COMPLEX NUMBERS

Since no real number x satisfies $x^2<0$, we introduce an imaginary number i with the convenient property

$$i^2=-1.$$

We assume that i can be manipulated using standard rules of algebra. For example,

$$(4i+5)(2-i)=8i-4i^2+10-5i=8i+4+10-5i=14+3i.$$

Suppose that x and y are real numbers. Then

$$z=x+iy$$

is called a **complex number** with **real part** x and **imaginary part** y.

The **modulus** of a complex number $z=x+iy$ is given by

$$|Z| = \sqrt{x^2 + y^2}$$

and its **complex cunjugate** by

$$\bar{z} = x - iy \ .$$

We have

$$z\bar{z} = |\mathrm{z}|^2 \ ,$$

because

$$z\bar{z} = (x+iy)(x-iy)=x^2-ixy+iyx-i^2y^2=x^2+y^2=|\mathrm{z}|^2 \ .$$

Exercise: Let z_1 and z_2 be complex numbers. Show that

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$,

(ii) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$, A1

(iii) $\overline{\overline{z_1}} = z_1$.

Solution of (i):

If $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, then

$$\overline{z_1 + z_2} = \overline{(x_1 + iy_1) + (x_2 + iy_2)}$$
$$= \overline{(x_1 + x_2) + i(y_1 + y_2)}$$
$$= (x_1+x_2)-i(y_1+y_2)=(x_1-iy_1)+(x_2-iy_2)$$
$$= \overline{(x_1 + iy_1)} + \overline{(x_2 + iy_2)}$$
$$= \overline{z_1} + \overline{z_2}$$

Exercise: Show that

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$,

(ii) $e^{cx} = 1 + \frac{cx}{1!} + \frac{c^2x^2}{2!} + \frac{c^3x^3}{3!} + \frac{c^4x^4}{4!} + \frac{c^5x^5}{5!} + \dots$,

(iii) $e^{ix} = 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \dots$, A2

(iv) $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, A3

and (v) $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ A4

Solution of (i):

The Taylor series of $f(x)=e^x$ around $x_0=0$ is given by

$$f(x)= \frac{f(x_0)}{0!}(x-x_0)^0+ \frac{f'(x_0)}{1!}(x-x_0)^1+ \frac{f''(x_0)}{2!}(x-x_0)^2+\dots$$
$$= \frac{e^0}{0!}(x-0)^0+ \frac{e^0}{1!}(x-0)^1+ \frac{e^0}{2!}(x-0)^2+\dots$$
$$=1+ \frac{1}{1!}x+ \frac{1}{2!}x^2+\dots$$

The **Euler relation**

$$e^{ix} = \cos(x) + i\sin(x)$$

can be obtained by comparing the Taylor series of e^{ix} with the Taylor series of $\sin(x)$ and $\cos(x)$.

$$e^{ix} = 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= \cos(x) + i\sin(x)$$

The function e^{ix} is therefore periodic with period 2π.

Moreover,

$$\left|e^{ix}\right| = \left|\cos(x) + i\sin(x)\right| = \sqrt{\cos^2(x) + \sin^2(x)} = 1$$

and

$$e^{i(2k\pi)} = \cos(2k\pi) + i\sin(2k\pi) = 1 + i0 = 1.$$

Exercise: Use the Euler relation to show that

$$\cos(\omega) = \frac{e^{i\omega} + e^{-i\omega}}{2} \qquad \text{A5}$$

and $$\sin(\omega) = \frac{e^{i\omega} - e^{-i\omega}}{2i}. \qquad \text{A6}$$

Exercise: Show that $\overline{e^{ix}} = e^{-ix}$. AE

Exercise: Show that

$$\sin^2(x) + \cos^2(x) = 1, \qquad \text{A7}$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y). \qquad \text{A8}$$

and $$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y). \qquad \text{A9}$$

<u>Exercise:</u> Derive the formula AG

$$\sum_{t=0}^{n} q^t = \frac{1-q^{n+1}}{1-q}$$

for the sum of a geometric series.

Hint: Multiply each side of the equation

$$\sum_{t=0}^{n} q^t = 1+q+q^2+\ldots+q^n$$

by (1-q).

<u>Exercise:</u> Show that A0

$$\sum_{t=1}^{n} e^{i\omega_k t} = \sum_{t=1}^{n} \cos(\omega_k t) = \sum_{t=1}^{n} \sin(\omega_k t) = 0\ ,$$

if $0 < \omega_k = \frac{2\pi k}{n} < 2\pi$.

Solution:

$$\sum_{t=1}^{n} e^{i\omega_k t} = \sum_{t=1}^{n} (e^{i\omega_k})^t = e^{i\omega_k} \sum_{t=0}^{n-1} (e^{i\omega_k})^t$$

$$= e^{i\omega_k} \frac{1-e^{i\omega_k n}}{1-e^{i\omega_k}} = e^{i\omega_k} \frac{1-e^{i(2\pi k)}}{1-e^{i\omega_k}} = e^{i\omega_k} \frac{1-1}{1-e^{i\omega_k}} = 0$$

$$\Rightarrow \quad 0 = \sum_{t=1}^{n} e^{i\omega_k t} = \sum_{t=1}^{n} \{\cos(\omega_k t) + i \sin(\omega_k t)\}$$

$$= \sum_{t=1}^{n} \cos(\omega_k t) + \mathrm{i} \sum_{t=1}^{n} \sin(\omega_k t)$$

$$\Rightarrow \quad \sum_{t=1}^{n} \cos(\omega_k t) = 0, \ \sum_{t=1}^{n} \sin(\omega_k t) = 0$$

Exercise: Show that AQ

(i) $\sum_{t=1}^{n} \cos^2(\omega_k t) = \frac{n}{2}$ and (ii) $\sum_{t=1}^{n} \sin^2(\omega_k t) = \frac{n}{2}$,

if $0 < \omega_k = \frac{2\pi k}{n} < \pi$.

Solution of (i):

$$\sum_{t=1}^{n} \cos^2(\omega_k t) = \sum_{t=1}^{n} \tfrac{1}{4}(e^{i\omega_k t} + e^{-i\omega_k t})^2$$

$$= \tfrac{1}{4}\sum_{t=1}^{n}(e^{i(2\omega_k)t} + 2e^0 + e^{-i(2\omega_k)t})$$

$$= \tfrac{1}{4}\sum_{t=1}^{n} e^{i\frac{2\pi(2k)}{n}t} + \tfrac{n}{2} + \overline{\tfrac{1}{4}\sum_{t=1}^{n} e^{i\frac{2\pi(2k)}{n}t}}$$

$$= 0 + \tfrac{n}{2} + \overline{0} = \tfrac{n}{2}.$$

Exercise: Show that

$$\sum_{t=1}^{n} \sin(\omega_j t)\cos(\omega_k t) = 0,$$ AO

$$\sum_{t=1}^{n} \cos(\omega_j t)\cos(\omega_k t) = 0, \text{ if } j \neq k,$$ AC

and $$\sum_{t=1}^{n} \sin(\omega_j t)\sin(\omega_k t) = 0, \text{ if } j \neq k,$$ AS

where $0 < \omega_j = \frac{2\pi j}{n}$, $\omega_k = \frac{2\pi k}{n} < \pi$.

For any ω we have AP

$$\left|\sum_{t=1}^{n} x_t e^{i\omega t}\right|^2 = \sum_{t=1}^{n} x_t e^{i\omega t}\ \overline{\sum_{t=1}^{n} x_t e^{i\omega t}} = \sum_{t=1}^{n} x_t e^{i\omega t}\ \sum_{t=1}^{n} x_t e^{-i\omega t}$$

$$=(x_1 e^{i\omega 1} + x_2 e^{i\omega 2} + x_3 e^{i\omega 3} + \ldots)(x_1 e^{-i\omega 1} + x_2 e^{-i\omega 2} + x_3 e^{-i\omega 3} + \ldots)$$

$$=(x_1^2 + x_2^2 + \ldots)e^{i\omega 0}$$

$$+(x_1x_2+x_2x_3+\ldots)e^{-i\omega 1}+(x_1x_3+x_2x_4+\ldots)e^{-i\omega 2}+\ldots$$

$$+(x_2x_1+x_3x_2+\ldots)e^{i\omega 1}+(x_3x_1+x_4x_2+\ldots)e^{i\omega 2}+\ldots$$

$$=(x_1^2 + x_2^2 + \ldots)e^{i\omega 0}$$

$$+(x_1x_2+x_2x_3+\ldots)(e^{-i\omega 1}+ e^{i\omega 1})+(x_1x_3+x_2x_4+\ldots)(e^{-i\omega 2}+ e^{i\omega 2})+\ldots$$

$$=(x_1^2 + x_2^2 + \ldots)$$

$$+(x_1x_2+x_2x_3+\ldots)2\cos(\omega 1)+(x_1x_3+x_2x_4+\ldots)2\cos(\omega 2)+\ldots$$

For a Fourier frequency $0<\omega_k=\frac{2\pi k}{n}<2\pi$ we have

$$\frac{1}{n}\left|\sum_{t=1}^{n} x_t e^{i\omega_k t}\right|^2 = \frac{1}{n}\left|\sum_{t=1}^{n} x_t e^{i\omega_k t} - \bar{x}\sum_{t=1}^{n} e^{i\omega_k t}\right|^2$$

$$= \frac{1}{n}\left|\sum_{t=1}^{n} (x_t - \bar{x}) e^{i\omega_k t}\right|^2$$

$$= \underbrace{\frac{1}{n}\sum_{t=1}^{n}(x_t-\bar{x})^2}_{\hat{\gamma}(0)} + \sum_{j=1}^{n-1}\Big(\underbrace{\frac{1}{n}\sum_{t=1}^{n-j}(x_t-\bar{x})(x_{t+j}-\bar{x})}_{\hat{\gamma}(j)}\Big)2\cos(\omega_k j),$$

hence

$$\underbrace{\frac{1}{2\pi n}\left|\sum_{t=1}^{n} x_t e^{i\omega_k t}\right|^2}_{I(\omega_k)} = \frac{1}{2\pi}\left(\hat{\gamma}(0)+2\sum_{j=1}^{n-1}\hat{\gamma}(j)\cos(\omega_k j)\right).$$

Exercise: Show that AR

$$I(\omega_k) = \frac{1}{2\pi} \sum_{j=-(n-1)}^{n-1} \hat{\gamma}(j) e^{-i\omega_k j},$$

where for negative j, $\hat{\gamma}(j)$ is defined by $\hat{\gamma}(|j|)$.