Peirce’s Search for a Graphical Modal Logic  
(Propositional Part)

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Received 1 May 2010  Revised 6 November 2010  Accepted 11 November 2010

This paper deals with modality in Peirce’s existential graphs, as expressed in his gamma and tinctured systems. We aim at showing that there were two philosophically motivated decisions of Peirce’s that, in the end, hindered him from producing a modern, conclusive system of modal logic. Finally, we propose emendations and modifications to Peirce’s modal graphical tinctured systems and to their underlying ideas that will produce modern modal systems.

1. Introduction

Our concern is modality in Peirce’s existential graphs (EGs), as expressed in his \(\gamma\) and tinctured systems. EGs as restricted to propositional and first-order predicate logic have experienced a small renaissance with Zeman’s 1964 dissertation (Zeman 1964) and with Roberts’s 1973 monograph and again, more recently, with works, namely, of Hammer (1998), Roberts (1997) and Shin (2002). While these publications have found a certain following since (we think of the works on knowledge representation in computer science and explicitly name Sowa’s conceptual graphs), there is still little research on Peirce’s modal logic and especially on the formal aspects of his tinctured EGs.

We restrict our investigations to Peirce’s propositional modal logic, that is, to the subset of his \(\gamma\), and later tinctured, system whose non-modal components are solely those of his propositional \(\alpha\) system, leaving the first-order \(\beta\) components, normally also embedded in the \(\gamma\) respectively tinctured graphs, for future research.\(^1\) Another restriction is that we deal with Peirce’s system purely syntactically, that is, without giving a formal interpretation for his graphs. For formal semantics of EGs and Peirce’s contributions to the development of possible-world semantics, see the papers of Hammer (1998) and Pietarinen (2006a). For a game theoretic approach to Peirce’s logic, consult Pietarinen’s (2006b) Signs of Logic.

What we aim at is, first, showing that there were two philosophically motivated decisions of Peirce’s (see Section 4.1 and 4.2) that, in the end, hindered him from producing a modern, conclusive syntactical system of modal logic (Section 4.3). To do so, we have to describe the development that led Peirce to the tinctured system, which is the one he finally preferred (Sections 2 and 3).

Second, we intend to propose emendations and modifications to Peirce’s modal graphical tinctured systems and to their underlying ideas that will produce modern modal systems (Section 5). For \(\gamma\) graphs, this has already been done or at least initiated by Zeman in his 1964 dissertation. For tinctured graphs, this has not been done yet. Roberts makes some suggestions, but adds, ‘A full scale comparison of the tinctured graphs with contemporary modal logic awaits separate treatment’ (Roberts 1973, p. 105) and furthermore ‘[T]his

\(^1\) Although Peirce consistently uses the terms Alpha, Beta and Gamma (most of the time capitalized), we prefer using the Greek letters for brevity.
material is put forward provisionally. Further experimentation will show that additional restrictions are necessary to preserve the consistency of the system. [...] By publishing these results I hope to elicit help in the development of the tinctured EG’ (p. 107).

Peirce’s EGs may be considered his most advanced system of graphical logic insofar as they offer correct and complete sets of derivation rules for both propositional and first-order predicate logic. Peirce gives a complete set of derivation rules (in the heading of CP 4.414, he calls them rules of transformation) for the α and β parts of his existential systems in his 1906 Monist article ‘Prolegomena to an apology for pragmaticism’ (Peirce 1906 pp. 535–541; reprinted as CP 4.530–572), there called ‘the four permissions’. We base our consideration on the α rules as laid out in this tinctured period article, because there they are given in their most mature, and final, form. For readers not familiar with these rules, we refer to Appendix 1.

The set of well-formed α graphs and, with that, the concept of the level of a graph are defined in the following way (we will need this later on):

1. Each propositional constant is an α graph. Its level is 0.
2. The empty space (sheet of assertion or Phemic sheet) is an α graph of level 0, too.3
3. If ϕ is an α graph of level n, then

![α graph level n+1]

is an α graph of level n + 1.
4. If ϕ and ψ are α graphs of levels n and m, respectively, their concatenation ϕψ is an α graph whose level is the maximum of n and m.
5. Nothing else is an α graph.

Furthermore, we define the following: a graph is said to be evenly/oddly enclosed if its level is an even/odd number. The area of a cut is the ‘surface within the Cut, continuous with the parts just within it’ (CP 4.556). Generally, the area of a (sub-)graph ϕ is the area of the cut with the lowest level such that this cut fully contains ϕ. Note that by this definition, the whole sheet of assertion is an area, too.

2. Modal components of the γ graphs

2.1. Introducing γ

The γ part of EGs has been said to correspond to higher order logic, to abstractions and qualities, to a meta-theory about graphs themselves and to modal logic, at times. From a modern standpoint, one might even feel that γ mixes concepts so different – different kinds of modalities, and even meta-theory – that it would be advantageous to split up the whole system. And this is something we have in mind when directing our investigations towards the modal account.

To account for modality, Peirce finds it sufficient to add a new kind of cut to his α and β systems, respectively. Visually, this broken cut is exactly what its name implies: a cut not

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2 For our purposes, it is important to know that at different stages of the development of his systems, Peirce has different concepts of what operation actually makes up a cut, that is, negation. These range from the cut being a simple closed line separating true statements from statements that are false to sophisticated operations such as first writing false propositions at the bottom surface of the sheet (Peirce calls it the verso) and then making them visible by cutting them out and turning around the cut out part of the sheet.

3 Note that this clause makes the grammatical structure of a given graph ambiguous, but this does not matter for our purposes.

4 Roberts considers it corresponding ‘roughly, to second (and higher) order functional calculi, and to modal logic’ (Roberts 1973, p. 64). Zeman concentrates on γ’s modal power (Zeman 2002).
made up from a straight line, but from a broken line:

The broken cut indicates a weaker kind of negation, not asserting absolute falsehood of the area enclosed, but only possible falsehood – in other words, expressing the enclosed area possibly to be false and, hence, equivalent to the modern symbolic ‘\(\diamond\neg\)’. Conceding both this equivalence and the fact that modalities may be expressed by one another (i.e. that \(\square P \iff \neg \diamond \neg P\) and that \(\diamond P \iff \neg \square \neg P\)), it is easy to see that Peirce’s broken cut suffices (together with the standard cut) to express all kinds of modern, alethic modalities:

\[
\begin{align*}
\text{necessity} & : \diamond P \iff \neg \diamond \neg P \\
\text{possibility} & : \diamond P \iff \neg \square \neg P
\end{align*}
\]

2.2. \(\gamma\) derivation rules

Peirce seems to take it for granted that for \(\gamma\) there must be a system of derivation rules producing all true propositions. He, nevertheless, gives only three of these rules, considering the system ‘still in its infancy’ (CP 4.511), admitting that ‘[he] was as yet able to gain mere glimpses, sufficient only to show [him] its reality, and to rouse [his] intense curiosity, without giving [him] any real insight into it’ (CP 4.576) – and predicting that it ‘will be many years before [his] successors will be able to bring it to the perfection to which the alpha and beta parts have been brought’ (CP 4.511).

\(\Gamma_1\)  ‘In a broken cut already on the sheet of assertion any graph may be inserted.’ (CP 4.516)

(This obviously must be taken literally, requiring the broken cut to be the outermost cut, that is, a cut of level 0, and the insertion to take place directly within the broken cut – or some other restriction of a similar kind. It should be noted that Roberts (1973) does not explicitly mention this rule.)

\(\Gamma_2\)  ‘An evenly enclosed alpha cut may be half erased so as to convert it into a broken cut, and an oddly enclosed broken cut may be filled up to make an alpha cut. Whether the enclosures are by alpha or broken cuts is indifferent.’ (CP 4.516)

\(\Gamma_3\)  If, at some state of information, a proposition P is known to be true (indicated by a selective), we may infer its necessity for any later state of information, as indicated by a higher number of selectives (CP 4.518):

From \(\vdash P\), we can infer \(\square P\).

\(\Gamma_3\) may at first look like an equivalent to the modern rule of necessitation, also known as the Gödel rule. This would pose no practical restrictions on which modern system to

\[\text{loc. cit.}, \text{the graphs being of the form of Peirce’s figures 188 and 189, respectively.}\]
use for interpreting $\gamma$, since the necessitation rule is part of every system that may come to mind. What makes things much more difficult are those small dashes (sometimes called cross-marks) on the lower side of the broken cut. Peirce calls them *selectives*, and they indicate that the necessity we are dealing with is not an absolute, logical necessity of the modern kind, but a necessity relative to the Graphist’s state of knowledge: something is necessary if known to be true, whether for purely logical reasons or as a matter of empirical fact: if our state of information is such that we know a proposition $P$ to be true, it becomes impossible for $P$ to become false – and this, by definition, means that it is necessarily true.

The second rule, $\Gamma 2$, comes down to $\neg \Diamond \neg P \rightarrow P$, the modal axiom $T$, indicating that what Peirce had in mind was at least a T-type system.

2.3. Giving up the modal part of $\gamma$?

Let us start with some terminological clarifications.

For the rest of this paper, by ‘the modal part of the $\gamma$ graphs’, we exclusively mean the use of the broken cut (in addition to the $\alpha$ and $\beta$ rules).

‘Did Peirce give up the modal part of $\gamma$?’ can be understood as one of two different questions: (1) Did Peirce stop using the broken cuts? (2) Did he still name the system $\gamma$ after having introduced the tinctures?

The first one will be the topic of the next section, and the second one is only a question of terminology, which cannot be decided by textual evidence, as far as we know: on behalf of the tinctures, Peirce once said:

This improvement gives substantially, as far as I can see, nearly the whole of that Gamma Part which I have been endeavoring to discern. (CP 4.578)

Statements such as this one leave it open if the improvement is such that Peirce wants the result still to be called $\gamma$. We name those of Peirce’s attempts which contain broken cuts the ‘$\gamma$ system’ (no matter whether tinctures are also included) and those which contain only tinctures the ‘tinctured system’.

2.3.1. What could have been the reasons for giving it up? Peirce’s view on the sheet of assertion does not fit nicely with modal propositions. Peirce considers the sheet to be a representation of all individuals existent (from now on for short: to contain all individuals existent), with true facts being certain kinds of individuals (other kinds of individuals coming into play only with $\beta$). Writing down a proposition on the sheet of assertion points out its being one of them: a true fact. While this strictly assertional sheet is compatible with the turn-around view of the standard cut (since each cut gives view to the back of the sheet of assertion, its front is unharmed in its strictly positive character), this is not the case with modality and the broken cut: when uttering a modal proposition, we are no longer dealing with existent individuals, but with individuals possible. But, at least initially, Peirce’s sheet of assertion was not meant to bear merely possible individuals (he considered the so-called tinctured sheets later on to express different modes of modalities).

Two other issues are pointed out by Roberts (1973, p. 87):

[Peirce] was not satisfied with Gamma. In the first place, [Peirce] was sure that there were rules of inference yet to be discovered […]. In the second place, he had been unable to develop the purely syntactical exposition of Gamma that he had aimed

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Footnote 7: The derivation will start with the assertion of $\neg \Diamond \neg P$, which in $\gamma$ notation is a $P$ within a broken cut, enclosed by a standard cut. Now the inner cut, that is, the inner negation, will be enclosed by an odd number of cuts (i.e. by one cut), licensing our filling it up to become a standard cut. This operation leaves us with $\neg P$, which is the EG equivalent of $P$. 
for; the use of his new symbols depended too heavily on their significations. He said this much in the 1903 Syllabus. (Ms 478, p. 157)

Finally, we consider a peculiarity of the broken cut severely incompatible with Peirce’s view on the semantics of the sheet of assertion. Half erasing a double cut (i.e. applying the first part of the derivation rule we call $\Gamma^2$) is passing from an assertorial proposition to a modal one – what is the case is also possible (more precisely: what is not the case is also possibly not the case). Unfortunately, this means that applying a derivation rule alters the state of the individuals to whom the cut is applied: whereas when the standard cut is applied to existing individuals (existing facts), the broken cut must be applied to possible individuals (possible facts). A system whose derivation rules force us to change the semantical or the metaphysical state of an individual to whom the involved propositions are referring seems inadequate.

2.3.2. Did Peirce indeed give it up? In ‘Prolegomena’, where Peirce is very much concerned with tinctures, he does not mention the broken cuts anymore. The question, therefore, arises whether he thought of the tinctures as substituting the broken cut or as just adding further, new facilities to his EG.

Neither in the formulation of his rules nor in his explanations of the examples does one find any hints on ideas related to those of the broken cut; he only refers to tinctures. He bothers a lot about the rules – so why should he have left aside exactly the broken cuts, especially as they would have been a central part of a treatment of modalities, which is an important aim of the paper.

One could argue that he could simply have refrained from including the $\gamma$ part, restricting himself to $\alpha$ and $\beta$, or one could conjecture that he just tacitly imported them, presupposing what he had written elsewhere – but we think that he then would at least have told his readers about this decision.

Moreover, there is a seemingly rather obvious reason for thinking that Peirce kept the broken cuts as an element of EG, when he developed the tinctured graphs: broken lines appear in his examples given in ‘Prolegomena’ – but they are not broken cuts in Peirce’s sense (for a detailed argument, see Appendix 2), they only look alike; in fact, they simply indicate that after turning a piece twice, it is again the recto of the sheet what we see.

Anyway, it becomes clear not only from the examples he gives in ‘Prolegomena’ but also from the simple systematic considerations that one does not need the broken cut to express ‘possibility’, instead one can use a tincture ‘locally’ (Peirce 1906, p. 527: ‘we are to imagine that the Graphist always finds provinces where he needs them’):

\[
\diamond\neg
\]

and

\[
\text{A}
\]

both say $\diamond\neg$.

Our conclusion is that though one cannot definitely decide whether Peirce abandoned the modal part of $\gamma$ and it seems clear that he could have stuck to the broken cuts within the tinctured system, he, first, could have equally well dropped them, since there was no need for them anymore and, second, did not use them anymore later on, especially in ‘Prolegomena’.
3. The tinctured graphs

3.1. Peirce’s presentation of the tinctured graphs

Important text sources are Ms 295, the ‘Prolegomena’ and Ms 670. Which passages the Peirce scholar has to consult depends very much upon whether he or she is interested in semantic or syntactic considerations. For semantic questions, for example, the papers on continua are obligatory, because the constitution of the universe of discourse has technical implications. Zeman (1997) works out Peirce’s semantic ideas. We will from now on concentrate on syntax as far as possible.

Peirce’s elaboration of the tinctured graphs (cf. Roberts 1973, pp. 88 ff.) keeps quite an experimental character. We divide it into three different stages.

3.1.1. The layered sheet of assertion (broken cut, within the $\gamma$ part, 1903) Peirce’s first modification to the $\alpha-\beta$ system lies in replacing the single sheet of assertion, representing all individuals existent, by a layered sheet. While the top layer (Peirce calls it the $\alpha$-sheet) retains its assertorial function, the (coloured) layers below, becoming visible where there is a hole in the top layer, are ‘areas of conceived propositions which are not realized’ ($\textit{CP}$ 4.512).

It is not per se clear if the broken cut is the only kind of see-through hole in the top layer and if the standard cut retains its function of turning the sheet of assertion (now the top layer), also retaining the $\textit{recto/verso}$ distinction, or if the standard cut, too, becomes an actual see-through hole, giving view to some layer of false propositions. We do not think that there is conclusive evidence in either direction. On the one hand, one may take Peirce’s remarks in $\textit{CP}$ 4.512 that ‘[a]t the cuts we pass into other areas’ and that in $\gamma$ ‘all kinds of signs take new forms’ as indications for the former view. On the other hand, when taking this view, certain things stop matching awfully well. Consider, for example, double negation, how could we ensure that cutting a hole in a layer visible through another hole should always give view to a layer where the initial proposition is true again? We need to ensure this if we want to keep the rules of the double cut, and there is no evidence whatsoever that Peirce intended to drop them.

Putting aside the problem of negation, those layers visible through broken cuts make up conceivable states of affairs – possible worlds (Peirce literally uses the term ‘imaginary worlds’ in $\textit{CP}$ 4.512). Each such possible world may itself be cut into, thereby giving view to yet another possible world. Again, it is not strictly clear if this remark considers broken cuts, standard cuts or even both, but we consider it quite natural for the broken cut to lie within the extension of this remark, because, after all, why should the broken cut behave differently when applied to an imaginary world than when applied to the actual world?

Provided that broken cuts are see-through holes when applied to imaginary worlds, too, we here have some kind of ordering of worlds – each world is able to ‘see’ worlds lying below it, but none lying above it (as long as we are not confronted with cuts of a worm-hole-like nature, or as the miscellaneous layers are not arranged in a Rolodex-like fashion).

Although we did not find conclusive evidence whether standard cuts, like their broken counterparts, are see-through holes in $\gamma$, if they are, this would introduce a number of problems non-trivial to deal with. Besides the already mentioned problem with double negation, what happens with a straight cut within a broken cut, and vice versa? We consider it not unlikely that it was questions such as these that caused Peirce finally to settle with the $\textit{recto/verso}$ view in his 1906 Monist account for his tinctured graphs: ‘Should the Graphist desire to negative a Graph, he must scribe it on the $\textit{verso}$, and then […] must make an

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incision [...] round the Graph instance to be denied, and must then turn over the excised piece (Peirce 1906, p. 528, and CP 4.556).

However, getting too deeply involved into studying multi-layered \( \gamma \) sheets here would certainly not follow Peirce’s line of thought: although clearly exposing the multi-layered view in CP 4.512–4.513, with the next paragraph, CP 4.514, he completely drops it for \( \gamma \) – and hence for three years – reinterpreting both kinds of cuts as simple lines on a single sheet of assertion.

3.1.2. Negation and modality coupled (April 1906, talk ‘For the National Academy of Science’, CP 4.573) In the preparations for his 1906 talk, Peirce reintroduces tinctures (colours). Colour expresses possibility and is placed on the verso of the sheet of assertion.\(^9\)

A cut indicates the following: cut out the piece and afterwards turn it around. The result has to be interpreted as follows: ‘It is not possible that …’ (more exactly: ‘[It] cuts off something from our list of subjective possibilities’, CP 4.574). All this can be iterated for several layers, each having a tint on its verso.

A major change is that instead of ‘possibly not’ – the broken cut – Peirce now uses an expression for ‘not possible’. The great disadvantage of the described procedures is that negation and possibility are joined in such a way that neither negation nor possibility can be expressed alone. Whereas the broken cut and the standard cut together are well able, as we saw, to express all the usual combinations of (alethic modal) operators and negation, the tincture with the standard cut – handled in the described way! – does not provide that. Using a cut always means turning around the piece and hence expresses ‘not possible’; one cannot get a view on the verso in any other way; therefore, ‘possibility’ can only be said as something not being the case. In other words, one only has \( \neg \Box \) at one’s disposal,\(^10\) neither \( \neg \) nor \( \Box \) (and no other operator), and \( \{\neg \Box, \land\} \) is, of course, not a functionally complete set for the propositional modal logic.

Note that there is no mention of different modes of modalities (other than possibilities) at that time; Peirce is only concerned with possibility. And this possibility is epistemologically determined (as ‘subjective’), not neutral.

3.1.3. Negation and modality separated (about April/May 1906 and later on, Ms 295, Ms 670) Peirce tries out different versions of combining cuts and tinctures and suggests several improvements. In Ms 670 (see Roberts 1973, p. 98), for example, the borders of sheets are made up by a cut and have the function of a cut, that is, they express negation, and hence modalities always appear in negated form. Another version (in ‘Prolegomena’) is as follows: there are tinctured sheets at one’s disposal, each one having its own verso, a cut is always connected with turning around the piece within the cut and is to be interpreted as negation (in the mode expressed by the respective sheet). The second alternative will be taken as the basis for the following section.

3.2. How did Peirce use tinctures? Examples and general considerations

The main change between using the broken cut and using tinctures is that the tinctures enable Peirce to distinguish between different modes of modalities (cf. Pietarinen 2006a,

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\(^9\) Actually, the verso usually is of bluish grey, but it may as well be yellow, or rose, or green, and the recto is of cream white – according to CP 4.573.

\(^10\) Of course, one could express ‘possible’ as ‘not-necessary not’ and use a different tincture for necessity, but then there is no connection between the tincture that is used for ‘not-possible’ and that which is used for ‘not-necessary’.
In ‘Prolegomena’, Peirce describes it like this (we omit all conventions concerning the March):

But certain parts of other sheets not having the significance of the Phemic sheet, but on which Graphs can be scribbled and erased, shall be sometimes inserted in the Phemic sheet and exposed to view [...]. Every part of the exposed surface shall be tinctured in one or another of twelve tinctures. These are divided into three classes of four tinctures each, the class-characters being called Modes of Tincture, or severally, Color, Fur, and Metal. (Peirce 1906, p. 526 sq.)

For our purpose, it is important to see that Peirce treats tinctured sheets as objects of syntax: they are inserted in the Phemic sheet, they are scribbled on, and turned around, so they are means for expressing and hence part of the ‘language’ (for Peirce, syntax in logic is manipulating signs in a certain way). The number four for each of the modes is easily explained if Peirce held the four-colour theorem to be true. This theorem states, roughly, that four colours are necessary and sufficient to colour every map (of countries, for example) in such a way that no two adjacent regions have the same colour. And it is very likely that Peirce thought, together with most mathematicians of that time, that the theorem was correct. Kempe, for example, was supposed to have given a valid proof for some time (then it turned out to be wrong), and Peirce himself tried to figure out a proof (see Biggs et al. 1977).

There is so much about the formal aspect of the tinctures, but they can also be characterized by the content they are designed to cope with:

[T]he Mode of Tincture of the province […] shows whether the Mode of Being which is to be affirmatively or negatively attributed to the state of things described is to be that of Possibility, when Color will be used; or that of Intention, indicated by Fur; or that of Actuality shown by Metal. Special understandings may determine special tinctures to refer to special varieties of the three genera of Modality. (Peirce 1906, p. 527)

What is written on a sheet is to be understood as being in the mode indicated by the colour or material of the sheet. For example, if we suppose that blue indicates possibility, every sentence with $A$ written on the blue paper is to be read as ‘It is possible that $A$’.

The special tinctures are to be understood, Peirce tells us, as a refinement of the modes of modalities. But besides this, even if one restricts the use of tinctures to one colour, let us say, there are remarkable differences between the use of tinctures and that of the broken cut. One is that tinctures allow you to express a modality in a non-negated form; the broken cut does not (at least, Peirce never uses it like that, because the very idea of the broken cut is ‘weak negation’).

Another difference is topological (in the mathematical sense): whereas putting together two sheets of the same tincture (both supposed to be topologically equivalent to a disk) gives a new sheet of the same topological class, joining two cuts to become one new cut does not work without a break of continuity.

There are three pieces of information that Peirce gives concerning the formation rules – he calls them ‘Conventions’ – involving tinctures. First, tinctures are at disposal wherever they are needed. Second, as we saw, a cut is the border of a province in some of Peirce’s drafts and is made within a province in others (as one can see in ‘Prolegomena’, tinctures are not, in general, to be understood as filling the enclosure of a cut), but in none of his examples does a cut lie on two different provinces (compare our remarks in Section 4).

Hence, we might state as a formation rule that a cut must not ‘leave’ a province (in contrast to the lines of identity of $\beta$ (CP 4.579)) and that means that it must not be situated partly
within, partly outside a province. Third, if we allow cuts to be not only at the border of a
tinctured sheet, negation and modality can be separated: a cut may be drawn on or around
a tinctured sheet, making it possible to express ♦¬ as well as ¬♦.

Peirce does not give any derivation rules (Roberts 1973, p. 98, says this, too) for a
tinctured system, but from γ, we know (Section 2.2) that he was thinking about rules such
as ‘possibility follows from actuality’ or, more generally, about inferring a proposition in a
certain mode from the same proposition in a different mode. (In his examples, he frequently
examines the difference between a graph and the same graph supplemented by a tincture at
some place of the graph, but he does not describe this in terms of inferences.)

As far as we know, he never mentions a situation in which one may write two propositions,
initially written on different sheets of the same mode, on one and the same sheet.

4. Why was Peirce not led to a syntactical system of modal logic
in the modern sense?

The following considerations are based on tinctured systems not using broken cuts, as are
laid down in the late manuscripts (of what we called the third period, see our Section 3.1.2).
The question why the modal γ did not lead to a modern system of modal logic will not be
treated here, and we doubt that it could be settled, as Peirce turned away from the broken
cut (according to our arguments in Section 2.3.2) and put his interest in the tinctures for the
rest of his considerations on EGs and modality.

We will try to show that there were two decisions which taken together build a reason
for failing to develop a modal logic in the modern sense. By ‘failure’ we neither imply that
Peirce was aiming at developing a syntactical system in the modern sense nor imply that he
should have, but only that it did not happen for these reasons.

4.1. EGs instead of entitative graphs

The first decision was something that happened much before the tinctured graphs had
been invented: Peirce had given up entitative graphs in favour of EGs.

The entitative graphs are simply the dual system to EGs, or, in Peirce’s own words:
‘Existential Graphs […] are merely entitative graphs turned inside out’.11 Translation
to and from an entitative graph and its existential counterpart is, therefore, a strictly
mechanical matter. Peirce knew this,12 but had, nevertheless, reasons for strictly preferring
the EGs.

His main reason for giving up entitative in favour of existential was probably (cf. Roberts
1973, p. 29) that whereas writing down a single proposition on the sheet of assertion means
asserting it, adding a second proposition does not – and what is worse – make even the first
proposition cease to be asserted.

As another reason for abandoning entitative in favour of existential, Peirce mentions that
entitatively the blank sheet of assertion would have to be interpreted as an absurdity, while
with EGs, it simply expresses what is taken for granted (CP 4.434) – truth itself. Here, too,
the entitative approach seems less natural, because uttering nothing at all generally means
refraining from uttering the absurd.

We may take Peirce’s remarks on representing the conditional as another reason for
abandoning the entitative graphs. If the most iconic representation of a conditional A → C

11 Ms 280, pp. 21–22, cit. Roberts (1973, p. 27). They are sometimes considered equivalent to the much later Laws of Form
system, most directly stated by Robert Burch (2006): ‘A version of the entitative graphs later appeared in G. Spencer Brown’s
Laws of Form, without anything remotely like proper citation of Peirce’.

12 ‘Any entitative graph may be converted into the equivalent existential graph by, first, enclosing each spot separately and, second,
enclosing the whole graph’ (Ms 485, p. 1, cit. Roberts 1973, p. 28).
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actually happens to be \((A(C))\) (cf. \(CP\ 4.435\)), there is no way to stick with the entitative system where \(A \rightarrow C\) cannot be expressed but \((A)C\) or, at best, \(((A)((C)))\).

Peirce’s choice has lasting consequences insofar as the iconicographic precedence of the conjunction remains obvious throughout all deductions within the system. In a calculus of natural deduction, no such precedence needs to exist, and in an axiomatic calculus, after having defined conjunction by disjunction (and negation) or \textit{vice versa}, which one was first cannot be seen anymore. But in the EGs, disjunctions always stay visually more complex than conjunctions. This is not, in general, a flaw of Peirce’s system, but it causes the mentioned incompatibilities, as we shall see.

4.2. The predominance of ‘possibility’

The term ‘predominance’ has to be explained to avoid misunderstandings. Neither beyond the tinctures nor as a philosophical concept is ‘possibility’ more important or somehow privileged over ‘necessity’. Only as a guideline, a leading idea \textit{when developing EGs}, it is, we claim, pivotal for Peirce.

Let us have a look at the explanation of an example\(^{13}\) that Peirce gives in ‘Prolegomena’ (one does not need to know what the example is concerned with to understand the point we want to make):

\[\text{T}\]he proposition amounts merely to asserting that there is a married woman who will commit suicide if every married man fails in business. The equivalence of these two propositions is the absurd result of admitting no reality but the existence. If, however, we suppose that to say that a woman will commit suicide if her husband fails, means that every possible course of events would either be one in which the husband would not fail or one in which the wife would commit suicide, then, to make that false it will not be requisite for the husband actually to fail, but it would suffice that there are possible circumstances under which he would fail, while yet his wife would not commit suicide (Peirce 1906, p. 516).

Peirce does not give a diagram at \textit{this} point.\(^{14}\) But this passage can be read as a hint that Peirce uses possibility, not necessity, as a key idea in his contributions to modal logic (cf. also Section 3.1.2). Crucial for this position is that there are \textit{real} possibilities (\(CP\ 4.579\)):

\[\text{[A]lthough I have always recognized that a possibility may be real, that it is sheer insanity to deny the reality of the possibility of my raising my arm, even if, when the time comes, I do not raise it [...]; yet whenever I have undertaken to develop the logic of relations, I have always left these references out of account, notwithstanding their manifest importance, simply because the algebras or other forms of diagrammatization which I employed did not seem to afford me any means of representing them. I need hardly say that the moment I discovered in the \textit{verso} of the sheet of Existential Graphs a representation of a universe of possibility, I perceived that a reference would be represented by a graph which should cross a cut, thus subduing a vast field of thought to the governance and control of exact logic (Peirce 1906, p. 516).}

\(^{13}\) We discuss this example in Appendix 2.

\(^{14}\) Roberts (1973, p. 96) finds this astonishing because Peirce had already made several diagrams for similar examples in his manuscripts. Roberts offers quite a simple explanation: in the weeks before the publication of the ‘Prolegomena’, the development of Peirce’s logical theory was progressing so rapidly that he did not manage to complete the examples in time. Roberts (1973) shows the missing diagram (p. 97), but it differs from the one Peirce gives later in ‘Prolegomena’. Roberts’ version is the one coinciding with the one we suggested as being ‘what Peirce has actually meant or should have meant at least’.
This quotation well illustrates what we mean by ‘possibility as a leading idea in the development of tinctured graphs’.

But why does reality of the possible matter so much for the logical system? Because it gives possibility a clear ‘place’ within logic: possible individuals (facts) may be considered as what the dots and lines of identity point at, just as it holds for (actual) individuals.

So, Peirce introduces ‘possibility’ as a semantic notion (compare also Section 3.1, especially 3.1.1). All his semantic considerations stay just loosely connected with his much further developed syntactical accounts. (As it leads us too far away from our topic, we just briefly mention that Peirce (1983, p. 84 sq.) already uses the term ‘universe’, but keeps his distance from Boole’s and De Morgan’s ‘universes of discourse’ (see Peirce 1906, p. 514, footnote; CP 4.544). For the history of the term ‘universe of discourse’ (see Iliff 1997, p. 203). The transition from ‘possible circumstances’ to ‘possible individuals’ would require some further study.) But though ‘possibility’ is introduced as a semantic concept in the EGs, it is, as we saw in Section 3.2, used as a part of syntax.

The modern logician’s attitude towards ‘necessity’ and ‘possibility’ is neutral: neither of the concepts is considered superior or prior to the other in any sense (besides that, one is — as the result of an arbitrary decision — defined in terms of the other). Not so with Peirce — philosophical reasons lead him to put an emphasis on possibility in some places, on necessity in others. Although necessity bothers Peirce a lot in epistemology, it is neither a central topic nor the starting point for his contributions to modal logic — this is what we try to show in the following paragraphs. We will not discuss the question of ‘subjective’ versus ‘objective’ possibility and necessity.

The first concept of necessity that we deal with is necessity as a property of inferences. Unlike ‘possibility’, which is throughout defined as a property of objects, individual acts, states, etc., ‘necessity’ in its main use is ascribed to inferences. This difference manifests itself in Peirce’s logic: possibility is something that is put on the sheet of assertion or that is a part of it. Necessity, on the other hand, denotes a way of dealing with the sheet of assertion or of estimating connections in reality, respectively. For Peirce, necessary reasoning concerning real objects corresponds to manipulations on the sheet of assertion. Logic is experiments with diagrams (cf. Peirce 1906, p. 493). ‘Necessity’ — necessary reasoning — is the whole subject of logical studies. It is, what logic is all about, not a single part of it (cf. CP 4.431).

It is a simple observation that in the large majority of cases, where Peirce states or uses a connection between possibility and necessity, he defines ‘necessity’ by ‘possibility’ (and ‘negation’), and not vice versa (see, e.g. CP 4.519, 4.546, Roberts 1973, p. 102). This priority of ‘possibility’ is not by chance or only for convenience, but there is a theoretical justification in Peirce’s philosophy. He insists that possibility and necessity are of different quality and that they belong to different categories:

Necessity is an idea of Thirdness. This word is equivocal: it is here taken in the sense of rational, i.e., general, necessity. It is not mere denial of Possibility. For Possibility, in the sense of Firstness, is not a subject of denial. The absence of any given

16 We are concerned with concepts involving logical properties only. Interpretative terms such as ‘physical necessity’ are beyond the scope of our investigations.
17 In the technical part of his γ system, Peirce very strongly takes the view that possibility and necessity are subjective concepts and that they are ‘relative to the state of information’ (CP 4.517 from the Lowell Lectures of 1903) of a certain individual, the Graphist.
18 For our purposes, problems concerning the epistemic status, reliability, etc., of necessary reasoning may be put aside; it is just the role it plays in logic that is important for us.
possibility is, of course, a possibility; but to leave a character standing and remove from it its possibility is nonsense, unless one means to speak of a representamen of the quality, in which case the element of Thirdness is the predominant one (Peirce 1992–1998, 2, p. 271).

A thing is – on its own – as it is. Having capabilities or potentials is part of the genuine nature of a thing, it is firstness. Lacking a possibility affords an interpreter attributing the absence of this property to the thing. Therefore, possibility belongs to the thing, but necessity depends on the thing and the interpreter. We cannot discuss this sufficiently but must be content with the statement that this may be why possibility is a more fundamental concept than necessity for Peirce in the context of logical systems. (Compare, for example, Ms 670, German translation Peirce (2000, p. 460): oddly enclosed graphs express necessity if the logical universe is that of metaphysical probability.)

Admittedly, there are stand-alone occurrences of necessity as a part of syntax in Peirce’s writings. For example, in an early list of Ms 295 (cf. Roberts 1973, p. 90), he suggests necessity to be brought in by putting a piece of ermine on the sheet of assertion. Peirce calls propositions written on ermine ‘metaphysically’, or ‘rationally’, or ‘secondarily’ necessitated. But such occurrences are very rare, and they are not coordinated with any other considerations.

To sum it up: neither of these occurrences of ‘necessity’ is a rival for ‘possibility’ regarding the position of advancing the development of modal logic.

4.3. How did these presuppositions hinder him?

Let us use the following axioms to define some of the usual propositional modal logic systems:

\begin{align*}
(K) \text{ (or } (R)) & \vdash \Box (A \land B) \leftrightarrow (\Box A \land \Box B) \\
(T) & \vdash \Box A \rightarrow A \\
(D) & \vdash \Box A \rightarrow \Diamond A \\
(4) & \vdash \Box A \rightarrow \Box \Box A \\
(5) & \vdash \Diamond A \rightarrow \Box \Diamond A
\end{align*}

Additionally, we need the rule of necessitation and the rule of inference:

\begin{align*}
(RN) & \quad \text{From } \vdash A, \text{ it may be inferred that } \vdash \Box A. \\
(RI) & \quad \text{From } \vdash A \text{ and } \vdash A \rightarrow B, \text{ it may be inferred that } \vdash B.
\end{align*}

We consider the systems K, T, D, S4 and S5, based on propositional logic (PL):

\begin{align*}
K: & \quad \text{PL + (K), (RN), and (RI)} \\
T: & \quad \text{K + (T)} \\
D: & \quad \text{K + (D)} \\
S4: & \quad \text{T + (4)} \\
S5: & \quad \text{T (+ (4)) + (5)}
\end{align*}

Peirce’s considerations, and also Roberts’ reconstruction (Roberts 1973, p. 107 sq.), confine to statements of a type such as (T), (4) or (5), that is, they state inferential relations between a proposition with a modal prefix (a concatenation of a number of modal operators) and the very same proposition with a different modal prefix. An exposition of (K), or

19 That is, sign.

20 The axiom we call (K) here is usually called (R) – a combination of (M) and (C) (cf. Chellas 1995, p. 114) – but it does the job of (K) in our context.
more generally of connections between several different propositions, is almost completely missing – and this is exactly the place where Peirce’s troubles with developing a modal logic may be localized.

\((K)\) – or some equivalent – is the very basis of every system of modal logic. \((K)\) guarantees that *modus ponens* stays valid for necessitated propositions. Now, if Peirce would have been engaged with necessity and have used a sheet of the sort

![Diagram](image1)

let us say, he would have very easily come to formulate a rule \((K_P)\) which suggests itself:

![Diagram](image2)

In words: iff two graphs are written on provinces of the same sort, they may be written on one province of this sort instead (or the provinces may be joined together). The formula corresponding to this scheme is, of course, \((K)\).

But as Peirce, in fact, concentrates on the representation of *possibility* in his tinctured graphs, the situation is different and we ought to have a look at versions of \((K)\) dealing with possibilities:

\[
\begin{align*}
(K^{\Diamond,\lor}) & \vdash \Diamond(A \lor B) \iff \Diamond A \lor \Diamond B \\
(K^{\Diamond,\land}) & \vdash \Diamond(\neg A \land \neg B) \iff \neg(\neg \Diamond A \land \neg \Diamond B)
\end{align*}
\]

Reformulated as a transformation rule, we get in Peircean notation (we use the representation for azure from ‘Prolegomena’ to express possibility):

![Diagram](image3)

This rule is not quite self-evident.\(^22\)

So – to conclude – what are the reasons for Peirce’s contributions to modal logic staying fragmentary even though he seems to have had all the fundamental ideas?

Two basic and independent decisions, which unluckily interfere, impede the development of a modal logic system: on the one hand, for reasons of iconicity and of simplicity, Peirce decides to give up the entitative graphs in favour of the EGs, meaning that he chooses to give the simplest representation to conjunctions, not to disjunctions. On the other hand, he makes a philosophically motivated decision for using possibility (the possibility operator), and against using necessity, as the basic conception of modal logic. These two decisions do not fit together well, because in modal logic \(\land\) and \(\Diamond\) do not ‘harmonize’ – in Peirce’s system even less than in the usual ones: any axiom taking over the task of \((K)\) has a simple form exactly if it contains \(\Box\) and chooses \(\land\) to have the simplest representation, or \(\Diamond\) is contained and \(\lor\) has the simplest form, but not if \(\land\) and \(\Diamond\) or \(\lor\) and \(\Box\) are combined in this manner.\(^23\)

\(^{21}\) From now on, we will substitute axioms stating implications by corresponding transformation rules.

\(^{22}\) If you take the usual version of \((K)\) using \(\to\), you do not get a self-evident scheme, either.

\(^{23}\) In the entitative system, the version of \((K)\) using possibility would look like \((K_P)\) in the EGs.
($\vdash \Box (A \land B) \leftrightarrow (\Diamond A \land \Diamond B)$ is not equivalent with $(K)$!) So, some of Peirce’s philosophical principles prevented him from formulating an analogue to the basic rule $(K)$ of modal logic (and in consequence from modifying the $\alpha$ (and $\beta$) rules, for without anything corresponding $(K)$, there is no way of establishing any connection between graphs on different sheets and, therefore, no strong motive to modify the $\alpha$ rules). That Peirce did not think of anything like two graphs being necessarily true together if they are both necessarily true can be seen from a passage in ‘Prolegomena’: ‘Two graph instances not in the same Province, though on the same Mode of Tincture are only insofar connected that both are in the same Universe’ (Peirce 1906, p. 532).

Another circumstance could have obstructed Peirce’s view. For Peirce, modal logic emerged from some parallels between negation and modality (‘not being’ and ‘not being possible’ in $\gamma$, for example), but negation and modality behave quite opposite to each other. Inserting a proposition is admissible in a cut, but not a priori on a tinctured sheet.24 And while it is indispensable for a modal logic in the modern sense that a formula such as $(K)$ or $(K \Diamond, \lor)$ is taken as an axiom, there is, of course, no corresponding axiom for negation – the connection between $\square (A \land B)$ and $\neg A \land \Box B$ (or between $\Diamond (A \lor B)$ and $\Diamond A \lor \Diamond B$ or the like) is a fundamental axiom for all usual modal logical systems, whereas it need not be stated as an axiom or rule that $\neg (A \land B)$ can be deduced from $\neg A \land \neg B$ (as it follows by applying the rule of insertion and the rule of erasure once). This might explain how it is possible that the idea of formulating such a rule did not occur to Peirce’s mind.

5. Peirce aside: a variant of tinctured graphs generating modal propositional logic

Hence, to get a system of modal propositional logic (MPL), we have to turn away from Peirce a bit, but this is no reason for wondering; for example, Roberts already wrote: ‘I fully expect that new rules will be called for’ (Roberts 1973, p. 107).

5.1. The notation for necessity

In the following, we deviate from Peirce’s ideas insofar as we take ‘necessity’ instead of ‘possibility’ as the primitive notion. But we try to stick to his ideas as far as possible. We call the sheets used to indicate necessity red sheets and represent them like this:

What does a red-coloured area on which there is a cut containing an $A$ mean? We have to choose whether it shall mean $\Box \neg A$ or $\neg \Box A$. Since both turn out to do the job equally well, we arbitrarily choose $\Box \neg A$. So, the combinations of necessity and negation will be translated according to the following scheme:

\[\begin{array}{c|c|c|c|c}
A & \Box A & \Box \neg A & -\Diamond A & -\Box A \\
\hline
\neg A & -\neg A & \Diamond \neg A & \Diamond \Box A & \Diamond A \\
\end{array}\]

24 If the tincture expresses necessity, inserting would be wrong; if the tincture expresses possibility, an axiom is needed.
With these conventions, one may simply adopt Zeman’s translations from \( \alpha \) graphs to PL. The \( \alpha \) graphs may be mapped one to one onto the formulae of PL, as Zeman 2002 shows.\(^{25}\) It is immediate that such a one-to-one correspondence still exists between the tinctured \( \alpha \) graphs (\( \alpha \) graphs plus tintured sheets on which the graphs may be written) and MPL. We just have to make sure for (sub-)graphs not to lie only partially within a red area. Sheets are not allowed to have holes, except perhaps – depending on the definition of the cuts – those produced by cuts.

5.2. The system \( K_p \)

The difficulties of our venture do not lie in the translations of the modal logical axioms to Peirce’s graphs, but in the modifications of the rules, we recommend for the \( \alpha \) graphs. The set of transformation rules that we study is meant to be ‘admissive’ in the sense that we allow more things to be inserted, iterated, etc., than inevitable for getting a complete system. The permissions of the Rules of Iteration, Insertion, etc., could as well be restricted to the (white) Phemic sheet, because every theorem can be derived using \( (RNP) \). But the system would be clumsier this way. And although Peirce left the \( \alpha \) rules almost unchanged (compare Roberts 1973, p. 87), he, on the other hand, complained about not finding suitable transformation rules – had he tried to answer, for example, the question whether or not a red sheet may be iterated, this would most probably have made it much easier for him to find satisfying rules. And not accidentally, his rules for the \( \alpha \) graphs are ‘admissive’ in this sense, too.

We study the system \( K_p \) constituted by the following rules, and we will show that \( K_p \) is equivalent to the modal logical system \( K \).

\[\begin{align*}
\text{(R1)} & \quad \text{Rule of Erasure: Any graph (including red sheets) whose level is even may be erased.} \\
\text{(R2)} & \quad \text{Rule of Insertion: On an area of odd level, any graph (including red sheets) may be inserted.} \\
\text{(R3)} & \quad \text{Rule of Iteration: Every graph } \varphi \text{ (including red sheets) occurring on an area may be iterated in (i.e. copied to) the same area or to any of its sub-areas, but only on the same red sheet (not on an additional one) and not on the area of } \varphi \text{ itself.} \\
\text{(R4)} & \quad \text{Rule of Deiteration: Provided that the first part of the Second Permission, Iteration, licenses deriving a graph } \varphi \text{ from a graph } \psi \text{, it is valid to derive } \psi \text{ from } \varphi. \\
\text{(R5)} & \quad \text{Rules of Double Cut: If there are two cuts such that (a) one of them is completely contained within the area of the other and (b) there is nothing in between them except empty space and (c) it is not the case that they lie on different coloured sheets, then they may be removed at will. Furthermore, such double cuts may be inserted at any place in or around a graph or a sub-graph, provided that this is done such that nothing gets in between them and they lie on the same coloured sheet.} \\
\text{(RNP)} & \quad \text{Rule of Necessitation: Any red sheet bearing any graph attained from the empty Phemic sheet by using the rules may be put anywhere on the (white) Phemic sheet.} \\
\text{(KP)} & \quad \text{Any two red sheets not separated by a cut may be joined, and any red sheet may be torn into two pieces, provided no cut will be destroyed.}
\end{align*}\]

\(^{25}\) There are, of course, different such mappings depending on the calculus used to formulate the logical system. Zeman (1964) uses two of them.
One advantage of these rules over a usual axiomatic calculus or calculus of natural deduction is – as already in the case of the $\alpha$ graphs (without modality) – that they allow manipulations inside of formulae, which makes deductions more easy.

5.3. Proof of the equivalence of $\mathbf{K}$ and $\mathbf{K}_p$

In what follows, we denote by $F_i$ ($i = 0, 1, 2, \ldots$) the formula of PL which corresponds to the $\alpha$ graph $G_i$. Further designations are to be understood accordingly (cf. Zeman 1964, for example). We will use the same letter for an elementary proposition in Peirce’s notation and in the (axiomatic) modal logic, capitalizing it in the graph, and using its lower case for example). We will use the same letter for an elementary proposition in Peirce’s notation

All formulae are supposed to contain $\land$ and $\neg$ as the only connectives (which corresponds well with the graphs). Empty cuts are mapped onto some contradiction ($p \land \neg p$, for example).

As a first step, we prove that applying the rules of $\mathbf{K}_p$ transforms any graph $G_1$ into a graph $G_2$ such that the corresponding formula $F_2$ may be inferred from $F_1$ in MPL.

We start with ($\mathbf{K}_p$). Suppose $G_1$ and $G_2$ to be written (within an arbitrary graph) on a piece of red sheet each. Then, the corresponding formula is $\Box F_1 \land \Box F_2$. Joining the two pieces yields a graph whose translation is $\Box (F_1 \land F_2)$. As substituting $\Box (F_1 \land F_2)$ for $\Box F_1 \land \Box F_2$ is admissible in $\mathbf{K}$, we gain the desired result for ($\mathbf{K}_p$).

We turn to ($R2$). To succeed with a proof by induction, we have to claim a bit more than actually required, namely that for any graph $G^X$, which is obtained from $G$ by inserting a proposition $X$ in $G$, the following holds.

If the number of $\neg$-signs in whose scope $X$ lies within $F$ (the formula corresponding to $G$) is even, $F^X \vdash F$ holds. If the number is odd, one gets $F \vdash F^x$.

We now give the proof by induction. Let $n$ be the level of a graph $G$. If $n = 1$, the formula $F$ corresponding to $G$ may be written as $\neg (\Delta_1 F_1' \land \Delta_2 F_2' \land \cdots \land \Delta_k F_k') \land F'$, where $\Delta_i \in \{\Box \Box \cdots \Box : j \geq 0\}$ for $i = 1, \ldots, k$, each of the $F_i'$ is an elementary proposition and $F'$ does not contain a $\neg$-sign. If inserting $X$ in the graph corresponds to replacing some of the $F_i$'s by $F_i \land x$, we may suppose w.l.o.g. $i = 1$. Then, as $\vdash (F_1 \land x) \rightarrow F_1$, we find $\vdash \Box((F_1 \land x) \rightarrow F_1$) by ($RN$). Hence, by ($K$), we derive $\vdash \Box (F_1 \land x) \rightarrow \Box F_1$. Repeating this argument as often as necessary yields $\vdash \Delta_1 (F_1 \land x) \rightarrow \Delta_1 F_1$ and hence $\neg (\Delta_1 F_1' \land \Delta_2 F_2' \land \cdots \land \Delta_k F_k') \land F' \rightarrow (\Delta_1 (F_1' \land x)) \land \Delta_2 F_2' \land \cdots \land \Delta_k F_k') \land F'$.

Therefore, in the case $n = 1$, we found that $F \vdash F^x$, if $x$ is in the scope of an odd number of $\neg$-signs (i.e. in the scope of one $\neg$-sign). Analogously, we get $F^x \vdash F$, if $x$ is inserted within $F'$.

Suppose now the claim holding for all numbers $\leq n$, and let $G$ be a graph within which each proposition is enclosed by at most $n + 1$ cuts.

The corresponding formula $F$ can be represented as $\Delta_1 F_1 \land \cdots \land \Delta_k F_k \land F_{k+1}$, where $\Delta_1, \ldots, \Delta_k$ are of the form $\Box \Box \cdots \Box \neg$ ($j \geq 0$) and $F_{k+1}$ does not contain any $\neg$-sign. Let $G_i$ be the sub-graph in which a proposition $X$ is to be inserted, $F_i$ the corresponding sub-formula of $F$, $G^X_i$ the resulting graph after inserting $X$, $F^X_i$ the resulting formula and $p$ the proposition in $F_i$ which is thereby replaced by $p \land x$.

To be precise: to get an expression of this form, you have to transform the graph by ($\mathbf{K}_p$) before the translation so that any two elementary propositions are separated, which means that they lie on different pieces of red sheet. Another proof by induction is required, but since there is neither a problem in principle nor any doubt in the correctness of the result of this sub-proof, we omitted it here. It would use the same sort of arguments as that follow in the next lines.
If the number of $\neg$-signs in whose scope $p$ lies within $F$ is odd, $\vdash F_i^x \rightarrow F_i$ holds either by the induction hypothesis or, in case $i = k + 1$ (in which case the number of $\neg$-signs is 0), by arguments such as those used above. It follows that $\vdash \neg F_i \rightarrow \neg F_i^x$ and hence by (RN) $\vdash \square(\neg F_i \rightarrow \neg F_i^x)$. So by (KP), we gain $\vdash \square \neg F_i \rightarrow \square \neg F_i^x$, and so on, until we get $\vdash \triangle_i F_i \rightarrow \triangle_i F_i^x$.

If the number of $\neg$-signs in whose scope $p$ lies within $F$ is even, the induction hypothesis says that $\vdash F_i \rightarrow F_i^x$. Analogously to the case of $n$ being even, we achieve $\vdash \triangle_i F_i^x \rightarrow \triangle_i F_i$.

Hence, the claim is true for $n + 1$.

From this, we infer as a special case: inserting in an oddly enclosed region transforms a graph into another in such a way that for the corresponding formulae, the desired implication holds.

The proof for the Rule of Erasure (R1) does not afford any new arguments; one only needs to reverse the directions of some implications. Just note that there is a special case that one has to pay attention to: when removal of a proposition leaves an empty cut in the corresponding formula, the ‘empty brackets’ have to be substituted by the false or by some contradiction $p \land \neg p$ (to get a well-formed formula).

Next, we give the proof for the Rule of Iteration (R3).

Let $X/x$ be the graph/proposition that we want to iterate ($x$ possibly contains one or more instances of the necessity operator), $G$ the graph on the area of $X$, except $X$ itself, and $F$ the corresponding formula. Then, the formula corresponding to the whole graph on the area of $X$ is of the form $f(x \land F)$, where $f$ is some well-formed composition of $\neg$, $\land$ and $\square$-signs. Let now $X$ be iterated on an admissible part of $G$, which means not on a different piece of red sheet. We denote the resulting formula by $F^x$. Next, we substitute in $F$ and in $F^x$ every proposition of the form $\square(\ldots)$ by a simple letter and obtain $F'$ and $F'^x$, respectively. Because of the assumption, $x$ is preserved in $F'^x$. If we now ignore the inner structure of $x$ (if there is any), we gain formulae $x \land F'$ and $x \land F'^x$ of PL, for which, the Rules of Iteration and Deiteration ‘hold’, as Zeman (1964, p. 53 sqq.) has shown, which means that $x \land F'$ and $x \land F'^x$ are equivalent. Now, as equivalence (not just implication in one direction) holds between these formulae and as (K) holds, $f$ does not change the situation: $f$ may be built step by step, using at each step, either $\neg$ or $\square$ or $\ldots \land p$ (for some proposition $p$), and the equivalence is preserved at each of these steps.

The Rule of Necessitation (RN$_{F}$) says that the white Phemic sheet, beyond any graph deducible from the empty Phemic sheet, may be replaced by a red sheet. This means for PL that $\vdash \square A$ may be inferred from $\vdash A$ for every theorem $A$ – which is guaranteed in K by (RN).

The Rules of Double Cut (R5) do not cause any problems, so we omit the argumentation.

Now, it remains to be shown that the axioms and rules of K may be derived from the rules (R1)–(R5), (RN$_{F}$) and (KP). But there is almost nothing to do. As Zeman (1964) has already proven that the corresponding graph to any formula of PL can be derived by the $\alpha$ rules, and as the modified $\alpha$ rules of KP allow us to use every well-formed graph in applying the rules (there is no restriction on what graph to insert, iterate, etc., just on where to), the rules provide all we want. The fulfilment of (K) and (RN) is guaranteed straightforwardly by (KP) and (RN$_{F}$).

5.4. The systems $T_{P}$, $D_{P}$, $S4_{P}$ and $S5_{P}$

In this section, we just study some representations of modal axioms in Peirce’s notation, and we formulate them, in accordance with Peirce, as rules. The proofs that these ‘Peircean’ versions are equivalent to the axioms are easy and we omit them, but we will prove in our notation a connection between two of the axioms. The notation that we suggest behaves exactly as the usual MPL.
Without further assumptions, $\vdash \Box A$ does not entail $\vdash A$, so one cannot simply remove a red sheet. A system not including $(T)$ is a very poor system anyway, so it is near at hand to seek an analogue for

$$(T) \vdash \Box A \rightarrow A$$

This is easy to find:

$(T_P)$ Any red sheet within an evenly enclosed area may be removed.

$K_P$ together with $(T_P)$ gives a system equivalent to $T$.

The system $D$, using the axiom $(D) \vdash \Box A \rightarrow \Diamond A$, may be translated by either using two different tinctures or otherwise expressing $\Diamond A$ by $\neg \Box \neg A$, in which case, it allows a double cut to be inserted in such a way that one cut is on a red sheet and the other one on a white sheet (or the two being on different red sheets). We choose the second:

$(D_P)$ Within an evenly enclosed area, a double cut may be placed in such a way that the inner cut lies within a sheet and the outer on a different sheet.

We show that $(D_P)$ follows if we assume $(T_P)$ by showing the stronger claim that $\neg \Box \neg A$ can be deduced from $A$ if $(T_P)$ is assumed.

![Diagram of steps (1) to (6)]

Justifications of the steps by the rules:

1. $(R5)$
2. $(R2)$
3. $(R3)$
4. $(T_P)$
5. $(R5)$
6. $(R4), (R5), (R1)$ (i.e. $A, A \rightarrow B$ then $B$)

All steps can be done not only on the Phemic sheet, but in every evenly enclosed area also.

From $(T_P)$, it follows that ‘double sheets’ may be removed in evenly enclosed areas, but in oddly enclosed areas, one is not allowed to do so. Therefore, piling of sheets cannot, in general, be avoided in $T_P$; to get rid of heaps of sheets, one needs the additional axiom (which is corresponding to $\vdash \neg \Box \neg A \rightarrow \neg \Box A$, equivalent to $(4) \vdash \Box A \rightarrow \Box \Box A$):

$(4_P)$ Within an oddly enclosed area, any red sheet lying on a red sheet and being enclosed by the same number of cuts may be removed.

Characteristic for the Peircean version of S4 is, therefore, that red sheets on red sheets may within their area be piled up and removed at will, as long as they are not separated by a cut.
If one prefers a rule in a formulation closer to the usual (4), one can put it like this:

\( (4'_p) \) Within an evenly enclosed area, a red sheet may be laid on another red sheet, provided that the added sheet lies completely on the other one and contains the whole graph that was contained by the first sheet.

That the rules \((4_p)\) and \((4'_p)\) are really equivalent can be seen by looking at the corresponding conditionals:

\[
\vdash \neg \Box \Box A \rightarrow \neg \Box A
\]

is the translation of \( \vdash \neg \Box \Box A \rightarrow \neg \Box A \) and means much the same as (just eliminate the double cuts)

\[
\vdash \Box A \rightarrow \Box \Box A
\]

which is the translation of \( \vdash \Box A \rightarrow \Box \Box A \).

\( (5_p) \) allows the stacking of sheets without minding the cuts. There are several ways of giving a rule appropriate for taking the part of \((5)\) in the tinctured system: like \((4_p)\), just omitting the restriction concerning the enclosure by the same number of cuts, or as a permission to put a red sheet under any graph which is an (evenly or oddly) enclosed red sheet. \((4_p)\) can be read as allowing to put a red sheet under another red sheet, and now the rule is extended to enclosed sheets.) \((5_p)\) actually means that layers of red sheets are superfluous.

We do not give any diagrams in connection with \(S5P\), because complex combinations of piled sheets are difficult to represent. We just want to remark that Peirce also seems to pay tribute to this fact as he mentions the convention that sheets further away from the March are supposed to overlie those nearer to the March (Peirce 1906, p. 527).

To sum up: \(K_P\) together with \((T_P), (D_P), (T_P) + (4_p), \) respectively, \((T_P) + (4_p) + (5_p)\) give the systems \(T_P, D_P, S4_p, \) respectively, \(S5_P\), which are equivalent to the usual systems. Hence, Peirce’s diagrams qualify very well for representing propositional modal logic – provided that colour is understood as expressing necessity.

6. Which of the systems suits best Peirce’s own ideas?

6.1. Syntax and iconicity

For several reasons, none of the systems presented here, \(K_P, D_P, S4_P\) and \(S5_P\), exactly fits with Peirce’s ideas: Peirce clearly had a variety of modalities in mind when developing the tinctured graphs; even where Peirce’s considerations are restricted to one sort of modality only, this modality is possibility not like that in our proposal necessity; Peirce does not give any derivation rules for the tinctured graphs, so we lack a criterion for saying whether our system corresponds to what Peirce had in mind, and his systems are a mixture of syntax and semantics, whereas we concentrated our presentation on syntactical systems only.
Nevertheless, we think that our system can be seen as an emendation of a part of Peirce’s tinctured system. It has the great advantage of achieving what one expects it to. Peirce definitely wanted derivation rules for all his logical systems; there are some hints (in his \( \gamma \) system) of what they should look like, and as he had already given up a system (entitative graphs) in favour of one that ‘worked better’ (EGs) once, he could probably have been convinced of starting his system of modal logic from a sheet representing necessity instead of possibility.

But, one could argue, it was not ‘pragmatic’ reasons alone that made him give up entitative graphs, but mainly this was due to the EGs being more iconic in character. To examine the situation of the tinctures, a few words about iconicity have to be said.

While there is, of course, a widespread literature on icons and iconicity, we shall restrict ourselves to some of Peirce’s remarks. First one has to confirm that Peirce stays uneasy with his tinctured graphs and that the representation he finds best is not optimal – it is not iconic:

I may as well, at once, acknowledge that, in Existential Graphs, the representation of Modality (possibility, necessity, etc.) lacks almost entirely that pictorial, or Iconic, character which is so striking in the representation in the same system of every feature of propositions \( \text{de inesse} \). Perhaps it is in the nature of things that it should be so in such wise that for Modality to be iconically represented in the same ‘pictorial’ way in which the other features are represented would constitute a falsity in the representation (CP 4.553, note 1).

In this remark in ‘Prolegomena’, he hesitates over whether or not the representation of possibility should or could be iconic.27

We have to make our ideas on what he means by ‘iconic’ in this context clearer. As an example for what Peirce calls ‘falsity in the representation’, one could think of the following: if you write down in the notation of the EGs \( \neg (A \land \neg B) \), then you cannot claim \( \neg (A \rightarrow B) \), because you see that \( A \rightarrow B \) holds. But what is it that should be seen in or mapped by a representation of (a mode of) modality?

Still, I confess I suspect there is in the heraldic representation of modality as set forth a defect capable of being remedied. If it be not so, if the lack of ‘pictorialness’ in the representation of modality cannot be remedied, it is, because modality has, in truth, the nature which I opined it has […] Modality is not, properly speaking, conceivable at all, but the difference, for example, between possibility and actuality is only recognizable much in the same way as we recognize the difference between a dream and a waking experience […] (CP 4.553, note 1).

The representation of modality has a differentiating function, and it must distinguish between different modes of modalities by separating areas on the Phemic sheet. But as the difference between different modes of modalities is not conceivable according to Peirce, there is nothing that can be mapped (represented) iconically.

Though Peirce never arrives at a definite position on this question, there is an aspect of the problem which can be settled satisfactorily. Suppose tinctured sheets represent modes of modalities (iconically or not), then in some cases, ‘iconic’ can be given a rather articulate meaning by comparing the situation with writing down two propositions, which means asserting them both and, hence, their conjunction. In CP 4.433, this is considered ‘a highly

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27 It is not clear whether Peirce wants to say that representation by tinctures at all is not iconic or if he just sees no way for making the special choices of tinctures iconic. But most likely he means the former. If you consider the relation between ‘possibility’ and ‘icon’ the other way around, his position is beyond doubt: an icon is a possibility (see Peirce 1983, p. 73).
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iconic mode of representation to understand […], where [two propositions] are written on different parts of the sheet, as the assertion of both’. Now, this understanding of ‘iconicity’ seems very similar to what could be stated about ($K_P$): two necessary propositions can be said to be necessary together, and this can be seen when we move sheets together until they touch or overlap.

Hence, supposing the presuppositions that we made for the modal system $K_P$, we assume that Peirce would have accepted ($K_P$). What about other rules? From our considerations about the $\gamma$ graphs (Section 2.2), it seems clear anyway that Peirce thought of a T-type system. And there is at least one argument that speaks in favour of $S5_P$: there are no hints that Peirce might have considered it meaningful to stack Phemic sheets of the same sort.

Of course, further emendations would be desirable. Pietarinen noted that, ‘[w]ith tinctures, Peirce anticipated not only alethic modal logic, but also epistemic logic, erotetic logic, deontic logic, belief-desire-intention logic, and the logic of imperatives’ (Pietarinen 2006b, p. 127). Multimodal logic should be developed to do justice to Peirce’s aims, but we must leave this for future work on the subject.

6.2. Syntax versus semantics

There still remains the question if it is semantics or syntax in the modern sense that comes closer to Peirce’s attitudes towards modality and his tinctured graphs.

For Peirce, the Phemic sheet is semantical in nature. It ‘bears’ the (representations of) the objects that we are talking about; it is an icon of the totality of them. Entities like the cuts clearly belong to what we now call syntax. The tinctured sheets are situated in between. Peirce took them to be bearers of ‘possible individuals’ and, therefore, to be part of semantics like the Phemic sheet. In this paper, we used them as syntactical objects. We shifted them around, moved them and removed them, and so on – Peirce did so himself, but he leaves the impression of not being totally happy with it. He was torn between the two modes as the sheets themselves are torn between belonging to what they are put on and what is put on them.

According to the usual semantical definition, $\Diamond P$ is true in a world $\omega$ iff it is true in some world accessible from $\omega$, and $\Box P$ is true in a world $\omega$ iff it is true in every world accessible from $\omega$. This gives Peirce an additional reason for preferring possibility to necessity: a proposition about an alternative world is a conception easier to grasp than a proposition about all possible alternatives, especially it is easier to draw one world (in whatever way) than to draw ‘arbitrarily many’. Now, one could try to understand the tinctured sheets as possible worlds (as Peirce indicates in several places, cf. Pietarinen 2006a) for the technical purposes of formal semantics. Then, one ought to say what ‘accessible’ means. For example, one could define that a (tinctured) sheet is accessible from another if the first lies at least partially above the second. Transitivity of the accessibility relation would then mean that the sheets must not be stacked like this:

But though it seems possible to express some (or perhaps most) of the ideas of modern possible-world semantics by use of tinctured sheets, one faces an insurmountable obstacle: Peirce does not develop a semantics that is clearly separated from syntax, but we know because of Carnap, Gödel and Tarski that the semantics of first-order logic cannot be expressed by means of the language of first-order logic only. Accordingly, the semantics which Hammer (1998) provides for EGs uses an interpretation function creating a semantics in the modern sense, but this is not grounded in Peirce’s work. Hammer does not make (essential) use of the sheet of an assertion as containing (representations of) individuals,
so fading out an idea that was important for Peirce. We are confronted with the analogous situation concerning modal logic: we have to choose between the tinctured sheets being either syntactical or semantical objects. But this decision cannot be made in accordance with Peirce – so that we have to choose to follow one of the lines of his thoughts, neglecting the other.28 And our choice for this paper was the syntactical part.

References


Appendix 1: derivation rules for the α graphs

We give the transformation rules for the α graphs in the version of Peirce (1906).

First Permission – Deletion (CP 4.565): Any graph whose level is even may be erased.29

First Permission – Insertion: On an area of odd level, any graph whatsoever may be inserted.

Second Permission – Iteration (CP 4.566): Every graph ϕ occurring in an area may be iterated in (i.e. copied to) the same area or to any of its sub-areas, except in the area of ϕ itself.30

28 Neither did we take the game theoretic aspects of Peirce’s logic into consideration, though we see that it is an important motivational factor for Peirce’s logic.

29 At an earlier stage, in his 1903 A Syllabus of Certain Topics of Logic, CP 4.415, Peirce makes an exception to this rule by adding that an empty cut must not be deleted from the sheet of assertion. We do not know why. He does not always make this exception, though – cf., for example, CP 4.377 (‘anything written down may be erased’) or, in a way, CP 4.489 (‘[a]ny partial graph may be erased’).

30 The exception to the rule is explicitly stated only by Roberts (1973, p. 42 sq.), not by Peirce himself. Omitting it would falsely license, for example, the derivation of P(Q(Q)) from (P(Q)), that is, the derivation of (P → (Q ∧ ¬Q)) from P → Q.
Second Permission — Deiteration: Provided that the first part of the Second Permission, Iteration, licenses deriving a graph \( \varphi \) from a graph \( \psi \), it is valid to derive \( \psi \) from \( \varphi \). This does not require \( \varphi \) being the result of actually applying Iteration.31

Third Permission — Double Cut (CP 4.567):32 If there are two cuts such that (a) one of them is completely contained within the area of the other and (b) this area does not contain anything but the former, they may be called ‘double cut’ and may be removed at will. Furthermore, such double cuts may be inserted at any place in or around a graph or a sub-graph, provided that this is done such that nothing gets in between them.

Appendix 2

Note that the example that we will discuss in the following is beyond the scope of our paper, as it needs the \( \beta \) part of Peirce’s graphs; we only mention it, because it could be seen as an objection against our conclusions.

To make it clear that the broken lines in ‘Prolegomena’ (Peirce 1906) are not broken cuts, we need to say a few more words about tinctures. Tinctures express a mode of modality, which means — in ‘Prolegomena’ — metals stand for actualities, colours for possibilities and furs for intentions. They are four each (colours, furs and metals), and they can be used ‘locally’ as well as ‘globally’ (our terminology, not Peirce’s), which means that the tinctured area can cover the whole sheet or a big part of it (which is suggested by the word ‘province’, Peirce uses), but also only a part of a graph may lie on it, namely only a part of a line of identity, as can be seen in an example on page 540. Just below this example, one finds the figure that we want to discuss now:

Peirce explains:

[...] I shall now scribe Fig. 14 [the one above] all in one province. This may be read ‘There is some married woman who will commit suicide in case her husband fails in business.’ This evidently goes far beyond saying that if every married man fails in business some married woman will commit suicide. Yet note that since the Graph is on Metal it asserts a conditional proposition de inesse and only means that there is a married woman whose husband does not fail or else commits suicide. That, at least, is all it will seem to mean if we fail to take account of the fact, that being all in one Province, it is said that her suicide is connected with his failure. (Peirce 1906, p. 540)

It would be a failure, according to Peirce, to understand the proposition as

\[
\exists x \exists y (W x y \land \neg (F y \land \neg S x))
\]

(with the obvious correlations between the words and the predicate letters).33 Neglect the contribution of the tincture for the moment and suppose the broken line were a broken cut, then the corresponding formula would be

\[
\exists x \exists y (W x y \land \neg (F y \land \neg S x)) \lor \exists x \exists y (W x y \land (F y \to \Box S x))
\]

— whatever kind of ‘possibility’ might be meant. But this formula would still already be true if there were any man who does not fail in business, and therefore, the broken cut (as it would be used here, if it were a broken cut) would not solve the problem at all; moreover, it would not change anything relevant to the problem at all. Now let us include the tincture again: this does not change matters, as it has become clear that if the tincture manages to change the situation, then it manages it independently of the broken cut, and therefore, the broken cut is superfluous.

If we try to interpret the broken line in the above example as a broken cut, we face the following additional problem: what sort of possibility is it, of which the broken cut says that it is not the case? There are three options:

1. If the tincture below were a colour, expressing itself a sort of possibility, the standard (not the broken) cut

31 Peirce’s wording of this rule in CP 4.566, ‘[a]ny Graph [...] (if already Iterated) may be Deiterated’, might be understood this way. That this was not his intention shows his comment that ‘[t]o deiterate a Graph is to erase a second Instance of it’ (loc. cit.) and, even more clearly, the remark that ‘[t]he operation of deiteration consists in erasing a replica which might have illatively resulted from an operation of iteration’ (CP 4.506, italics by us).

32 In the much simplified spirit of Roberts’ R5, p. 44.

33 Edgington (2006) gives an overview of the discussions on material implication.
would be the negation of it, making the broken cut superfluous. (2) If the tincture below were expressing a different mode of modality (as it is, in fact, in Peirce’s description: he says it is Argent, that means actuality), why should Peirce not have used the ‘right’ tincture, the one that expresses the desired possibility? (3) What if the tincture did not express modality itself at all, but just specified the mode of modality expressed by the broken cut? This would presuppose that there is some modally, ontologically and epistemologically neutral concept of ‘possibility’ (expressed by the broken cut) Peirce is committed to – but this is not the case (see Section 3.1, especially 3.1.2). Hence, all options for understanding the broken lines as broken cuts are ruled out.