

Let  $x$  be a general **linear process** represented by PL

$$x_t = \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} = \left( \sum_{j=-\infty}^{\infty} \psi_j L^j \right) u_t,$$

where  $u$  is white noise with mean 0, variance  $\sigma^2$  and spectral density

$$f_u(\omega) = \frac{\sigma^2}{2\pi}.$$

The stationarity of  $x$  follows from

$$Ex_t = E \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} = \sum_{j=-\infty}^{\infty} \psi_j E u_{t-j} = 0$$

and  $\gamma(k) = \text{Cov}(x_t, x_{t-k}) = Ex_t x_{t-k} = E \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \sum_{j=-\infty}^{\infty} \psi_j u_{(t-k)-j}$

$$= \sum_{r=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \psi_r \psi_j E u_{t-r} u_{t-(j+k)}$$

$$= \sum_{j=-\infty}^{\infty} \psi_{j+k} \psi_j \underbrace{E u_{t-(j+k)} u_{t-(j+k)}}_{\sigma^2}$$

The spectral density of  $x$  is given by

$$\begin{aligned} f_x(\omega) &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) e^{-i\omega k} \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left( \sigma^2 \sum_{j=-\infty}^{\infty} \psi_{j+k} \psi_j \right) e^{-i\omega k} \\ &= \frac{\sigma^2}{2\pi} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_{j+k} e^{-i\omega(j+k)} e^{i\omega j} \\ &= \frac{\sigma^2}{2\pi} \sum_{j=-\infty}^{\infty} \psi_j e^{i\omega j} \sum_{k=-\infty}^{\infty} \psi_{j+k} e^{-i\omega(j+k)} \\ &= \frac{\sigma^2}{2\pi} \sum_{j=-\infty}^{\infty} \psi_j e^{i\omega j} \sum_{k=-\infty}^{\infty} \psi_k e^{-i\omega k} \\ &= \frac{\sigma^2}{2\pi} \sum_{j=-\infty}^{\infty} \psi_j e^{-i\omega j} \sum_{k=-\infty}^{\infty} \psi_k e^{-i\omega k} \\ &= f_u(\omega) \left| \sum_{j=-\infty}^{\infty} \psi_j e^{-i\omega j} \right|^2. \end{aligned}$$

PD

## Spectral densities of ARMA processes:

- MA( $q$ ):  $x_t = u_t + \theta_1 u_{t-1} + \dots + \theta_q u_{t-q} = (1 + \sum_{j=1}^q \theta_j L^j) u_t$

$$f(\omega) = \frac{\sigma^2}{2\pi} \left| 1 + \sum_{j=1}^q \theta_j e^{-i\omega j} \right|^2$$

- AR(1):  $x_t = \sum_{j=-\infty}^{\infty} \phi_1^j u_{t-j} = (\sum_{j=1}^{\infty} \phi_1^j L^j) u_t = \frac{1}{1 - \phi_1 L} u_t$

$$f(\omega) = \frac{\sigma^2}{2\pi} \left| \sum_{j=0}^{\infty} \phi_1^j e^{-i\omega j} \right|^2 = \frac{\sigma^2}{2\pi} \left| \sum_{j=0}^{\infty} (\phi_1 e^{-i\omega})^j \right|^2 = \frac{\sigma^2}{2\pi} \left| \frac{1}{1 - \phi_1 e^{-i\omega}} \right|^2$$

- AR( $p$ ):  $x_t = \frac{1}{1 - \phi_1 L - \dots - \phi_p L^p} u_t$

$$f(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{1}{1 - \phi_1 e^{-i\omega} - \dots - \phi_p e^{-i\omega p}} \right|^2$$

- ARMA( $p, q$ ):  $x_t = (1 - \sum_{j=1}^p \phi_j L^j)^{-1} (1 + \sum_{j=1}^q \theta_j L^j) u_t$

$$f(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{1 + \sum_{j=1}^q \theta_j e^{-i\omega j}}{1 - \sum_{j=1}^p \phi_j e^{-i\omega j}} \right|^2$$

PA

If the denominator polynomial

$$\Phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$$

has a root  $z_0$  near the unit circle, i.e.,

$$\Phi(z_0) = 0 \text{ and } z_0 \approx 1 \cdot e^{-i\omega_0 j},$$

then the spectral density  $f$  will have a peak near frequency  $\omega_0$ . Similarly, if the numerator polynomial

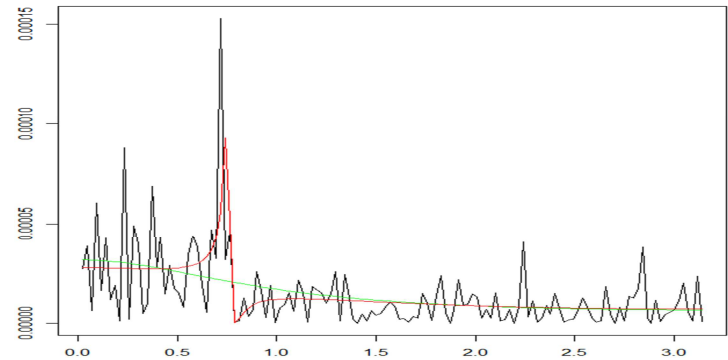
$$\Theta(z) = 1 + \theta_1 z + \theta_2 z^2$$

has a root near the unit circle, then  $f$  will have a trough.

**Exercise:** Plot the periodogram of the (demeaned) growth rates of the quarterly GDP together with the spectral densities implied by fitted ARMA models of order (1,0) and (3,2), which are the best ARMA models according to BIC and AIC, respectively.

```
arma.sp <- function(f,phi,p,theta,q,s2) { # ARMA spectr.
nf <- length(f); sp<-rep(s2/(2.*pi),nf)
if (q>0) for (i in 1:nf)
  sp[i]<-sp[i]*(Mod(1.+sum(theta*exp(-1i*f[i]*(1:q))))))^2
if (p>0) for (i in 1:nf)
  sp[i] <- sp[i]/(Mod(1.-sum(phi*exp(-1i*f[i]*(1:p))))))^2
return(sp) }
```

```
plot(f,pg,type="l"); COL <- c("green","red")
P <- c(1,3); Q <- c(0,2) # 1st model: p=1, q=0, 2nd: 3, 2
for (i in 1:2) { p <- P[i]; q <- Q[i]
h <- arima(r.dm,order=c(p,0,q),include.mean=F)
sp <- arma.sp(f,h$model$phi,p,h$model$theta,q,
              h$sigma)
lines(f,sp,col=COL[i]) }
```



The ARMA(1,0) model cannot reproduce the sudden decrease in the periodogram at about frequency 0.8. In contrast, the ARMA(3,2) model accomplishes this with a conjugate pair of roots of the denominator polynomial

$$\Phi(z)=1-\phi_1z-\dots-\phi_pz^p$$

at frequency 0.756 and another conjugate pair of roots of the numerator polynomial

$$\Theta(z)=1+\theta_1z+\dots+\theta_qz^q$$

at frequency 0.789.

**Exercise:** Find the roots of the polynomials

$$\Phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \phi_3 z^3$$

and

$$\Theta(z) = 1 + \theta_1 z + \theta_2 z^2.$$

```
p <- 3; q <- 2
```

```
h <- arima(r.dm, order=c(p,0,q), include.mean=F,  
          transform.pars=TRUE)
```

```
phi <- h$model$phi; theta <- h$model$theta
```

```
phi
```

```
1.7209425 -1.3825710 0.2790249
```

```
theta
```

```
-1.4086876 0.9999598
```

```
# roots of 1-phi[1]*z-phi[2]*z^2-phi[3]*z^3
```

```
z <- polyroot(c(1,-phi)); z
```

```
0.7385509+0.6964346i 0.7385509-0.6964346i
```

```
3.4779078+0.0000000i
```

```
# polar coordinates of first root:
```

```
c(Mod(z[1]),Arg(z[1])) # modulus & argument
```

```
1.015125 0.756057
```

```
# roots of 1+theta[1]*z+theta[2]*z^2
```

```
z <- polyroot(c(1,theta)); z
```

```
0.7043721+0.7098592i 0.7043721-0.7098592i
```

```
# polar coordinates of first root:
```

```
c(Mod(z[1]),Arg(z[1])) # modulus & argument
```

```
1.0000201 0.7892781# polar coordinates:
```

Exercise: Show that the spectral density of an MA( $q$ ) process can be written as

$$f(\omega) = \frac{\sigma^2}{2\pi} \left( \left| 1 + \sum_{j=1}^q \theta_j \cos(\omega j) \right|^2 + \left| \sum_{j=1}^q \theta_j \sin(\omega j) \right|^2 \right). \quad \text{PR}$$

Exercise: Show that the spectral density of an AR(1) process can be written as

$$f(\omega) = \frac{\sigma^2}{2\pi} \frac{1}{1 - 2\phi_1 \cos(\omega) + \phi_1^2}. \quad \text{P1}$$

Spectral densities of ARFIMA processes:

- ARFIMA( $p,d,q$ ):  $x_t = (1 - \sum_{j=1}^p \varphi_j L^j)^{-1} (1-L)^{-d} (1 + \sum_{j=1}^q \theta_j L^j) u_t$

$$f(\omega) = \frac{\sigma^2}{2\pi} \left| 1 - e^{-i\omega} \right|^{-2d} \left| \frac{1 + \sum_{j=1}^q \theta_j e^{-i\omega j}}{1 - \sum_{j=1}^p \varphi_j e^{-i\omega j}} \right|^2$$

If  $d > 0$ , then  $\left| 1 - e^{-i\omega} \right|^{-2d} \rightarrow \infty$  as  $\omega \rightarrow 0$ .

If  $d < 0$ , then  $\left| 1 - e^{-i\omega} \right|^{-2d} \rightarrow 0$  as  $\omega \rightarrow 0$ .

Exercise: Show that  $\left| 1 - e^{-i\omega} \right|^2 = 2 - 2 \cos(\omega)$  **PF**  
 and  $2 - 2 \cos(\omega) = 4 \sin^2(\frac{\omega}{2})$ . **P2**

Exercise: Show that  $\sin(\omega) \sim \omega$  as  $\omega \rightarrow 0$ . **PH**

Hint: Use l'Hospital's rule to show that  $\frac{\sin(\omega)}{\omega} \rightarrow 1$  as  $\omega \rightarrow 0$ .

In the neighborhood of frequency 0, we have

$$f(\omega) \sim C \left| 1 - e^{-i\omega} \right|^{-2d} = 4^{-d} C \sin^{-2d}(\frac{\omega}{2})$$

and

$$\log f(\omega) \sim \log 4^{-d} C + d(-2 \log(\sin(\frac{\omega}{2}))),$$

hence an estimate of  $d$  can be obtained by regressing the first log periodogram ordinates on  $-2 \log(\sin(\frac{\omega}{2}))$  and a constant. Of course, this estimate depends on the number of log periodogram ordinates used in the regression.

Exercise: Show that  $\int_0^\varepsilon \omega^{-2d} d\omega$  exists only if  $d < \frac{1}{2}$ . **PI**

An estimate of  $d$  that is greater than or equal to  $\frac{1}{2}$  is therefore an indication that the integral

$$\int_0^\varepsilon f(\omega) d\omega \sim 4^{-d} C \int_0^\varepsilon \sin^{-2d}(\frac{\omega}{2}) d\omega \sim 4^{-d} C \int_0^\varepsilon (\frac{\omega}{2})^{-2d} d\omega$$

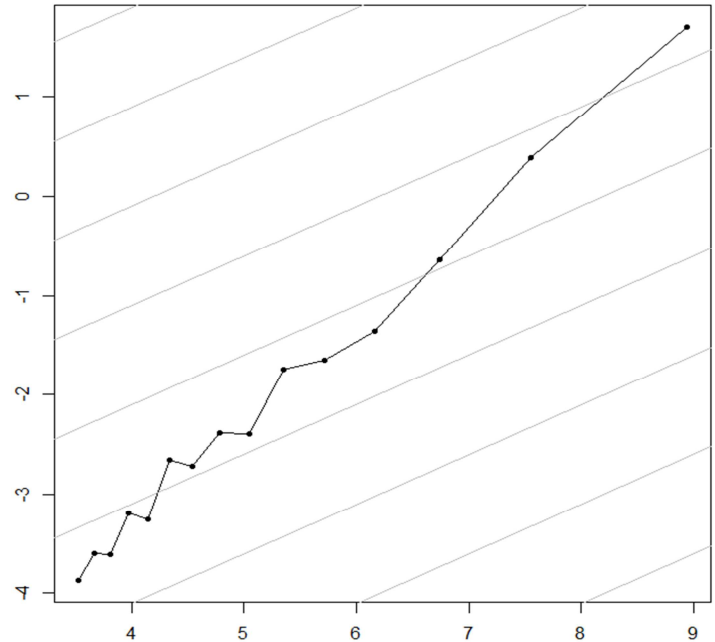
does not exist, which is inconsistent with stationarity.

**Exercise:** Check the stationarity of the log GDP by examining the slope  $d$  of the relationship between the log periodogram and  $-2 \log(\sin(\frac{\omega}{2}))$  in the neighborhood of frequency 0.

```
par(mfrow=c(1,1),mar=c(2,2,1,1))
M <- floor(N/2); F <- (2*pi/N)*(1:M)
FT <- fft(y)[2:(M+1)] # excl. k=0,M+1,M+2,...,N-1
PG <- (1/(2*pi*N))*(Mod(FT))^2

# only the 15 lowest frequencies:
plot(-2*log(sin(F[1:15]/2)),log(PG[1:15]),type="o",
      pch=20)

# several lines with slope 0.5:
for (a in seq(-10.10,0.5)) abline(a=a,b=0.5,col="gray")
```



The log GDP is nonstationary because the slope is much greater than 0.5. Note that the lower frequencies are to the right and the higher frequencies to the left.

It follows from

$$\begin{aligned} |e^{-i\omega} - b_j|^2 &= (e^{-i\omega} - b_j)(e^{i\omega} - \overline{b_j}) \\ &= e^{-i\omega} b_j \left( \frac{1}{b_j} - \frac{1}{e^{-i\omega}} \right) e^{i\omega} \overline{b_j} \left( \frac{1}{\overline{b_j}} - \frac{1}{e^{i\omega}} \right) = |b_j|^2 \left| e^{-i\omega} - \frac{1}{\overline{b_j}} \right|^2 \end{aligned}$$

that an ARMA spectral density

$$f(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{1 + \sum_{j=1}^q \theta_j e^{-i\omega j}}{1 - \sum_{j=1}^p \varphi_j e^{-i\omega j}} \right|^2 = \frac{\sigma^2}{2\pi} \left| \frac{\theta_q (e^{-i\omega} - b_1) \dots (e^{-i\omega} - b_q)}{\varphi_p (e^{-i\omega} - a_1) \dots (e^{-i\omega} - a_p)} \right|^2$$

remains the same when a root  $b_j$  of the polynomial

$$\Theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q$$

is replaced by  $1/\overline{b_j}$ , the factor  $\theta_q$  by  $\theta_q |b_j|^2$  (to make sure that the constant in the new MA polynomial is still 1), and  $\sigma^2$  by  $\sigma^2 / |b_j|^2$ .

Analogously, if a root  $a_j$  of the polynomial

$$\Phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$$

is replaced by  $1/\overline{a_j}$ , the factor  $\phi_p$  must be replaced by  $\phi_p |a_j|^2$  (to make sure that the constant in the new AR polynomial is still 1), and  $\sigma^2$  by  $\sigma^2 |a_j|^2$ .

**Exercise:** Find a causal and invertible representation of the ARMA(1,3) process given by

$$(1-1.6L)x_t=(1-2.9L+5.2L^2-2L^3)u_t, \sigma^2=1.$$

```
ARMA.CausInv <- function(phi,p,theta,q,v) {
  # calculates causal & invertible ARMA representation
  require(polynom) # the package "polynom" is loaded
  # poly.calc calculates polynomial with specified roots
  phi2 <- phi; theta2 <- theta; v2 <- v
  if (p>0) {
    a <- polyroot(c(1,-phi)) # roots of 1-phi[1]*z-...
    a2 <- a
    for (j in 1:p)
      if (Mod(a[j])<1) {
        a2[j] <- 1/Conj(a[j])
        v2 <- v2*Mod(a[j])^2 }
    phi2 <- poly.calc(a2)
    phi2 <- -phi2[2:(p+1)]/phi2[1] }
}
```

```
if (q>0) {
  b <- polyroot(c(1,theta)) # roots of 1+theta[1]*z+...
  b2 <- b
  for (j in 1:q)
    if (Mod(b[j])<1) {
      b2[j] <- 1/Conj(b[j])
      v2 <- v2/Mod(b[j])^2 }
  theta2 <- poly.calc(b2)
  theta2 <- theta2[2:(q+1)]/theta2[1] }
return(phi=phi2,theta=theta2,s2=v2) }
```

```
p <- 1; q <- 3; phi <- 1.6; theta <- c(-2.9,5.2,-2); s2 <- 1
polyroot(c(1,-phi)); polyroot(c(1,theta))
0.625+0i
0.3+0.4i 0.3-0.4i 2.0+0.0i
h <- ARMA.CausInv(phi,p,theta,q,s2)
c(h$phi,h$theta,h$s2)
0.625 -1.100 0.550 -0.125 6.25
polyroot(c(1,-h$phi)); polyroot(c(1,h$theta))
1.6+0i
1.2+1.6i 2.0+0.0i 1.2-1.6
```