Let $y_1,\ldots,y_n$ be a time series obtained by measuring a variable (e.g., the real US GDP) once in each of n sequential time periods $t_1,t_2,\ldots,t_n$.

Formally, we regard this time series as a function

$$y:\{t_1,\ldots,t_n\}\rightarrow\mathbf{R}$$

that assigns for each $i\in\{1,\ldots,n\}$ the real number y_i to the time period t_i. The set $T=\{t_1,\ldots,t_n\}$ is called the **domain** of the time series.

Time series operators transform one or more time series into a new time series. A simple example is the **multiplication operator**, which transforms a time series y with domain T into a new time series $\lambda\cdot y$ with the same domain by multiplying each value of y by a constant λ, i.e.,

$$(\lambda\cdot y)(t)=\lambda\cdot y(t)\quad\forall t\in T.$$

<u>Example:</u>

y	$10\cdot y$
2013 → 5.4	2013 → 54
2014 → 5.5	2014 → 55
2015 → 5.8	2015 → 58

The **addition operator** and the **subtraction operator** transform two series x and y that are defined over the same domain T into new time series $x+y$ and $x-y$, respectively:

$$(x+y)(t)=x(t)+y(t)\quad\forall t\in T,$$
$$(x-y)(t)=x(t)-y(t)\quad\forall t\in T.$$

<u>Example:</u>

x	y	$x+y$
2013 → 3.3	2013 → 5.4	2013 → 8.7
2014 → 3.0	2014 → 5.5	2014 → 8.5
2015 → 3.1	2015 → 5.8	2015 → 8.9

The **lag** (or **backward shift** or **backshift**) **operator** is denoted by L (or by B). It shifts a time series so that the shifted time series lags one time unit behind, i.e.,

$$(Ly)(t+1)=y(t)\quad\forall t\in T.$$

<u>Example:</u>

y	Ly
2013 → 5.4	
2014 → 5.5	2014 → 5.4
2015 → 5.8	2015 → 5.5
	2016 → 5.8

If the domain of y is finite, it cannot be equal to the domain of Ly. For example, if y has domain {2013,2014,2015}, Ly has domain {2014,2015,2016}.

Whenever y and Ly do not have the same domain, it is not possible to define $y+Ly$ or $y-Ly$. To resolve this problem, we will henceforth assume that all time series have domain

$$\mathbf{Z}=\{\ldots,-2,-1,0,1,2,\ldots\}.$$

The n observed values are viewed as a finite segment of a doubly infinite series.

Manipulating the lag operator

Indicating the double application of L by L^2, we have

$$(L^2y)(t)=(L(Ly))(t)=(Ly)(t-1)=y(t-2).$$

In general,

$$(L^ky)(t)=y(t-k).$$

If $k<0$, the direction of the shift is reversed. For example,

$$(L^{-4}y)(t)=y(t+4).$$

The operator L^{-1} is called **lead operator**.

<u>Example:</u>

$L^{-1}y$	y	Ly	L^2y
⋮	⋮	⋮	⋮
$2012 \to 5.4$	⋮	⋮	⋮
$2013 \to 5.5$	$2013 \to 5.4$	⋮	⋮
$2014 \to 5.8$	$2014 \to 5.5$	$2014 \to 5.4$	⋮
⋮	$2015 \to 5.8$	$2015 \to 5.5$	$2015 \to 5.4$
⋮	⋮	$2016 \to 5.8$	$2016 \to 5.5$
⋮	⋮	⋮	$2017 \to 5.8$
⋮	⋮	⋮	⋮

OL

The lag operator is linear, i.e.,

$$L(x+y)=Lx+Ly$$

and

$$L(\lambda\cdot y)=\lambda\cdot(Ly).$$

This follows from

$$\begin{aligned}L(x+y)(t)&=(x+y)(t-1)\\&=x(t-1)+y(t-1)\\&=(Lx)(t)+(Ly)(t)\\&=(Lx+Ly)(t)\end{aligned}$$

and

$$\begin{aligned}(L(\lambda\cdot y))(t)&=(\lambda\cdot y)(t-1)\\&=\lambda\cdot y(t-1)\\&=\lambda\cdot(Ly(t))\\&=(\lambda\cdot(Ly))(t).\end{aligned}$$

The **differencing operator** Δ is defined by

$$\Delta y=y-Ly.$$

The repeated application of Δ is indicated by Δ^k, e.g.,

$$\begin{aligned}\Delta^2y&=\Delta(\Delta y)\\&=\Delta y-L(\Delta y)\\&=(y-Ly)-L(y-Ly)\\&=y-Ly-Ly+L^2y\\&=y-2Ly+L^2y\end{aligned}$$

Exercise: Show that $\Delta^3y=y-3Ly+3L^2y-L^3y$. O3

It will turn out to be very convenient to write an expression such as

$$y-3Ly+3L^2y-L^3y$$

in the form

$$(1-3L+3L^2-L^3)y,$$

where 1 denotes the identity operator L^0, which does not have any effect. Expressions like

$$1-3L+3L^2-L^3$$

are called **lag operator polynomials**.

Example: $\Delta y = y - Ly = (1-L)y$

Manipulating lag operator polynomials in the same way as polynomial functions we can "solve" the last exercise immediately:

$$\begin{aligned}\Delta^3 y &= (1-L)^3 y \\ &= (1-3L+3L^2-L^3)y \\ &= y-3Ly+3L^2y-L^3y\end{aligned}$$

Exercise: (i) Show that the approximation of the nonlinear function $g(r)=\log(1+r)$ obtained by using the first two terms of its Taylor series is given by the linear function $f(r)=r$.
(ii) Use R to sketch $f(r)$ and $g(r)$ for $-0.1 \le r \le 0.1$. OT

Let y be a time series with domain **Z**. The quantity

$$R(t)=\frac{y(t)-y(t-1)}{y(t-1)}$$

is called the **return** (or **growth rate**) from time $t-1$ to time t.

For a small return $R(t)$ we have

$$R(t) \approx \log(1+R(t)) = \log\left(1+\frac{y(t)-y(t-1)}{y(t-1)}\right) = \log\left(\frac{y(t)}{y(t-1)}\right).$$

The quantity

$$r(t)=\log\left(\frac{y(t)}{y(t-1)}\right)=\log(y(t))-\log(y(t-1))$$

is called **log return**.

Exercise: Compare *R* and *r* for the log GDP.

• Create a times series object **Y.ts** and use the function **lag**, which is well defined for time series objects (but unfortunately acts like a lead operator), to calculate *R*.

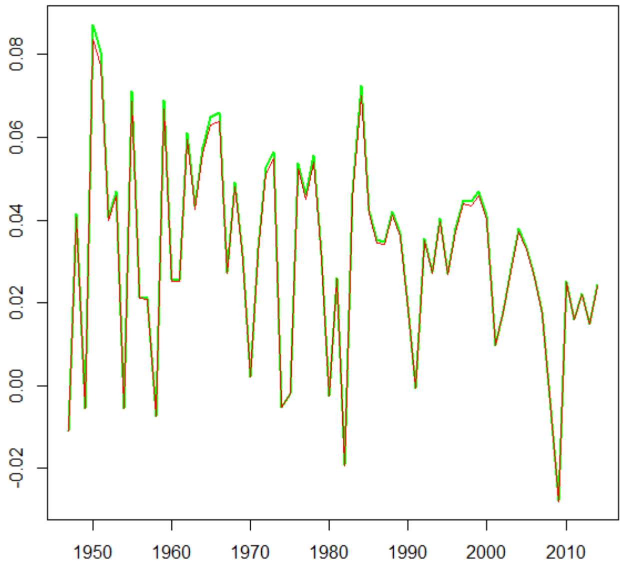
```
> Y.ts <- ts(Y,start=1946)
> R <- (Y.ts-lag(Y.ts,k=-1))/lag(Y.ts,k=-1)
>              # lead operator: k=1, lag operator: k=-1
```

• Calculate the differences of the log GDP.

```
> y.ts <- ts(y,start=1946)
> r <- y.ts-lag(y.ts,k=-1)
```

• Plot *R* and *r*.

```
> plot(R,type="l",xlab=" ",ylab="",col="green",lwd=2)
>                                            # double line width
> lines(r,col="red")
```



There is practically no difference between *R* and *r*.