A process ε is called an **autoregressive conditional heteroskedastic process of order 1** (ARCH(1) process) if it can be represented as

$$\varepsilon_t = \sigma_t e_t,$$

where the random variables e_t are i.i.d. with zero mean and unit variance and where σ_t satisfies

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2, \ \alpha_0 > 0, \ \alpha_1 > 0.$$

We have

$$\varepsilon_t^2 = \sigma_t^2 \ e_t^2 = (\alpha_0 + \alpha_1 \varepsilon_{t-1}^2) e_t^2 .$$

Substituting $(\alpha_0+\alpha_1 \varepsilon_{t-1}^2) e_t^2$ for ε_t^2,

$(\alpha_0+\alpha_1 \varepsilon_{t-2}^2) e_{t-1}^2$ for ε_{t-1}^2,

$(\alpha_0+\alpha_1 \varepsilon_{t-3}^2) e_{t-2}^2$ for ε_{t-2}^2, …

gives

$$\begin{aligned}
\varepsilon_t^2 &= (\alpha_0+\alpha_1 \varepsilon_{t-1}^2) e_t^2 \\
&= \alpha_0 e_t^2 + \alpha_1(\alpha_0+\alpha_1 \varepsilon_{t-2}^2) e_{t-1}^2 \ e_t^2 \\
&= \alpha_0(e_t^2 + \alpha_1 e_t^2 \ e_{t-1}^2) + \alpha_1^2 \ \varepsilon_{t-2}^2 \ e_{t-1}^2 \ e_t^2 \\
&= \alpha_0(e_t^2 + \alpha_1 e_t^2 \ e_{t-1}^2) + \alpha_1^2 (\alpha_0+\alpha_1 \varepsilon_{t-3}^2) e_{t-2}^2 \ e_{t-1}^2 \ e_t^2 \\
&= \alpha_0(e_t^2 + \alpha_1 e_t^2 \ e_{t-1}^2 + \alpha_1^2 \ e_t^2 \ e_{t-1}^2 \ e_{t-2}^2) + \alpha_1^3 \ \varepsilon_{t-3}^2 \ e_{t-2}^2 \ e_{t-1}^2 \ e_t^2 \\
&\ \ \vdots \\
&= \alpha_0 \sum_{j=0}^{k} \alpha_1^j e_t^2 \dots e_{t-j}^2 + \alpha_1^{k+1} \ \varepsilon_{t-(k+1)}^2 \ e_t^2 \dots e_{t-k}^2 .
\end{aligned}$$

For the existence of the causal representations

$$\varepsilon_t^2 = \alpha_0 \sum_{j=0}^{\infty} \alpha_1^j e_t^2 \dots e_{t-j}^2 \ , \ \varepsilon_t = e_t \sqrt{\alpha_0 \Big(1 + \sum_{j=1}^{\infty} \alpha_1^j e_{t-1}^2 \dots e_{t-j}^2 \Big)}$$

it is required that $|\alpha_1| < 1$. GR

Using the conditional expectation properties GC

- $E(E(y|x))=Ey$,
- $E(g(x)y|x)=g(x)E(y|x)$,
- x, y independent $\Rightarrow E(y|x)=Ey$,

we obtain for the ARCH(1) process ε with representation

$$\varepsilon_t=e_t\sqrt{\alpha_0(1+\sum_{j=1}^{\infty}\alpha_1^j e_{t-1}^2\dots e_{t-j}^2)}\,,$$

$$\begin{aligned}E(\varepsilon_t|e_{t-1},e_{t-2},\dots) &= E(e_t\sqrt{\alpha_0(1+\sum_{j=1}^{\infty}\alpha_1^j e_{t-1}^2\dots e_{t-j}^2)}\,|e_{t-1},e_{t-2},\dots)\\ &= \sqrt{\alpha_0(1+\sum_{j=1}^{\infty}\alpha_1^j e_{t-1}^2\dots e_{t-j}^2)}\,E(e_t|e_{t-1},e_{t-2},\dots)\\ &= \sqrt{\alpha_0(1+\sum_{j=1}^{\infty}\alpha_1^j e_{t-1}^2\dots e_{t-j}^2)}\;\underbrace{Ee_t}_{=0}\\ &= 0\end{aligned}$$

and

$$E\varepsilon_t = E(\underbrace{E(\varepsilon_t|e_{t-1},e_{t-2},\dots)}_{=0}) = 0.$$

Moreover, GV

$$\begin{aligned}\mathrm{Var}(\varepsilon_t) = E\varepsilon_t^2 = E\,\alpha_0\sum_{j=0}^{\infty}\alpha_1^j e_t^2\dots e_{t-j}^2 &= \alpha_0\sum_{j=0}^{\infty}\alpha_1^j\underbrace{Ee_t^2}_{=1}\dots\underbrace{Ee_{t-j}^2}_{=1}\\ &= \alpha_0\sum_{j=0}^{\infty}\alpha_1^j\\ &= \frac{\alpha_0}{1-\alpha_1}\end{aligned}$$

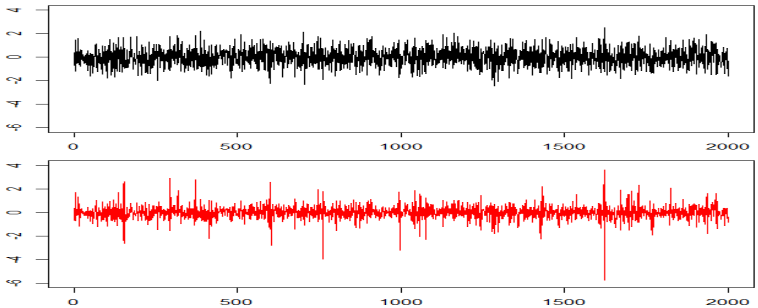
and the variance conditional on information up to time t-1 is given by

$$\begin{aligned}\mathrm{Var}(\varepsilon_t|e_{t-1},e_{t-2},\dots) &= E(\varepsilon_t^2|e_{t-1},e_{t-2},\dots)\\ &= E((\alpha_0+\alpha_1\varepsilon_{t-1}^2)e_t^2|e_{t-1},e_{t-2},\dots)\\ &= (\alpha_0+\alpha_1\varepsilon_{t-1}^2)E(e_t^2|e_{t-1},e_{t-2},\dots)\\ &= \sigma_t^2\underbrace{Ee_t^2}_{=1}.\end{aligned}$$

Exercise: (i) Simulate n=2000 observations from an ARCH(1) process with parameters α_0=0.1 and α_1=0.8.

```
n <- 2000; e <- rnorm(n)  # 2000 N(0,1) random numbers
a <- c(0.1,0.8)             # model parameters
s <- sqrt(a[1]/(1-a[2]))    # unconditional standard dev.
es <- e[1]*s                # first observation
for (i in 2:n) es[i] <- e[i]*sqrt(a[1]+a[2]*es[i-1]^2)
```

(ii) Compare the distributions of $\varepsilon_t=\sigma_t e_t$ and σe_t, which have the same unconditional variance σ^2.



The sample from the ARCH(1) model (red) exhibits not only serial correlation of volatility but also tail thickness.

(iii) Estimate the parameters α_0 and α_1.

```
install.packages("tseries")  # install package "tseries"
library(tseries)  # load package "tseries"
h <- garch(es,order=c(0,1))  # fit ARCH(1) model
a.est <- h$coef  # estimated model parameters
a.est
a0        a1
0.1042485 0.7492072
```

(iv) Estimate the unconditional standard deviation σ.

```
s2.est <- a.est[1]/(1-a.est[2])  # estimated variance
s.est <- sqrt(s2.est)  # estimated standard deviation
```

(v) Estimate the conditional standard deviations σ_t.

```
s.c.est <- s.est
for (i in 2:n) s.c.est[i] <- sqrt(a.est[1]+a.est[2]*es[i-1]^2)
s.c.est[1:4]
0.6447293 0.6833438 1.2869898 1.4910399 …
```

Alternatively:

```
h$fitted.values[,1]
NA 0.6833438 1.2869898 1.4910399 …
```

Exercise: Let x be an AR(p) process represented by $x_t=\phi_1 x_{t-1}+\ldots+\phi_p x_{t-p}+u_t$. Show that the AR($p$) process y with nonzero mean, which is obtained by adding a constant μ to each x_t, satisfies $y_t=\lambda+\phi_1 y_{t-1}+\ldots+\phi_p y_{t-p}+u_t$, where $\lambda=(1-\phi_1-\ldots-\phi_p)\mu$. GP

Exercise: Let a causal ARCH(1) process be given by

$$\varepsilon_t=\sigma_t e_t,\ \ \sigma_t^2=\alpha_0+\alpha_1\varepsilon_{t-1}^2 .$$

(i) Show that $E\varepsilon_t\varepsilon_{t-k}=0$ if $k\neq 0$. GE

(ii) Show that $Ew_t=0$ and $Ew_t w_{t-k}=0$ if $k\neq 0$, where $w_t=\sigma_t^2(e_t^2-1)$. GW

Hints: $E\varepsilon_t\varepsilon_{t-k}=E(E(\varepsilon_t\varepsilon_{t-k}\,|\,e_{t-1},e_{t-2},\ldots))$

$Ew_t w_{t-k}=E(E(w_t w_{t-k}\,|\,e_{t-1},e_{t-2},\ldots))$

Note: It appears from

$$\varepsilon_t^2=\sigma_t^2+\sigma_t^2(e_t^2-1)=\sigma_t^2+w_t=\alpha_0+\alpha_1\varepsilon_{t-1}^2+w_t$$

that the squares ε_t^2 follow an AR(1) process with nonzero mean.

A zero-mean white noise ε is called an **ARCH process of order *p*** (ARCH(p) process) if it can be represented as

$$\varepsilon_t=\sigma_t e_t,$$

where the random variables e_t are i.i.d. with zero mean and unit variance and where σ_t^2 satisfies

$$\sigma_t^2=\alpha_0+\alpha_1\varepsilon_{t-1}^2+\ldots+\alpha_p\varepsilon_{t-p}^2,\ \ \alpha_0>0,\ \alpha_1,\ldots,\alpha_p\geq 0.$$

For causality it is required that ε_t is a function of $e_t,e_{t-1},\ldots$

It appears from

$$\varepsilon_t^2=\sigma_t^2+\sigma_t^2(e_t^2-1)=\sigma_t^2+w_t=\alpha_0+\alpha_1\varepsilon_{t-1}^2+\ldots+\alpha_p\varepsilon_{t-p}^2+w_t,$$

where $w_t=\sigma_t^2(e_t^2-1)$, that the squares ε_t^2 follow an AR(p) process with nonzero mean.

Exercise: Show that $Ew_t=0$. G0

Exercise: Show that the variance of ε_t is given by

$$\sigma_\varepsilon^2=\frac{\zeta}{1-\alpha_1-\ldots-\alpha_p}.$$

Hint: Start with $\varepsilon_t^2=\alpha_0+\alpha_1\varepsilon_{t-1}^2+\ldots+\alpha_p\varepsilon_{t-p}^2+w_t$ and take expectations. G2

A zero-mean white noise ε is called a **generalized ARCH process of order (*p*,*q*)** (GARCH(p,q) process) if it can be represented as

$$\varepsilon_t = \sigma_t e_t,$$

where the random variables e_t are i.i.d. with zero mean and unit variance and σ_t^2 satisfies

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \ldots + \alpha_p \varepsilon_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \ldots + \beta_q \sigma_{t-q}^2,$$

$$\alpha_0 > 0,\ \alpha_1, \ldots, \alpha_p, \beta_1, \ldots,\ \beta_q \geq 0.$$

Adding $w_t = \varepsilon_t^2 - \sigma_t^2$ to both sides of this equation we obtain

$$\begin{aligned}\varepsilon_t^2 &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \ldots + \alpha_p \varepsilon_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \ldots + \beta_q \sigma_{t-q}^2 + w_t \\ &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \ldots + \alpha_p \varepsilon_{t-p}^2 + \beta_1 \varepsilon_{t-1}^2 + \ldots + \beta_q \varepsilon_{t-q}^2 + w_t \\ &\qquad - \beta_1(\varepsilon_{t-1}^2 - \sigma_{t-1}^2) - \ldots - \beta_q(\varepsilon_{t-q}^2 - \sigma_{t-q}^2) \\ &= \alpha_0 + (\alpha_1 + \beta_1)\varepsilon_{t-1}^2 + (\alpha_2 + \beta_2)\varepsilon_{t-2}^2 + \ldots - \beta_1 w_{t-1} - \ldots - \beta_q w_{t-q} + w_t,\end{aligned}$$

hence it appears that the squares ε_t^2 follow an ARMA(s,q) process with s=max(p,q) and nonzero mean.

Exercise: Show that for a GARCH(1,1) model the unconditional variance is given by

$$E\varepsilon_t^2 = \frac{\alpha_0}{1-(\alpha_1+\beta_1)}.$$

GG

Exercise: Use a GARCH(1,1) model to forecast the conditional variance.

Hint: $\sigma_{t+1}^2 = \alpha_0 + \alpha_1 \varepsilon_t^2 + \beta_1 \sigma_t^2$

```
g <- garch(es,order=c(1,1))   # fit GARCH(1,1) model
a.est <- g$coef[1:2]; b.est <- g$coef[3]
s2.c.est <- g$fitted.values[n,1]^2
s2.c.fc <- a.est[1]+a.est[2]*es[n]^2+b.est*s2.est
```