorem. Let k be a natural number and $i_1, \ldots, i_k \in \{1, \ldots, \frac{n}{2}\} \cup \{\frac{n}{2} + m, \ldots, \frac{3n}{2} + m\} \cup \{\frac{3n}{2} + 2m, \ldots, 2(n+m)\}$, then

$$(D_{i_1} \circ \ldots \circ D_{i_k}) (\Psi_{()_{n+m}})_{|z_1 = \ldots = z_{2(n+m)} = 1}$$

is a polynomial in m of degree at most k.