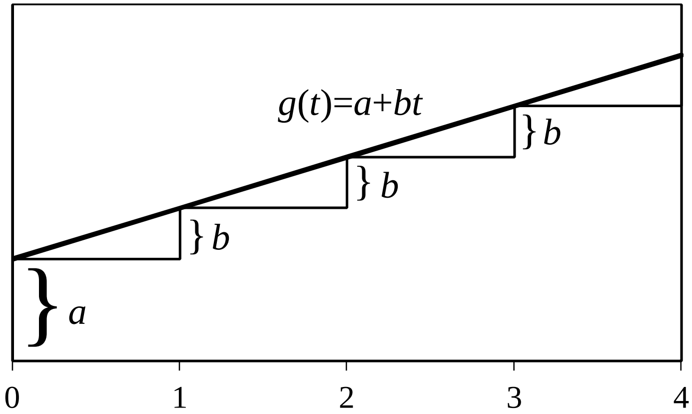
A **linear trend** g is an (inhomogeneous) linear function of time, i.e.,

$$g(t)=a+bt.$$

The parameters a and b are called intercept and slope, respectively.

The linear trend changes by b units every unit of time.

Its graph is a straight line.

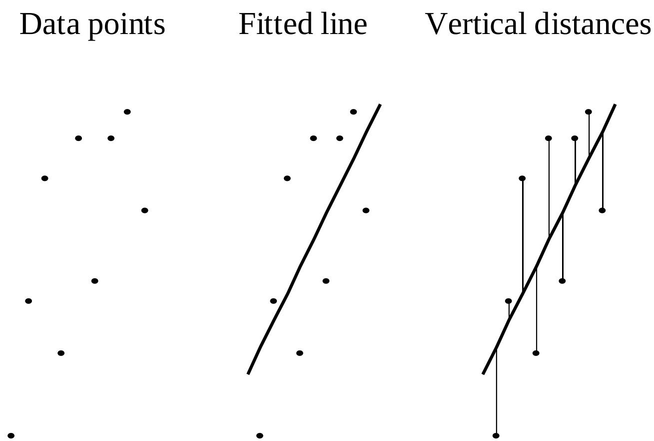


Our goal is to find the linear trend-line that best fits n given data points:

$$P_1=(1,y_1),\ P_2=(2,y_2),\ \ldots,\ P_n=(n,y_n)$$

For the evaluation of the goodness of fit we use the vertical distances of the data points from the line.

Example (n=9):



Minimizing the sum of squared errors

TL

When a time series $y_1,\ldots,y_n$ appears to grow at a roughly linear rate, we may describe the series as the sum of a linear trend and a series of deviations from the trend, i.e,

$$y_t=(a+bt)+u_t.$$

To find the parameters *a* and *b* that minimize the sum of squared deviations (errors)

$$SSE(a,b)=\sum_{t=1}^{n} u_t^2 = \sum_{t=1}^{n}(y_t-(a+bt))^2=\sum_{t=1}^{n}(y_t-a-bt)^2$$

we compute the partial derivatives of *SSE* with respect to *a* and *b* and set them equal to zero.

$$\frac{\partial}{\partial a} SSE(a,b) = \sum_{t=1}^{n} 2\,(y_t-a-bt)(-1) = 0$$

$$\Rightarrow a=\tfrac{1}{n}\sum_{t=1}^{n} y_t - b\,\tfrac{1}{n}\sum_{t=1}^{n} t = \bar{y} - b\,\bar{t}$$

$$\frac{\partial}{\partial b} SSE(a,b) = \sum_{t=1}^{n} 2\,(y_t-a-bt)(-t) = 0$$

$$\Rightarrow \sum_{t=1}^{n} y_t t - (\bar{y} - b\,\bar{t}) \sum_{t=1}^{n} t - b \sum_{t=1}^{n} t^2 = 0$$

$$\Rightarrow \mathrm{b}=\frac{\sum_{t=1}^{n} y_t t - \bar{y}\sum_{t=1}^{n} t}{\sum_{t=1}^{n} t^2 - \bar{t}\sum_{t=1}^{n} t} = \frac{\frac{1}{n}\sum_{t=1}^{n} y_t t - \overline{y}\overline{t}}{\frac{1}{n}\sum_{t=1}^{n} t^2 - \bar{t}^2} = \frac{s_{yt}}{s_t^2}$$

In **matrix notation** the **linear trend model**

$$y_t=a+bt+u_t,\ t=1,2,\ldots,n$$

can be written as

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a+b\cdot 1+u_1 \\ a+b\cdot 2+u_2 \\ \vdots \\ a+b\cdot n+u_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

or in short form as

$$y=X\beta+u,$$

where

$$y=\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},\ X=\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & n \end{pmatrix},\ \beta=\begin{pmatrix} a \\ b \end{pmatrix},\ u=\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

Those values

$$\hat{a}=\bar{y}-\hat{b}\,\bar{t}\,,\ \hat{b}=\frac{\frac{1}{n}\sum_{t=1}^{n} y_t t-\bar{y}\bar{t}}{\frac{1}{n}\sum_{t=1}^{n} t^2-\bar{t}^2},$$

that minimize the sum of squared errors

$$\sum_{t=1}^{n}(y_t-a-bt)^2=\sum_{t=1}^{n}u_t^2=u^Tu=(y-X\beta)^T(y-X\beta)$$

are called the **least squares (LS) estimates** of the parameters a and b.

<u>Exercise:</u> Show that $\begin{pmatrix} p & r \\ s & q \end{pmatrix}^{-1}=\frac{1}{pq-rs}\begin{pmatrix} q & -r \\ -s & p \end{pmatrix}$ TI

<u>Exercise:</u> Use the fact that X^TX is a 2×2 matrix to show that

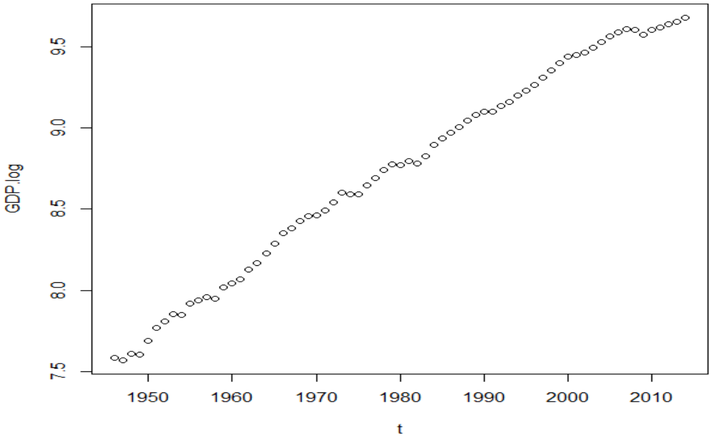
$$\hat{\beta}=\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}=(X^TX)^{-1}X^Ty.$$

TM

The log transformation

Exercise: Try to convert the roughly exponential trend of the annual US GDP (stored in **Y**; see Appendix C) into a roughly linear trend by taking logarithms.

- Enter **y <- log(Y)** to store the logarithms of **Y** into **y**.
- Enter **plot(D,y)** to plot the log GDP against time.



Exercise: Fit a linear trend to the log GDP.

First we approximate **y** by a linear function of time.

```
> LM <- lm(y~D)   # linear model
> LM   # print a brief report of the results
Call:
lm(formula = y ~ D)
Coefficients:
(Intercept)          t
-55.31934     0.03235
```

Next we get the names of the elements of the list returned by the function **lm**.

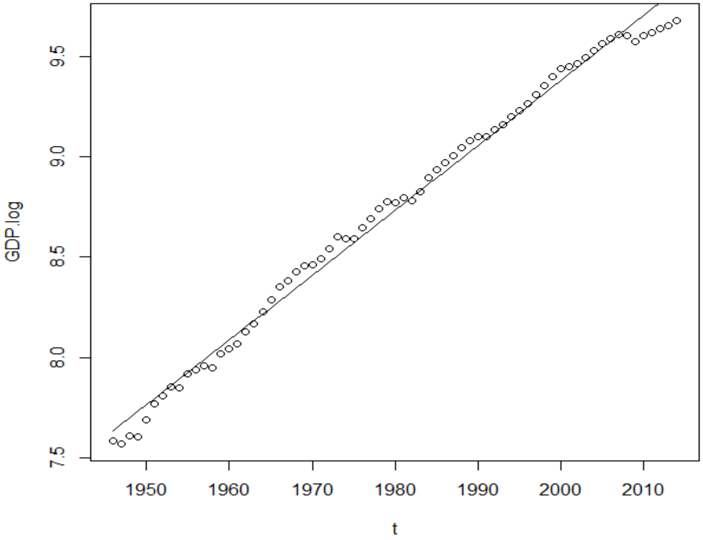
```
> names(LM)
[1] "coefficients"  "residuals" "effects" "rank"
[5] "fitted.values" "assign" "qr"  "df.residual"  …
```

The elements of the list **LM** can be referenced using double square brackets or via their names.

```
> a <- LM[[1]][1]; b <- LM$coefficients[2]
```

Exercise: Plot the linear trend fitted to the log GDP.

```
plot(D,y)   # plot y against D
lines(D,LM$fitted.values)   # add trend line
```

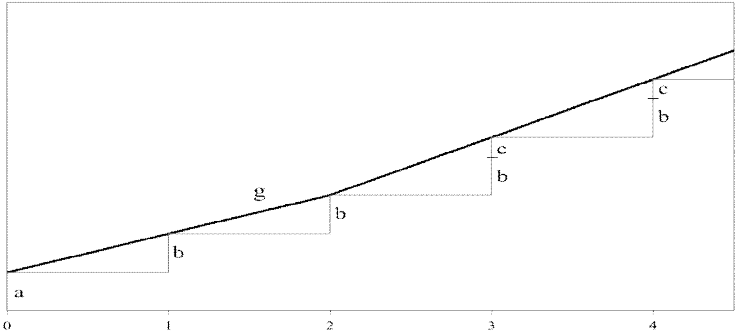


Exercise: Plot the residuals of the linear trend model with no axis labels.

```
plot(D,LM$residuals,xlab=" ",ylab=" ")
```

The graph of the trend residuals (deviations from the fitted trend) suggests that there could still be a trend left, possibly an upward trend until into the early seventies and a downward trend thereafter. Such a broken trend could be due to a slowdown in growth following the oil price shock in early 1973.

A broken linear trend g allows a sudden change (**structural break**) of the slope from b to $b+c$.



Introducing the variable $z_t=max(0,t-q)$, which takes a value of 0 up to period 2 and grows linearly afterwards, we have

$g(1)=a+b \quad = a+1b+0c = a+b\ 1+c\ z_1$

$g(2)=a+b+b \quad = a+2b+0c = a+b\ 2+c\ z_2$

$g(3)=a+b+b+(b+c) \quad = a+3b+1c = a+b\ 3+c\ z_3$

$g(4)=a+b+b+(b+c)+(b+c) \quad = a+4b+2c = a+b\ 4+c\ z_4$

$g(5)=a+b+b+(b+c)+(b+c)+(b+c) = a+5b+3c = a+b\ 5+c\ z_5$

$\vdots$

Perron's "changing growth" model

$$y_t=a+bt+cz_t+u_t$$

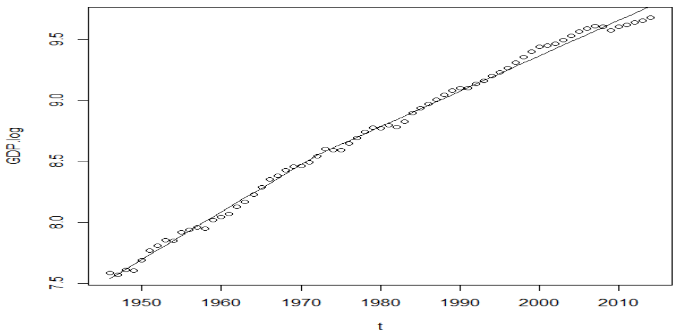
where $z_t=max(0,t-q)$ and q is the last time period of the first regime, can be written in matrix notation as

$$y=\begin{pmatrix} y_1 \\ \vdots \\ y_q \\ y_{q+1} \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & q & 0 \\ 1 & q+1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & n & n-q \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_q \\ u_{q+1} \\ \vdots \\ u_n \end{pmatrix} = X\beta + u.$$

It is a special case of the **multiple regression model**. In such a model, the matrix X is called **design matrix**. The columns of the design matrix are called **regressors** or **explanatory variables**. The elements of the vector β are called **regression parameters**.

Exercise: Fit a broken linear trend $g(t)=a+bt+cz_t$ to the log GDP.

```
> q<-27  # The 1st  regime is from 1 to q=27 (1946-72).
> z <- rep(0,q) # rep(0,q) replicates 0 q times.
> z <- c(z,1:( N-q)) # z: 0 … 0 1 … N-q
> LM.btrend <- lm(y~D+z) # a incl. by default
> LM.btrend$coefficients # print a, b, and c
  (Intercept)             t            z
-68.50502127   0.03907569  -0.01002573
> plot(D,y); lines(D,LM.btrend$fitted.values)
```



Exercise: Plot the residuals of the broken linear trend model with no axis labels.

```
> plot(D,LM.btrend$residuals,xlab=" ",ylab=" ")
```

There might be another slowdown in growth after the bursting of the housing bubble in 2007. To model this second break we would need another dummy variable of the form 0 … 0 1 2 3 … For the modeling of breaks in the intercept we would need dummy variables of the form 0 … 0 1 1 1 …