Stationary states of nonlinear Schrödinger equations with trapping potentials in supercritical dimensions

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Motivation

The Schrödinger-Newton-Hooke equation is a nonlinear Schrödinger equation with nonlocal nonlinearity and a trapping potentia:.

$$i\partial_t\psi = -\Delta\psi + |x|^2\psi - \left(\int_{\mathbb{R}^d} \frac{|\psi(t,y)|^2}{|x-y|^{d-2}}dy\right)\psi$$

It is usually encountered as a mathematical description of various quantum-mechanical systems. However, it can be also obtained as a nonrelativistic limit of weak scalar perturbations of anti-de Sitter spacetime:

$$G_{\mu\nu} + \Lambda_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \qquad g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{m^2c^2}{\hbar^2}\phi = 0$$

Ansätze $\phi(t,x) = e^{-i\frac{mc^2}{\hbar}t}u(t,x)$

$$ds^2 = -c^2\left(1+\frac{2A(t,x)}{c^2}\right)dt^2 + \left(1+\frac{2B(t,x)}{c^2}\right)\sum_{j=1}^{d}\left(dx^j\right)^2$$

together with the assumption $\lim_{c\to\infty}\Lambda c^2 = -\frac{d(d-1)}{2}\omega^2$

in the limit c→∞ give: $\begin{cases} i\partial_t u = -\Delta u + |x|^2 u + uv \\ \Delta v = |u|^2 \end{cases}$

v plays here the role of the potential and vanishes in ∞. This equation is equivalent to SNH system and gives us a motivation to investigate it in higher dimensions, including supercritical (d>6) where variational methods cease to work. From now on we assume d≥6.

ODE approach

We focus on stationary solutions: $u(t,x) = e^{-i\omega t}f(x)$

Besides, we assume spherical symmetry. In case of the ground states it leads to no loss in generality as positive solutions of this system are bound to be spherically symmetric (Busca and Sirakov, 2000).

Our goal are bound states with some fixed arbitrary central value f(0)=b. It is convenient then, to replace the potential by h(r)=-v(r)+ω. We get the following:

$$\begin{cases} f'' + \frac{d-1}{r}f' - r^2 f + fh = 0 \\ h'' + \frac{d-1}{r}h' + f^2 = 0 \end{cases} \qquad \begin{matrix} f(0)=b, \quad h(0)=c \\ f'(0)=h'(0)=0 \end{matrix}$$

We want to find such a value of c that the solution f is a bound state (it vanishes in infinity). Then one can obtain its frequency ω as $\lim_{r\to\infty} h(r) = \omega$

This limit exists as for large r the harmonic term dominates the system and the solution f behaves like

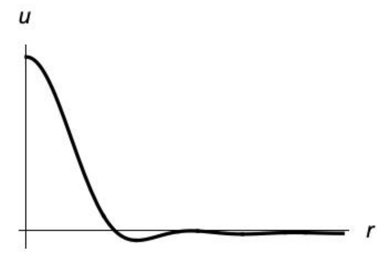
$$f(r) \approx C\,e^{-r^2/2}\,U\left(\frac{d-\omega}{4}, \frac{d}{2}, r^2\right)$$

The appropriate values of c can be found numerically by the shooting method.

Existence

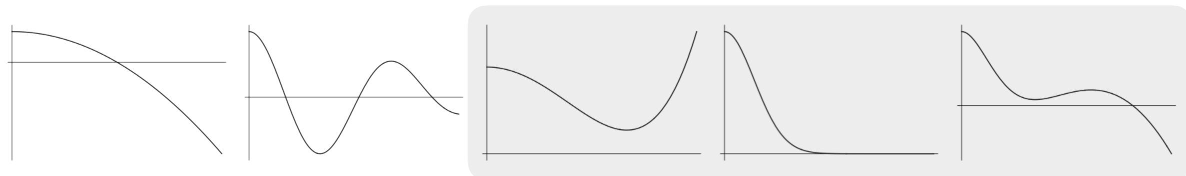
One can show that for any b>0:

- If c=0, the solution u is positive
- In the limit c→∞, u oscillates indefinitely (figure on the left)



It suggests the following definition:

$$I_n = \{c \geq 0 \,|\, f \text{ crosses zero } n \text{ times and has appropriate shape}\}$$



Then we define c_n = inf I_n and claim it gives us a bound state with n-1 zeros. The crucial observation is:

h is decreasing ⇒ f is monotonic for large r

Montonicity of f then gives us a trichotomy (f either diverges to one of the infinities or converges to zero), and a control on the creation of new zeros of f (they come from infinity separately as c changes).

These, together with some additional observations regarding stationary points of f, let us conclude via the continuous dependence on initial conditions theorem that for c_n the solution f cannot diverge to any of the infinities and must have n-1 zeros.

Uniqueness

Uniqueness of the obtained ground state (meaning that for a fixed b there exists exactly one c giving positive bound state) can be shown by assuming that there are two different f_1 and f_2 with $c_1>c_2$ and defining:

$$\rho(r) = f_1(r)/f_2(r) \qquad \mu(r) = r^{d-1}f_2(r)^2\rho'(r)$$

Then it holds $\mu'(r) = r^{d-1}f_2(r)^2\rho(r)\left[h_2(r) - h_1(r)\right]$

Investigation of this equation, together with equations for h_i, leads to ρ and μ being decreasing. In particular, for r>1 we have μ(r)<μ(1) and

$$-1 < \int_1^\infty \rho'(r)dr < f_2(1)^2\rho'(1)\int_1^\infty \frac{dr}{r^{d-1}f_2(r)^2} < 0$$

Hence, the integral on RHS is convergent. However, then from the CS inequality we get a contradiction:

$$\infty = \int_1^\infty dr \leq \left(r^{d-1}f_2(r)^2\right)^{1/2}\left(\frac{1}{r^{d-1}f_2(r)^2}\right)^{1/2} < \infty$$

The fact that f is positive was crucial here.

The uniqueness of excited solutions to NLS equations is a very challenging and in most cases open problem.

Similar methods can be used for other NLS equations, but the details may differ. For example, in the case of GP equation one gets existence of bound states, but the uniqueness cannot be proven this way.

Bibliography

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